An Explicit Formula of Subblock Occurrences for the p -Adic Expansion

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Let $b(n, w)$ be the number of occurrences of subblock w in the p -adic expansion of $n \in N$ and set $B(N, w) = \sum_{n=0}^{N-1} b(n, w)$ for $N \in N$. Properties of the value of $B(N, w)$ were investigated by Prodinger [8] (for $p = 2$) and by Kirschenhofer [3] (for a general p). In this paper we give a simple representation of $B(N, w)$ by means of previous result [5] on the explicit formula of generalized power and exponential sums of digital sums.

KEYWORDS: digital sum problem, singular function, multinomial measure

1 Introduction

Let p be a positive integer greater than 1 and denote the p -adic expansion of $n \in N$ by $n = \sum_{i \geq 0} \alpha_i(n)p^i$, where $\alpha_i(n) \in \{0, 1, \dots, p-1\}$. We set $s(n, l)_{(p)} = \sum_{i \geq 0} \mathbf{1}_{\{\alpha_i(n)=l\}}$ for $l = 1, 2, \dots, p-1$, and $s(n) = \sum_{l=1}^{p-1} ls(n, l)_{(p)}$. We define the power sum and the exponential sum of $s(n)$ by

$$S_k(N) = \sum_{n=0}^{N-1} s(n)^k, \quad k \in N,$$

$$F(\xi, N) = \sum_{n=0}^{N-1} e^{\xi s(n)}, \quad \xi \in R$$

for $N \in N$. The problems concerned with $S_k(N)$, $F(\xi, N)$ are called digital sum problems and investigated by many authors. For historical survey, see [10], [7]. Trollope [11] obtained an explicit formula of $S_1(N)$ for $p = 2$ and Delange [2] gave its elegant proof by use of the Takagi function. Coquet [1] studied an explicit formula of $S_k(N)$ for $k \geq 2$, $p = 2$ and obtained an explicit one. For an explicit formula of $F(\xi, N)$, Stein [9] gave a one. In [4], we have obtained explicit formulas of $S_k(N)$ and an explicit formula of $F(\xi, N)$ by use of a probabilistic method.

In [5], we have introduced a generalization of $S_k(N)$ and $F(\xi, N)$, which contain information per digit and obtained explicit formulas of them. We will apply these results to counting the number of occurrences of subblocks of digits.

Let $w = (a_{d-1}, a_{d-2}, \dots, a_0)$, $a_i \in \{0, 1, \dots, p-1\}$, $i = 0, 1, \dots, d-1$, be a subblock of digits (a word) with length $d > 1$. Set $q = p^d$ and $\tilde{w} = a_{d-1}p^{d-1} + a_{d-2}p^{d-2} + \dots + a_0$, that is, $\tilde{w}$ is a numeric number of w . For a given word w and $N \in N$, we set

$$(1) \quad B(N, w) = \sum_{n=0}^{N-1} b(n, w),$$

where

$$(2) \quad b(n, w) = \sum_{l \geq d} \mathbf{1}_{\{(\alpha_{l-1}(n), \alpha_{l-2}(n), \dots, \alpha_{l-d}(n)) = w\}}.$$

The following theorem was obtained by Prodinger [8] for $p = 2$ and by Kirschenhofer [3] for a general p .

Theorem 1.1 *Let $w = (a_{d-1}, a_{d-2}, \dots, a_0)$ be a word with length d and $a_{d-1} \neq 0$. Then we have*

$$\frac{1}{N} B(N, w) = \frac{\log_p N - (d-1)}{q} + H_w(\log_p N) + \frac{E_w(\log_p N)}{N}.$$

Here H_w is a continuous periodic function of period 1 with $H_w(0) = 0$ and E_w is a bounded function such that

$$-(1 - p^{1-d}) \frac{p^{-d}(p^d - \tilde{w} - 1)}{p-1} \leq E_w \leq (1 - p^{1-d}) \frac{p^{-d}\tilde{w}}{p-1}.$$

In this paper, we give a simple representation of H_w and E_w . If $w = (a_{d-1}, a_{d-2}, \dots, a_0)$ with $a_{d-1} = 0$, the

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above definition of $b(n, w)$ also counts the subblocks such that $(\alpha_{i-1}(n), \alpha_{i-2}(n), \dots, \alpha_{i-d}(n)) = w$ with $i > [\log_p n] + 1$. So we introduce the following definitions:

$$(3) \quad \bar{b}(n, w) = \sum_{[\log_p n] + 1 \leq i \leq d} \mathbf{1}_{\{(\alpha_{i-1}(n), \alpha_{i-2}(n), \dots, \alpha_{i-d}(n)) = w\}}$$

and

$$(4) \quad \bar{B}(N, w) = \sum_{n=0}^{N-1} \bar{b}(n, w),$$

which does not count such blocks. We also give an explicit formula of $\bar{B}(N, w)$.

