On Smooth $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ -Actions on S^{m+n-1} *

Fuichi UCHIDA†

Department of Mathematical Sciences, Faculty of Science, Yamagata University, Koshirakawa, Yamagata 990-8560, Japan

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We shall study smooth $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ actions on S^{m+n-1} , each of which is an extension of the standard $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action. We shall show such an action is characterized by a smooth $\mathbf{R}^2$ -action on the circle. Moreover, we shall show there are uncountably many topologically distinct such extensions.

KEYWORDS: Non-compact Lie group action, Sphere, Codimension one

0. Introduction

Consider the standard $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action on the $(m + n - 1)$ -sphere S^{m+n-1} . This action has codimension one principal orbits with $\mathrm{SO}(m - 1) \times \mathrm{SO}(n - 1)$ as the principal isotropy subgroup and just two singular orbits. Furthermore, the fixed point set F of the restricted $\mathrm{SO}(m - 1) \times \mathrm{SO}(n - 1)$ action is diffeomorphic to the circle S^1 .

In this paper, we shall study smooth $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ actions on S^{m+n-1} , each of which is an extension of the standard $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action. We shall show such an action is characterized by a smooth $\mathbf{R}^2$ -action on F . Moreover, we shall show there are uncountably many topologically distinct such extensions.

Let G be a non-compact semi-simple Lie group and K a maximal compact connected subgroup of G . For a smooth K action on a sphere S^N with codimension one orbits, we have studied smooth G actions each of which is an extension of the given K action. T. Asoh [1] has investigated smooth $\mathrm{SL}(2, \mathbf{C})$ actions on 3-manifolds. The maximal compact subgroup of $\mathrm{SL}(2, \mathbf{C})$ is $\mathrm{SU}(2)$ and it is known that the smooth action of $\mathrm{SU}(2)$ on 3-manifold is transitive or has a codimension one orbit. He has proposed certain new idea to investigate the latter case. His idea was improved and used by F. Uchida [6, 7, 8] and K. Mukōyama [2, 3].

In the case of these papers, G is a simple Lie group and the singular orbits of given K action are contained in open orbits of extended G action. On the contrary, $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ is not a simple Lie group and the singular orbits of the standard $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action on S^{m+n-1} are invariant under any extended smooth $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ action.

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1. Twisted linear actions

For any positive real number c , a smooth action Ψ_c of $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$ on S^{m+n-1} is defined by

$$\Psi_c((A, B), X \oplus Y) = e^\theta AX \oplus e^{c\theta} BY$$

where $A \in \mathrm{SL}(m, \mathbf{R})$, $B \in \mathrm{SL}(n, \mathbf{R})$, $X \in \mathbf{R}^m$, $Y \in \mathbf{R}^n$ such that $X \oplus Y \in S^{m+n-1}$ and θ is a certain real number. We call Ψ_c the twisted linear action [5]. The action Ψ_c has just three orbits and one of them is an open orbit and the others are compact orbits.

Each restricted $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action of these twisted linear actions is the standard action. Here, we call the action ψ of $\mathrm{SO}(m) \times \mathrm{SO}(n)$ on S^{m+n-1} standard if $\psi((A, B), \mathbf{u} \oplus \mathbf{v}) = A\mathbf{u} \oplus B\mathbf{v}$ for $A \in \mathrm{SO}(m)$, $B \in \mathrm{SO}(n)$ and $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{v} \in \mathbf{R}^n$ such that $\mathbf{u} \oplus \mathbf{v} \in S^{m+n-1}$.

2. Certain subgroups of $\mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R})$

Let $L(n)$, $L^*(n)$, $N(n)$ and $N^*(n)$ denote the closed connected subgroup of $\mathrm{SL}(n, \mathbf{R})$ consisting of matrices in the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}, \quad \begin{pmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & \cdots & 0 \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$$

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respectively. Moreover, we define a closed connected subgroup $N(a:b)$ of $N(m) \times N(n)$ consisting of the pairs of matrices in the form

$$\left(\begin{pmatrix} e^{a\theta} & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} e^{b\theta} & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \right)$$

where $\theta \in \mathbf{R}$ and a, b are real numbers such that $(a, b) \neq (0, 0)$. We see $N(\lambda a : \lambda b) = N(a:b)$ for any non-zero real number λ .

Under these notations, we can describe the isotropy subgroups of the twisted linear action Ψ_c . That is, the isotropy subgroup at the open orbit is $N(1:c)$ and the isotropy subgroups at the compact orbits are $\mathbf{SL}(m, \mathbf{R}) \times N(n)$ and $N(m) \times \mathbf{SL}(n, \mathbf{R})$, respectively.

Let σ be the automorphism of $\mathbf{SL}(n, \mathbf{R})$ defined by $\sigma(A) = {}^t A^{-1}$. The restriction $\sigma|_{\mathbf{SO}(n)}$ is the identity on $\mathbf{SO}(n)$. We see $\sigma(L(n)) = L^*(n)$ and $\sigma(N(n)) = N^*(n)$. By making use of the automorphism σ , we may define another type of twisted linear action of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ on S^{m+n-1} .

Now we may state the following lemma. The proof of the lemma is a simple routine work (cf. [4; Lemmas 2.3, 2.4]), so we omit the proof.

Lemma 2.1. *Suppose $m \geq 3$ and $n \geq 3$. Let S be a closed subgroup of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ such that the restricted action of $\mathbf{SO}(m) \times \mathbf{SO}(n)$ on the homogeneous space $(\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R}))/S$ has at most three isotropy types $\mathbf{SO}(m-1) \times \mathbf{SO}(n-1)$, $\mathbf{SO}(m) \times \mathbf{SO}(n-1)$ and $\mathbf{SO}(m-1) \times \mathbf{SO}(n)$.*

- (1) *If S contains $\mathbf{SO}(m) \times \mathbf{SO}(n-1)$, then $S = \mathbf{SL}(m, \mathbf{R}) \times N(n)$.*
- (2) *If S contains $\mathbf{SO}(m-1) \times \mathbf{SO}(n)$, then $S = N(m) \times \mathbf{SL}(n, \mathbf{R})$.*
- (3) *If S contains $\mathbf{SO}(m-1) \times \mathbf{SO}(n-1)$ but S does not contain $\mathbf{SO}(m) \times \mathbf{SO}(n-1)$ nor $\mathbf{SO}(m-1) \times \mathbf{SO}(n)$, then $S = N(m) \times N(n)$ or $S = N(a:b)$ for $(a,b) \neq (0,0)$.*

Here, the results are described up to an equivalence under automorphisms of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ which leave invariant the subgroup $\mathbf{SO}(m) \times \mathbf{SO}(n)$.

3. Smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ actions on S^{m+n-1}

Now we consider smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ actions on S^{m+n-1} whose restricted $\mathbf{SO}(m) \times \mathbf{SO}(n)$ action is standard.

Denote by $\mathbf{e}_1, \dots, \mathbf{e}_m, \mathbf{e}_{m+1}, \dots, \mathbf{e}_{m+n}$ the orthonormal standard basis of $\mathbf{R}^m \oplus \mathbf{R}^n$. If $m \geq 3$ and $n \geq 3$, then the isotropy subgroup at $a\mathbf{e}_1 \oplus b\mathbf{e}_{m+1}$ of the standard action ψ is $\mathbf{SO}(m-1) \times \mathbf{SO}(n-1)$ for any (a,b) such that $a^2 + b^2 = 1$ and $ab \neq 0$. Put $F = \{u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1} \mid u^2 + v^2 = 1\}$. Then the set F is the fixed point set of the restricted $\mathbf{SO}(m-1) \times \mathbf{SO}(n-1)$ action of the standard action ψ on S^{m+n-1} .

Let $\Phi: \mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R}) \times S^{m+n-1} \rightarrow S^{m+n-1}$ be a smooth action whose restricted $\mathbf{SO}(m) \times \mathbf{SO}(n)$ action is standard. Denote by $F(S)$ the fixed point set of restricted S action of Φ on S^{m+n-1} for any subgroup S of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$. Then, we see $F = F(\mathbf{SO}(m-1) \times \mathbf{SO}(n-1))$.