2 Preliminaries

For $n = \sum_{i \geq 0} \alpha_i(n)p^i$ and $l = 1, 2, \dots, p-1$, set

$$s(n, l)_{(p)} = \sum_{i \geq 0} \mathbf{1}_{\{\alpha_i(n) = l\}}$$

and

$$s(n)_{(p)} = (s(n, 1)_{(p)}, s(n, 2)_{(p)}, \dots, s(n, p-1)_{(p)}).$$

We define

$$(5) \quad S_k(N)_{(p)} = \sum_{n=0}^{N-1} s(n)_{(p)}^k = \sum_{n=0}^{N-1} \prod_{l=1}^{p-1} s(n, l)_{(p)}^{k_l}, \quad k = (k_1, \dots, k_{p-1}) \in N^{p-1},$$

and

$$(6) \quad F(\xi, N)_{(p)} = \sum_{n=0}^{N-1} e^{(\xi, s(n)_{(p)})} = \sum_{n=0}^{N-1} e^{\sum_{l=1}^{p-1} \xi_l s(n, l)_{(p)}}, \quad \xi = (\xi_1, \dots, \xi_{p-1}) \in \mathbf{R}^{p-1}.$$

For later convenience, we set $\xi_0 = 0$.

Let $I = I_{0,0} = [0, 1]$ and $I_{n,j} = [j/p^n, (j+1)/p^n]$, $j = 0, 1, \dots, p^n - 2$, $I_{n,p^n-1} = [(p^n - 1)/p^n, 1]$ for $n = 1, 2, 3, \dots$. Let $\mathbf{r} = (r_0, r_1, \dots, r_{p-2})$ be a vector such that $0 < r_l < 1$ for $l = 0, 1, \dots, p-2$ and $\sum_{l=0}^{p-2} r_l < 1$ and set $r_{p-1} = 1 - \sum_{l=0}^{p-2} r_l$. The probability measure μ_r on I defined by

$$\mu_r(I_{n+1,pj+l}) = r_l \mu_r(I_{n,j})$$

for $n = 0, 1, 2, \dots$, $j = 0, 1, \dots, p^n - 1$, $l = 0, 1, \dots, p-1$, is said to be a multinomial measure. We denote the distribution function of μ_r by $L(\mathbf{r}, \cdot)$:

$$L(\mathbf{r}, x) = \mu_r([0, x]), \quad x \in I.$$

For $N \in \mathbf{N}$, set $t = \log_p N$, and denote its integer part by $[t]$ and its decimal part by $\{t\}$. We now set

$$(7) \quad a(\mathbf{m}, x, \xi_0) = \frac{\partial^{|\mathbf{m}|}}{\partial \xi_1^{m_1} \cdots \partial \xi_{p-1}^{m_{p-1}}} \left(\frac{1 + e^{\xi_1} + \cdots + e^{\xi_{p-1}}}{1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}}} \right)^x \Big|_{\xi = \xi_0},$$

where $\mathbf{m} = (m_1, \dots, m_{p-1}) \in \mathbf{Z}_+^{p-1}$, $|\mathbf{m}| = m_1 + \cdots + m_{p-1}$, $x \in \mathbf{R}$ and $\xi_0 = (\xi_{0,1}, \dots, \xi_{0,p-1})$ and denote $(\partial^l / \partial x^l) a(\mathbf{m}, x, \xi_0)$ by $a^{(l)}(\mathbf{m}, x, \xi_0)$.