Lemma 3.1. *For any smooth action of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ on S^{m+n-1} whose restricted $\mathbf{SO}(m) \times \mathbf{SO}(n)$ action is standard, the set F coincides with one of the followings: $F(L(m) \times L(n))$, $F(L^*(m) \times L(n))$, $F(L(m) \times L^*(n))$ or $F(L^*(m) \times L^*(n))$.*

Proof. We see that the set F coincides with the union of the above four sets by Lemma 2.1. Put $F_0 = \{u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1} \mid u^2 + v^2 = 1, uv \neq 0\}$. Define $F_0(S) = F_0 \cap F(S)$. Then we see that the four sets $F_0(L(m) \times L(n))$, $F_0(L^*(m) \times L(n))$, $F_0(L(m) \times L^*(n))$ and $F_0(L^*(m) \times L^*(n))$ are mutually disjoint closed subsets of F_0 . Hence we obtain the desired result. q.e.d.

REMARK. By lemma 3.1 and the automorphism σ , we can assume $F = F(L(m) \times L(n))$ for the smooth action Φ without loss of generality.

Now we introduce the following notations each of which is a diagonal matrix:

$$\begin{aligned} D_1(\theta) &= \text{diag}(e^\theta, *, \dots, *) \in C_{\mathbf{SL}(m, \mathbf{R})}(\mathbf{SO}(m-1)), \\ D_2(\theta) &= \text{diag}(e^\theta, *, \dots, *) \in C_{\mathbf{SL}(n, \mathbf{R})}(\mathbf{SO}(n-1)), \\ j_1 &= \text{diag}(-1, -1, 1, \dots, 1) \in \mathbf{SO}(m), \\ j_2 &= \text{diag}(-1, -1, 1, \dots, 1) \in \mathbf{SO}(n). \end{aligned}$$

Here, $C_A(B)$ denotes the centralizer of B in A .

Put $G = \mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$, $K = \mathbf{SO}(m) \times \mathbf{SO}(n)$, $H = \mathbf{SO}(m-1) \times \mathbf{SO}(n-1)$ and $L = L(m) \times L(n)$. Moreover, we define M to be the identity component of $C_G(H)$. Then M is consisting of the pairs $(D_i(\theta_i))$,

$D_2(\theta_2))$.

Let $\Phi: \mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R}) \times S^{m+n-1} \rightarrow S^{m+n-1}$ be a smooth action whose restricted $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action is standard and $F = F(L(m) \times L(n))$. Clearly, F is M -invariant, since M is a subgroup of the normalizer of H in G . Now we consider the restricted M -action on F . By the correspondence $(\theta_1, \theta_2) \rightarrow (D_1(\theta_1), D_2(\theta_2))$, the group M is isomorphic to the additive group $\mathbf{R}^2$. So, the restricted M -action on F induces a smooth $\mathbf{R}^2$ -action Φ_M on F . We can describe

$$\Phi_M((\theta_1, \theta_2), (u, v)) = (a((\theta_1, \theta_2), (u, v)), b((\theta_1, \theta_2), (u, v)))$$

for certain real valued functions a, b , where $(u, v) = u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1}$. Since $j_s D_s(\theta) = D_s(\theta) j_s$ for $s = 1, 2$, we obtain the equations:

$$\begin{aligned} a((\theta_1, \theta_2), (u, -v)) &= a((\theta_1, \theta_2), (u, v)) = -a((\theta_1, \theta_2), (-u, v)), \\ b((\theta_1, \theta_2), (-u, v)) &= b((\theta_1, \theta_2), (u, v)) = -b((\theta_1, \theta_2), (u, -v)). \end{aligned}$$

We see that a is an odd function with respect to the variable u and an even function with respect to the variable v , on the other hand, b is an even function with respect to the variable u and an odd function with respect to the variable v . Therefore, there are real valued smooth functions α, β on $\mathbf{R}^2 \times F$ satisfying the equations:

$$\begin{aligned} a((\theta_1, \theta_2), (u, v)) &= u\alpha((\theta_1, \theta_2), (u, v)), \\ b((\theta_1, \theta_2), (u, v)) &= v\beta((\theta_1, \theta_2), (u, v)). \end{aligned}$$

Here, both functions α, β are even functions with respect to variables u and v , respectively.

REMARK. $\alpha((0, 0), (u, v)) = \beta((0, 0), (u, v)) = 1$.

Theorem 3.2. *Let $\Phi: \mathrm{SL}(m, \mathbf{R}) \times \mathrm{SL}(n, \mathbf{R}) \times S^{m+n-1} \rightarrow S^{m+n-1}$ be a smooth action whose restricted $\mathrm{SO}(m) \times \mathrm{SO}(n)$ action is standard and $F = F(L(m) \times L(n))$. Then Φ is characterized by the restricted M -action on F . That is, if the restricted M -actions on F are identical for two such actions Φ, Φ' , then the two actions Φ, Φ' are identical.*

Proof. Let (g_1, g_2) be any element of G . Then, there are $(k_1, k_2) \in K$, $(\theta_1, \theta_2) \in \mathbf{R}^2$ and $(\ell_1, \ell_2) \in L$ satisfying the equations:

$$g_s = k_s D_s(\theta_s) \ell_s \quad (s = 1, 2).$$

Then, we obtain

$$\begin{aligned} \Phi((g_1, g_2), u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1}) &= \psi((k_1, k_2), \Phi_M((\theta_1, \theta_2), u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1})) \\ &= \psi((k_1, k_2), \alpha u\mathbf{e}_1 \oplus \beta v\mathbf{e}_{m+1}) \\ &= (\alpha u k_1 \mathbf{e}_1 \oplus \beta v k_2 \mathbf{e}_{m+1}). \end{aligned}$$

By the above decomposition of g_s , we obtain

$$\theta_1 = \log \frac{\|g_1 u\mathbf{e}_1\|}{\|u\mathbf{e}_1\|} \quad (u \neq 0), \quad \theta_2 = \log \frac{\|g_2 v\mathbf{e}_{m+1}\|}{\|v\mathbf{e}_{m+1}\|} \quad (v \neq 0).$$

Thus we obtain

$$\Phi((g_1, g_2), u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1}) = \alpha \frac{\|u\mathbf{e}_1\|}{\|g_1 u\mathbf{e}_1\|} g_1 u\mathbf{e}_1 \oplus \beta \frac{\|v\mathbf{e}_{m+1}\|}{\|g_2 v\mathbf{e}_{m+1}\|} g_2 v\mathbf{e}_{m+1}$$

for $u > 0, v > 0$. Here,

$$\begin{aligned} \alpha &= \alpha\left(\left(\log \frac{\|g_1 u\mathbf{e}_1\|}{\|u\mathbf{e}_1\|}, \log \frac{\|g_2 v\mathbf{e}_{m+1}\|}{\|v\mathbf{e}_{m+1}\|}\right), (\|u\mathbf{e}_1\|, \|v\mathbf{e}_{m+1}\|)\right), \\ \beta &= \beta\left(\left(\log \frac{\|g_1 u\mathbf{e}_1\|}{\|u\mathbf{e}_1\|}, \log \frac{\|g_2 v\mathbf{e}_{m+1}\|}{\|v\mathbf{e}_{m+1}\|}\right), (\|u\mathbf{e}_1\|, \|v\mathbf{e}_{m+1}\|)\right). \end{aligned}$$

Let $\mathbf{u} \oplus \mathbf{v}$ be any element of S^{m+n-1} such that $\|\mathbf{u}\| \neq 0, \|\mathbf{v}\| \neq 0$. Then, $\mathbf{u} = k'_1 u\mathbf{e}_1, \mathbf{v} = k'_2 v\mathbf{e}_{m+1}$ for some $(k'_1, k'_2) \in K$ and positive real numbers u, v . Since $\Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) = \Phi((g_1 k'_1, g_2 k'_2), (u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1}))$, we obtain

$$\Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) = \alpha \frac{\|\mathbf{u}\|}{\|g_1 \mathbf{u}\|} g_1 \mathbf{u} \oplus \beta \frac{\|\mathbf{v}\|}{\|g_2 \mathbf{v}\|} g_2 \mathbf{v}$$

where

$$\alpha = \alpha\left(\left(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}\right), (\|\mathbf{u}\|, \|\mathbf{v}\|)\right),$$

$$\beta = \beta\left(\left(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}\right), (\|\mathbf{u}\|, \|\mathbf{v}\|)\right).$$

Finally, we see

$$\begin{aligned}\Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{0}) &= \|g_1 \mathbf{u}\|^{-1} g_1 \mathbf{u} \oplus \mathbf{0}, \\ \Phi((g_1, g_2), \mathbf{0} \oplus \mathbf{v}) &= \mathbf{0} \oplus \|g_2 \mathbf{v}\|^{-1} g_2 \mathbf{v}.\end{aligned}$$

Consequently, we see that the action Φ is described by the restricted M -action on F . q.e.d.