Theorem 2.1 ([5]) *We have*

$$\begin{aligned} \frac{\partial^{|\mathbf{k}|}}{\partial \xi_1^{k_1} \cdots \partial \xi_{p-1}^{k_{p-1}}} F(\xi, N)_{(p)} \Big|_{\xi = \xi_0} &= (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^t \\ &\quad \times \sum_{l=0}^{|\mathbf{k}|} H_{\mathbf{k},l}(t, \xi_0) \left(\frac{t}{1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}}} \right)^l. \end{aligned}$$

Here $H_{\mathbf{k},l}(x, \xi)$ is a continuous periodic function of period 1 with respect to x defined by

$$\begin{aligned} H_{\mathbf{k},l}(x, \xi_0) &= (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^l \\ &\quad \times \sum_{j_1=0}^{(|\mathbf{k}|-l) \wedge k_1} \cdots \sum_{j_{p-1}=0}^{(|\mathbf{k}|-l-(j_1+\cdots+j_{p-2})-l) \wedge k_{p-1}} \\ &\quad \times \binom{k_1}{j_1} \cdots \binom{k_{p-1}}{j_{p-1}} (1 + e^{\xi_{0,1}} + \cdots + e^{\xi_{0,p-1}})^{1-\{x\}} \end{aligned}$$

$$\times \frac{1}{l!} a^{(l)}(k-j, 1-\{x\}, \xi_0) \frac{\partial^{[j]}}{\partial \xi_1^{[j_1]} \cdots \partial \xi_{p-1}^{[j_{p-1}]}} L\left(r, \frac{1}{p^{1-\{x\}}}\right) \Big|_{\xi=\xi_0}$$

with $r_l = e^{\xi_l}/(1 + e^{\xi_1} + e^{\xi_2} + \cdots + e^{\xi_{p-1}})$, $l = 0, 1, \dots, p-1$. In particular, we have

$$S_k(N)_{(p)} = N \sum_{l=0}^{[k]} H_{k,l}(t, 0) \left(\frac{t}{p}\right)^l.$$

Corollary 2.1 For the unit vector $e_j = (0, \dots, 0, \underset{(j)}{1}, 0, \dots, 0)$, we have

$$(8) \quad S_{e_j}(N)_{(p)} = N \left(\frac{[t] + 1}{p} + p^{1-\{t\}} \frac{\partial}{\partial \xi_j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi=0} \right).$$

3 Results

Theorem 3.1 Let $w = (a_{d-1}, a_{d-2}, \dots, a_0)$ be a word with length d and $a_i \neq 0$ for some i . Then we have

$$\frac{1}{N} B(N, w) = \frac{\log_p N - (d-1)}{q} + H_w(\log_p N) + \frac{E_w(\log_p N)}{N}.$$

Here H_w is a continuous periodic function of period 1 with respect to t defined by

$$H_w(t) = -\frac{1}{q} \left(\sum_{i=0}^{d-1} \left\{ \frac{t-i}{d} \right\} - \frac{d-1}{2} \right) + \sum_{i=0}^{d-1} q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} L\left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) \Big|_{\xi=0} + \frac{d}{q}$$

for $\xi = (\xi_1, \xi_2, \dots, \xi_{q-1}) \in \mathbf{R}^{q-1}$, $r = (r_0, r_1, \dots, r_{q-2})$ with $r_l = e^{\xi_l}/(1 + e^{\xi_1} + e^{\xi_2} + \cdots + e^{\xi_{q-1}})$, $l = 0, 1, \dots, q-1$ and

$$E_w(t) = \frac{1}{q} \sum_{i=0}^{d-1} \left(\left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) - p^i \mathbf{1}_{\{p^d((N/p^i) - \lfloor N/p^i \rfloor) > \bar{w}\}} \right).$$

Theorem 3.2 We have

$$\frac{1}{N} \bar{B}(N, w) = \bar{H}_w(\log_p N) + \frac{\bar{E}_w(\log_p N)}{N}.$$

Here $\bar{H}_w$ is a continuous periodic function of period 1 with respect to t defined by

$$\bar{H}_w(t) = H_w(t) - \sum_{i=0}^{d-1} \left(\frac{q^{-\{(t-i)/d\}}}{q-1} + q^{-\{(t-i)/d\}} \{q^{\{(t-i)/d\}} \mathbf{1}_{\{[q^{\{(t-i)/d\}}] = \bar{w}\}} + q^{-\{(t-i)/d\}} \mathbf{1}_{\{[q^{\{(t-i)/d\}}] > \bar{w}\}} \right)$$

and

$$\bar{E}_w(t) = E_w(t) + \sum_{i=0}^{d-1} p^i \{q^{\{(t-i)/d\}} \mathbf{1}_{\{[q^{\{(t-i)/d\}}] = \bar{w}\}} + \frac{1}{p-1}.$$