4. Induced smooth $\mathbf{R}^2$ -action on F

The induced smooth $\mathbf{R}^2$ -action Φ_M on F is described by

$$\Phi_M((\theta_1, \theta_2), (u, v)) = (u\alpha((\theta_1, \theta_2), (u, v)), v\beta((\theta_1, \theta_2), (u, v)))$$

for certain real valued smooth functions α, β on $\mathbf{R}^2 \times F$, where both functions α, β are even functions with respect to variables u and v , respectively and $\alpha((0, 0), (u, v)) = \beta((0, 0), (u, v)) = 1$.

We obtain the following equations:

$$\begin{aligned}\alpha(\Theta' + \Theta, X) &= \alpha(\Theta', (\alpha(\Theta, X)u, \beta(\Theta, X)v))\alpha(\Theta, X), \\ \beta(\Theta' + \Theta, X) &= \beta(\Theta', (\alpha(\Theta, X)u, \beta(\Theta, X)v))\beta(\Theta, X),\end{aligned}$$

where $\Theta = (\theta_1, \theta_2)$ and $X = (u, v)$.

By these equations and the fact $\alpha(\mathbf{0}, X) = \beta(\mathbf{0}, X) = 1$ for any $X \in F$, we obtain $\alpha(\Theta, X) > 0, \beta(\Theta, X) > 0$ for any $\Theta \in \mathbf{R}^2$ and $X \in F$.

Now we define a mapping $\tilde{\Phi}: G \times S^{m+n-1} \rightarrow S^{m+n-1}$ by

$$\tilde{\Phi}((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) = \alpha \frac{\|\mathbf{u}\|}{\|g_1 \mathbf{u}\|} g_1 \mathbf{u} \oplus \beta \frac{\|\mathbf{v}\|}{\|g_2 \mathbf{v}\|} g_2 \mathbf{v}$$

where

$$\begin{aligned}\alpha &= \alpha\left(\left(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}\right), (\|\mathbf{u}\|, \|\mathbf{v}\|)\right), \\ \beta &= \beta\left(\left(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}\right), (\|\mathbf{u}\|, \|\mathbf{v}\|)\right)\end{aligned}$$

for $\mathbf{u} \oplus \mathbf{v} \in S^{m+n-1}$ such that $\|\mathbf{u}\| \neq 0, \|\mathbf{v}\| \neq 0$ and

$$\begin{aligned}\tilde{\Phi}((g_1, g_2), \mathbf{u} \oplus \mathbf{0}) &= \|g_1 \mathbf{u}\|^{-1} g_1 \mathbf{u} \oplus \mathbf{0}, \\ \tilde{\Phi}((g_1, g_2), \mathbf{0} \oplus \mathbf{v}) &= \mathbf{0} \oplus \|g_2 \mathbf{v}\|^{-1} g_2 \mathbf{v}.\end{aligned}$$

By a routine work, we obtain the following:

Proposition 4.1. $\tilde{\Phi}$ is an abstract G -action on S^{m+n-1} .

Proposition 4.2. If $\alpha \equiv \beta \equiv 1$, then $\tilde{\Phi}$ is not smooth.

Proof. In this case, the action $\tilde{\Phi}$ is given by

$$\tilde{\Phi}((g_1, g_2), (\mathbf{u}, \mathbf{v})) = \frac{\|\mathbf{u}\|}{\|g_1 \mathbf{u}\|} g_1 \mathbf{u} \oplus \frac{\|\mathbf{v}\|}{\|g_2 \mathbf{v}\|} g_2 \mathbf{v}.$$

Put $g_1 = \text{diag}(t, t^{-1}, 1, \dots, 1)$ and $\mathbf{u} = (x, y, 0, \dots, 0)$. Then we obtain

$$\frac{\|\mathbf{u}\|}{\|g_1 \mathbf{u}\|} g_1 \mathbf{u} = \sqrt{\frac{x^2 + y^2}{t^2 x^2 + t^{-2} y^2}} (tx, t^{-1}y, 0, \dots, 0).$$

We see that the function in the righthand side is not C^1 -differentiable at $x = y = 0$. q.e.d.

5. Certain smooth action of $\text{SL}(m, \mathbf{R})$ on $\mathbf{R}^m$

Define a mapping $\varphi_m: \text{SL}(m, \mathbf{R}) \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ by

$$\varphi_m(A, \mathbf{u}) = \frac{2}{1 - \|\mathbf{u}\|^2 + \sqrt{4\|A\mathbf{u}\|^2 + (1 - \|\mathbf{u}\|^2)^2}} A\mathbf{u}.$$

Then, we see that φ_m is a smooth $\text{SL}(m, \mathbf{R})$ -action on $\mathbf{R}^m$ and the three sets

$$\{\mathbf{u} \in \mathbf{R}^m \mid \|\mathbf{u}\| < 1\}, \{\mathbf{u} \in \mathbf{R}^m \mid \|\mathbf{u}\| = 1\}, \{\mathbf{u} \in \mathbf{R}^m \mid \|\mathbf{u}\| > 1\}$$

are invariant under the action φ_m . In particular, if $\|\mathbf{u}\| \neq 1$, then

$$\frac{1}{1 - \|\varphi_m(A, \mathbf{u})\|^2} \varphi_m(A, \mathbf{u}) = \frac{1}{1 - \|\mathbf{u}\|^2} A\mathbf{u},$$

and if $\|\mathbf{u}\| = 1$, then

$$\varphi_m(A, \mathbf{u}) = \frac{1}{\|A\mathbf{u}\|} A\mathbf{u}.$$

Put $D^m = \{\mathbf{u} \in \mathbf{R}^m \mid \|\mathbf{u}\| < 1\}$. Let $h: D^m \rightarrow \mathbf{R}^m$ be a diffeomorphism defined by

$$h(\mathbf{u}) = \frac{1}{1 - \|\mathbf{u}\|^2} \mathbf{u}.$$

Then we see that h is an equivariant diffeomorphism of D^m with the action φ_m onto $\mathbf{R}^m$ with the linear action. That is,

$$\varphi_m(A, \mathbf{u}) = h^{-1}(Ah(\mathbf{u})) \text{ for } \mathbf{u} \in D^m.$$

The action φ_m induces a smooth $\mathbf{R}$ -action ϕ_m on $\mathbf{R}$ by

$$\phi_m(\theta, t)\mathbf{e}_1 = \varphi_m(D_1(\theta), t\mathbf{e}_1).$$

We see that the $\mathbf{R}$ -action ϕ_m on $\mathbf{R}$ corresponds to the following vector field on $\mathbf{R}$:

$$t \frac{1 - t^2}{1 + t^2} \frac{d}{dt}.$$

6. Construction of smooth G -actions on S^{m+n-1}

First, we shall construct a smooth one-parameter group on the circle F .

Let k, ℓ be diffeomorphisms of $\mathbf{R} \times \{\pm 1\}$ onto open sets of F defined by

$$k(t, \varepsilon) = \left(\frac{t}{\sqrt{c^2 + t^2}}, \frac{c\varepsilon}{\sqrt{c^2 + t^2}} \right),$$

$$\ell(t, \varepsilon) = \left(\frac{c\varepsilon}{\sqrt{c^2 + t^2}}, \frac{t}{\sqrt{c^2 + t^2}} \right),$$

where c is a positive real number.

Let $\rho(t)$ be a smooth real valued function such that

$$\begin{aligned} \rho(t) &= 1 \text{ for } |t| \leq 1, \\ \rho(t) &= 0 \text{ for } |t| \geq 2, \\ 0 &< \rho(t) < 1 \text{ for } 1 < |t| < 2. \end{aligned}$$

Define a tangent vector field ξ on $F = \{(u, v) \in \mathbf{R}^2 \mid u^2 + v^2 = 1\}$ by

$$\xi = k_* \left(\rho(t^2) t \frac{1 - t^2}{1 + t^2} \frac{d}{dt} \right) + \ell_* \left(\rho(t^2) t \frac{1 - t^2}{1 + t^2} \frac{d}{dt} \right) + \rho(p(2u^2 - 1))uv \left(u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} \right),$$

where p is a positive real number. The vector field ξ is invariant under the involutions $J_1: (u, v) \rightarrow (-u, v)$ and $J_2: (u, v) \rightarrow (u, -v)$.