Remark 3.1 Delange [2] calculated the Fourier series of the periodic part of S_1 using the expansion of the Takagi function $T(x)$, that is,

$$(9) \quad T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi(2^{n-1}x)$$

for $x \in [0, 1]$ where $\psi(x) = |2x - 2[x + 1/2]|$. Noticing that H_w has an expansion similar to (9), Prodinger [8] (for $p = 2$) and Kirschenhofer [3] (for a general p) gave a representation of the Fourier series of H_w by use of the z -function of Hurwitz. We have already shown that $(\partial/\partial \xi_i) L(r, x)|_{\xi=0}$ has an expansion similar to (9) in [6, Section 5]. So we can also calculate the Fourier series of H_w via our representation by means of a technique of Delange.

4 Proof of Theorems

In the following we use the notations $t_q = \log_q N = t/d$, $t(i) = \log_q(1 + \lfloor N/p^i \rfloor)$ and $\overline{t(i)} = (\log_q \lfloor N/p^i \rfloor) \mathbf{1}_{\{N \geq p^i\}}$ for $N \in \mathbf{N}$ and $i = 0, 1, \dots, d-1$. We use the convention $\infty \cdot 0 = 0$.

Lemma 4.1 1) We have

- (i) $\lfloor \overline{t(i)} \rfloor = \left\lfloor \frac{t-i}{d} \right\rfloor$,
- (ii) $\left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) q^{1-t(i)} \mathbf{1}_{\{N \geq p^i\}} = \left\lfloor \frac{N}{p^i} \right\rfloor q^{[t(i)] - \lfloor \overline{t(i)} \rfloor + 1 - \{ \overline{t(i)} \}},$

$$(iii) \quad \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} = \left(\frac{N}{p^i} q^{1-\{(t-i)/d\}} \right) \mathbf{1}_{\{N \geq p^i\}}.$$

2) One of the following statements holds:

$$(i) \quad \widehat{[t(i)]} = [\overline{t(i)}] \text{ for any } i.$$

(ii) $\widehat{[t(i)]} = [\overline{t(i)}] + 1$ for some i . In this case, we have $\{\widehat{t(i)}\} = 0, 1 + [N/p^i] = q^{1+\{(t-i)/d\}}$ and $\widehat{[t(j)]} = [\overline{t(j)}]$ for any $j \neq i$.

Proof. 1) (i) It suffices to show the equality for $N \geq p^i$. Since $N = \sum_{k=0}^{[t]-i} \alpha_{[t]-k} p^{[t]-k}$, $\alpha_{[t]-k} \in \{0, 1, \dots, p-1\}$ with $k = 0, 1, \dots, [t]$ and $\alpha_{[t]} \neq 0$, we have

$$\left\lfloor \frac{N}{p^i} \right\rfloor = \sum_{k=0}^{[t]-i} \alpha_{[t]-k} p^{[t]-i-k}.$$

Then

$$\log_q \left\lfloor \frac{N}{p^i} \right\rfloor = \log_q \left(p^{[t]-i} \sum_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k} \right) = \frac{[t] - i + \log_p \sum_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k}}{d}.$$

As $0 \leq \log_p \sum_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k < 1$, we obtain

$$[\overline{t(i)}] = \left\lfloor \frac{[t] - i + \log_p \sum_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k}}{d} \right\rfloor = \left\lfloor \frac{[t] - i}{d} \right\rfloor = \left\lfloor \frac{t - i}{d} \right\rfloor.$$

1) (ii), (iii) are obviously from definitions of $\widehat{t(i)}$, $\overline{t(i)}$ and 1) (i).