Let ϕ be the one-parameter group on F corresponding to the vector field ξ . Then we see

$$J_s(\phi(\theta, (u, v))) = \phi(\theta, J_s(u, v)) \text{ for } s = 1, 2.$$

Put $c = 4, p = 3$ in the definitions of ξ, ϕ . Then we see

$$\begin{aligned} \xi &= k_* \left(t \frac{1 - t^2}{1 + t^2} \frac{d}{dt} \right) \text{ for } u^2 < \frac{1}{17}, \\ \xi &= \ell_* \left(t \frac{1 - t^2}{1 + t^2} \frac{d}{dt} \right) \text{ for } u^2 > \frac{16}{17}, \\ \xi &= 0 \text{ for } \frac{1}{9} < u^2 < \frac{1}{6} \text{ or } \frac{5}{6} < u^2 < \frac{8}{9}, \end{aligned}$$

$$\xi \neq 0 \text{ for } \frac{1}{6} < u^2 < \frac{5}{6}.$$

Now we define a smooth $\mathbf{R}^2$ -action φ on F by

$$\varphi((\theta_1, \theta_2), (u, v)) = \begin{cases} \phi(\theta_1, (u, v)) & \text{for } u^2 < \frac{1}{9} \\ \phi(\theta_2, (u, v)) & \text{for } u^2 > \frac{8}{9} \\ \phi(a\theta_1 + b\theta_2, (u, v)) & \text{for } \frac{1}{9} \leq u^2 \leq \frac{8}{9}. \end{cases}$$

Here, a, b are any real numbers. The smoothness of φ is assured by the condition:

$$\xi = 0 \text{ for } \frac{1}{9} < u^2 < \frac{1}{6} \text{ or } \frac{5}{6} < u^2 < \frac{8}{9}.$$

By the smooth $\mathbf{R}^2$ -action φ on F , we may define an abstract G -action $\tilde{\Phi}$ on S^{m+n-1} as in the section 4. Now we show that $\tilde{\Phi}$ is smooth on neighborhoods of $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$.

Define a diffeomorphism $\tilde{k}$ of $\mathbf{R}^m \times S^{n-1}$ onto an open set of S^{m+n-1} by

$$\tilde{k}(\mathbf{u}, \mathbf{v}) = \frac{1}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{u} \oplus \frac{4}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{v}.$$

Then we see

$$\tilde{\Phi}((A, B), \mathbf{u} \oplus \mathbf{v}) = \tilde{k}\left(\varphi_m\left(A, \frac{4}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{u}\right), \frac{1}{\|B\mathbf{v}\|} B\mathbf{v}\right),$$

for $\|\mathbf{u}\|^2 < 1/17$. Here, φ_m is the action of $\mathbf{SL}(m, \mathbf{R})$ on $\mathbf{R}^m$ defined in the section 5. This shows that $\tilde{\Phi}$ is smooth on a neighborhood of $\mathbf{u} = \mathbf{0}$. Similarly, we may show the smoothness of $\tilde{\Phi}$ on a neighborhood of $\mathbf{v} = \mathbf{0}$.

Therefore, we see that the G -action $\tilde{\Phi}$ on S^{m+n-1} is smooth.

By this G -action, we obtain the following result.

Theorem 6.1. *There exists a smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ action on S^{m+n-1} such that it has five open orbits and the isotropy types of them are the following: $L(m) \times N(n)$ (two orbits), $N(b: -a)$ and $N(m) \times L(n)$ (two orbits) for any real numbers a, b such that $(a, b) \neq (0, 0)$.*

REMARK. Let $\tilde{\Phi}_{(a,b)}$ be the above action. Then, the action is topologically distinct from any twisted linear action discussed in the first section, by the difference of the numbers of open orbits. Moreover, if the ratio $a: b$ and $c: d$ are different, then the actions $\tilde{\Phi}_{(a,b)}$ and $\tilde{\Phi}_{(c,d)}$ are topologically distinct, because the isotropy groups $N(b: -a)$ and $N(d: -c)$ are not conjugate. Consequently, there are uncountably many topologically distinct smooth actions of $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ on S^{m+n-1} , each of which is topologically distinct from any twisted linear action.

REMARK. By an exchange of the third term of the vector field ξ , we may obtain another smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ action on S^{m+n-1} . For example, we can construct smooth actions which have many open orbits.

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