2) As $\log_q(1 + [N/p^i]) = \log_q(p^{[t]-i}(\sum_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k + 1/(p^{[t]-i})))$, we have

$$\widehat{[t(i)]} = \left\lfloor \frac{[t] - i + \log_p \left(\sum_{k=0}^{[t]-i} \frac{\alpha_{[t]-k}}{p^k} + \frac{1}{p^{[t]-i}} \right)}{d} \right\rfloor.$$

If $\widehat{[t(i)]} = [\overline{t(i)}] + 1$ for some i , we know that $\log_p(\sum_{k=0}^{[t]-i} (\alpha_{[t]-k})/p^k + 1/(p^{[t]-i})) = 1$, and $[t] - i + 1$ divides by d . Then we have $\alpha_k = p - 1$ for $k = [t], [t] - 1, \dots, i$ and $\{\widehat{t(i)}\} = 0$. Moreover, as

$$\left\lfloor \frac{[t] - i + 1}{d} \right\rfloor = \left\lfloor \frac{[t] - i}{d} \right\rfloor + 1 = \frac{[t] - i + 1}{d},$$

we obtain

$$1 + \left\lfloor \frac{N}{p^i} \right\rfloor = p^{[t]-i+1} = q^{1+\{([t]-i)/d\}} = q^{1+\{(t-i)/d\}}.$$

Since $[t] - j + 1$ does not divide by d for $j \neq i$, we derive the last assertion of (ii).

Proof of Theorem 3.1. Since

$$b(n, w) = \sum_{i=0}^{d-1} s\left(\left\lfloor \frac{n}{p^i} \right\rfloor, \tilde{w}\right)_{(q)},$$

we have

$$(10) \quad B(N, w) = \sum_{i=0}^{d-1} \sum_{n=0}^{N-1} s\left(\left\lfloor \frac{n}{p^i} \right\rfloor, \tilde{w}\right)_{(q)}.$$

We now calculate the right-hand side of (10). By Corollary 2.1, we can easily derive

$$(11) \quad S_{e_*}(N)_{(q)} = N \left(\frac{[t_q] + 1}{q} + q^{1-\{t_q\}} \frac{\partial}{\partial \xi_{\tilde{w}}} L\left(r, \frac{1}{q^{1-\{t_q\}}}\right) \Big|_{\xi=0} \right).$$

By use of (11) and Lemma 4.1, we obtain for $N \geq p^i$

$$\begin{aligned} \sum_{n=0}^{N-1} s\left(\left\lfloor \frac{n}{p^i} \right\rfloor, \tilde{w}\right)_{(q)} &= p^i S_{e_*} \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right)_{(q)} - \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) s\left(\left\lfloor \frac{N}{p^i} \right\rfloor, \tilde{w}\right)_{(q)} \\ &= p^i S_{e_*} \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right)_{(q)} - \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(S_{e_*} \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right)_{(q)} - S_{e_*} \left(\left\lfloor \frac{N}{p^i} \right\rfloor \right)_{(q)} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) S_{e^*} \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right)_{(q)} + \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) S_{e^*} \left(\left\lfloor \frac{N}{p^i} \right\rfloor\right)_{(q)} \\
&= \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(\frac{[\widehat{t(i)}] + 1}{q} + q^{1-\{t(i)\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{t(i)\}}}\right) \Big|_{\xi=0} \right) \\
&\quad + \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor \left(\frac{[\overline{t(i)}] + 1}{q} + q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \Big|_{\xi=0} \right) \\
&= \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right) \frac{[\widehat{t(i)}] + 1}{q} + \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor \frac{[\overline{t(i)}] + 1}{q} \\
&\quad + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right) q^{1-\{t(i)\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{t(i)\}}}\right) \Big|_{\xi=0} \\
&\quad + \left(p^i - N + p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \Big|_{\xi=0} \\
&= \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \frac{N}{q} \left(\frac{t-i}{d} - \left\{ \frac{t-i}{d} \right\} + 1 \right) \\
&\quad + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor \left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{t(i)\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{t(i)\}}}\right) \Big|_{\xi=0} \right. \\
&\quad \left. - q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1 + \{(t-i)/d\}}}\right) \Big|_{\xi=0} \right) \\
&\quad + N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) \Big|_{\xi=0} \\
&\quad - N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left(L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) - L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \right) \Big|_{\xi=0} \\
&\quad + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left(L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1 + \{(t-i)/d\}}}\right) - L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \right) \Big|_{\xi=0}.
\end{aligned}$$

For $N < p^i$, since $[N/p^i] = [\widehat{t(i)}] = [\overline{t(i)}] = 0$ and $L(r, 1/q^{1-\{(t-i)/d\}})|_{\xi=0} = -1/q^2$, same equality holds.

By the definition of L (for details, see [6, Section 2]), we have

$$\begin{aligned}
&N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left(L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}}\right) - L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \right) \Big|_{\xi=0} \\
&= N q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left((r_0 + r_1 + \cdots + r_{k(i)}) \left(L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1 + \{(t-i)/d\}}}\right) - L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \right) \right) \Big|_{\xi=0} \\
&= p^i \frac{\partial}{\partial \xi_{\bar{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \Big|_{\xi=0} \\
&\quad + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left(L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1 + \{(t-i)/d\}}}\right) - L \left(r, \frac{1}{q^{1-\{\overline{t(i)}\}}}\right) \right) \Big|_{\xi=0}
\end{aligned}$$

where $k(i) = p^d((N/p^i) - [N/p^i]) - 1$. Therefore we have

$$\begin{aligned}
B(N, w) &= \sum_{i=0}^{d-1} \left(\frac{N}{q} \left(\frac{t-i}{d} - \left\{ \frac{t-i}{d} \right\} + 1 \right) + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor\right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} \right. \\
&\quad \left. + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor\right) \left\lfloor \frac{N}{p^i} \right\rfloor \left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{t(i)\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{t(i)\}}}\right) \Big|_{\xi=0} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1+\{\overline{t(i)}\}}} \right) \Big|_{\xi=0} \\
& + Nq^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}} \right) \Big|_{\xi=0} \\
& - p^i \frac{\partial}{\partial \xi_{\bar{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \Big|_{\xi=0} \Bigg).
\end{aligned}$$

We now break the proof up into two cases.

Case (i) Let $\widehat{[t(i)]} = [\overline{t(i)}]$ for any i . Then, since

$$\frac{1}{q^{1-\{\overline{t(i)}\}}} = \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1+\{\overline{t(i)}\}}} = \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1+\{(t-i)/d\}}},$$

we have

$$\begin{aligned}
& \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left\lfloor \frac{N}{p^i} \right\rfloor \left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}} \right) \Big|_{\xi=0} \right. \\
& \left. - q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1+\{(t-i)/d\}}} \right) \Big|_{\xi=0} \right) = 0
\end{aligned}$$

for any i .

Case (ii) $[\widehat{t(i)}] = [\overline{t(i)}] + 1$ for some i . Then, by lemma 4.1, we have

$$\begin{aligned}
& \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) \frac{[\widehat{t(i)}] - [\overline{t(i)}]}{q} + \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left\lfloor \frac{N}{p^i} \right\rfloor \left(q^{[\widehat{t(i)}] - [\overline{t(i)}] + 1 - \{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{\widehat{t(i)}\}}} \right) \Big|_{\xi=0} \right. \\
& \left. - q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1 + \left\lfloor \frac{N}{p^i} \right\rfloor}{q^{1+\{(t-i)/d\}}} \right) \Big|_{\xi=0} \right) \\
& = \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(\left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) \frac{1}{q} + \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} \left(qL \left(r, \frac{1}{q} \right) - L(r, 1) \right) \Big|_{\xi=0} \right) \\
& = \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(\left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) \frac{1}{q} + \left\lfloor \frac{N}{p^i} \right\rfloor q^{1-\{\overline{t(i)}\}} \frac{\partial}{\partial \xi_{\bar{w}}} (qr_0 - 1) \Big|_{\xi=0} \right) \\
& = \left(N - p^i \left\lfloor \frac{N}{p^i} \right\rfloor \right) \left(\left(1 + \left\lfloor \frac{N}{p^i} \right\rfloor \right) \frac{1}{q} - \left\lfloor \frac{N}{p^i} \right\rfloor q^{-\{\overline{t(i)}\}} \right) = 0
\end{aligned}$$

for i .

In any case, we have

$$\begin{aligned}
B(N, w) &= \sum_{i=0}^{d-1} \left(\frac{N}{q} \left(\frac{t-i}{d} - \left\lfloor \frac{t-i}{d} \right\rfloor + 1 \right) + Nq^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}} \right) \Big|_{\xi=0} \right. \\
&\quad \left. - p^i \frac{\partial}{\partial \xi_{\bar{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \Big|_{\xi=0} \right) \\
&= N \left(\frac{t - (d-1)}{q} - \frac{1}{q} \left(\sum_{i=0}^{d-1} \left\lfloor \frac{t-i}{d} \right\rfloor - \frac{d-1}{2} \right) + \sum_{i=0}^{d-1} q^{1-\{(t-i)/d\}} \frac{\partial}{\partial \xi_{\bar{w}}} L \left(r, \frac{1}{q^{1-\{(t-i)/d\}}} \right) \Big|_{\xi=0} + \frac{d}{q} \right) \\
&\quad - \sum_{i=0}^{d-1} p^i \frac{\partial}{\partial \xi_{\bar{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \Big|_{\xi=0}.
\end{aligned}$$

As

$$\begin{aligned} \left. \frac{\partial}{\partial \xi_{\tilde{w}}} (r_0 + r_1 + \cdots + r_{k(i)}) \right|_{\xi=0} &= \left. \frac{\partial}{\partial \xi_{\tilde{w}}} \left(\frac{1 + e^{\xi_1} + \cdots + e^{\xi_{k(i)}}}{1 + e^{\xi_1} + \cdots + e^{\xi_q}} \right) \right|_{\xi=0} \\ &= -\frac{1}{q} \left(\frac{N}{p^i} - \left[\frac{N}{p^i} \right] \mathbf{1}_{\{p^i((N/p^i) - [N/p^i]) > \tilde{w}\}} \right), \end{aligned}$$

we obtain our formula.

Proof of Theorem 3.2. Let

$$c(n, w) = \sum_{i > [\log_p n] + 1} \mathbf{1}_{\{(\alpha_{i-1}(n), \alpha_{i-2}(n), \dots, \alpha_{i-d}(n)) = w\}},$$

and set

$$C(N, w) = \sum_{i=0}^{N-1} c(n, w).$$

Then, by an easy calculation, we have

$$C\left(\left[\frac{N}{p^i}\right], w\right) = \sum_{k=0}^{[\overline{r(i)}]-1} q^k + \left(\left[\frac{N}{p^i}\right] - q^{[\overline{r(i)}]} \left[\frac{\left[\frac{N}{p^i}\right]}{q^{[\overline{r(i)}]}} \right] \right) \mathbf{1}_{\{[q^{(t-i)/d}] = \tilde{w}\}} + q^{[\overline{r(i)}]} \mathbf{1}_{\{[q^{(t-i)/d}] > \tilde{w}\}}.$$

Therefore, we have

$$\begin{aligned} C(N, w) &= \sum_{i=0}^{d-1} p^i C\left(\left[\frac{N}{p^i}\right], w\right) \\ &= \sum_{i=0}^{d-1} p^i \left(\sum_{k=0}^{[\overline{r(i)}]-1} q^k + \left(\left[\frac{N}{p^i}\right] - q^{[\overline{r(i)}]} \left[\frac{\left[\frac{N}{p^i}\right]}{q^{[\overline{r(i)}]}} \right] \right) \mathbf{1}_{\{[q^{(t-i)/d}] = \tilde{w}\}} + q^{[\overline{r(i)}]} \mathbf{1}_{\{[q^{(t-i)/d}] > \tilde{w}\}} \right). \end{aligned}$$

As $[[N/p^i]/q^{[\overline{r(i)}]}] = [(N/p^i)/q^{(t-i)/d}] = [q^{(t-i)/d}]$, we have

$$\begin{aligned} p^i \left(\left[\frac{N}{p^i}\right] - q^{[\overline{r(i)}]} \left[\frac{\left[\frac{N}{p^i}\right]}{q^{[\overline{r(i)}]}} \right] \right) &= p^i ([q^{(t-i)/d}] - q^{[\overline{r(i)}]} [q^{(t-i)/d}]) \\ &= Nq^{-\{(t-i)/d\}} \{q^{\{(t-i)/d\}}\} - p^i \{q^{(t-i)/d}\}. \end{aligned}$$

Moreover, as

$$\sum_{i=0}^{d-1} \sum_{k=0}^{[\overline{r(i)}]-1} q^k p^i = \frac{N}{q-1} \sum_{i=0}^{d-1} q^{-\{(t-i)/d\}} - \frac{1}{p-1},$$

we obtain our formula.

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