Online Algorithms for Convex Case Capital Investment

Toshiya ITOH

Global Scientific Information and Computing Center Tokyo Institute of Technology 2-12-1 O-okayama,
Meguro-ku, Tokyo 152-8550, Japan

Received May 14, 2001; final version accepted November 9, 2001

In a factory, we need to make capital investments in machines for manufacturing a product. In this paper, we deal with the convex case capital investment such that more expensive machines have cheaper production costs. What we wish to achieve is to design a good *online* algorithm that minimizes the sum of the production and capital costs when the production request and investment opportunities in the future are unknown. Azar, et al. proposed an (online) algorithm CONVEX for the convex case capital investment and showed that it is $(4 + 2\sqrt{2})$ -competitive. In this paper, we investigate the competitive ratio of the convex case capital investment more precisely and show that (1) for the convex case capital investment, the competitive ratio of the algorithm CONVEX is at least $4 + 2\sqrt{2} - \varepsilon$ for any $\varepsilon > 0$ (Theorem 3.3). In the practical point of view, we introduce a notion of “ γ -restricted” to the convex case capital investment and show that (2) for the γ -restricted convex case capital investment, the competitive ratio of the algorithm CONVEX is at most $5 + 4/(\gamma - 4)$ for any $\gamma \geq 6$ (Theorem 4.3); (3) for the γ -restricted convex case capital investment, the competitive ratio of the algorithm CONVEX is at least $5 - \varepsilon$ for any $\varepsilon > 0$ (Theorem 4.4). Finally, we also show that the competitive ratio of the γ -restricted convex case capital investment is at least $2 - \varepsilon$ for any $\varepsilon > 0$ (Theorem 5.4).

KEYWORDS: Capital Investment, Convex Case, Online Algorithms, Competitive Ratio, γ -Restricted

1 Introduction

1.1 Background and Motivation

The *online capital investment problem* is initiated by Azar, et al. [2]. Let us consider the following situation: A factory uses machines to produce units of some products. When producing each unit of the products by those machines, it requires some fixed costs, e.g., electricity, water, etc. A machine m_i available at time t_i is characterized by the *capital cost* c_i and the *production cost* p_i , where the capital cost c_i of m_i is the cost to buy the machine m_i and the production cost p_i of m_i is the cost to produce a unit of the product by the machine m_i . Over time, new machines would become available, which could replace old ones. Thus the factory may have opportunities to invest in those new machines. If many requests for productions would arrive in the future, the factory could buy a new machine with less production cost than that of the current one. Who knows in advance what happens in the future? Thus the factory must determine whether to invest in buying new machines to reduce production costs while future requests for productions and future investment opportunities are unknown.

This is a typical online financial problem and the competitive analysis [6] is one of the most standard ways to measure the performance of online algorithms, e.g., currency exchange problems [1,3] and asset allocation [5]. Azar, et al. [2] defined two classes of (online) capital investment problems, one is the convex case and the other is the nonconvex case. We say that the capital investment problem is *convex* if for any two machines $m_i = (c_i, p_i)$ and $m_j = (c_j, p_j)$ (available at time $t_i \geq t_j$, respectively), $p_i < p_j$ implies that $c_i \geq c_j$; *nonconvex* otherwise (the convexity may be a reasonable restriction for short term capital investments, but for long term capital investments, the nonconvexity may be more realistic). Informally, we say that an online algorithm is c -competitive if its cost is at most c times the cost of the optimal offline algorithm (the formal definition will be given in Section 2). For the convex case, Azar, et al. [2] proposed an online algorithm CONVEX that is $(4 + 2\sqrt{2})$ -competitive. For the nonconvex case, Azar, et al. [2] presented an online algorithm NONCONVEX that is $O(\min \{\log C, \log \log P, \log M\})$ -competitive, and also showed that any online algorithm is $\Omega(\max \{\log C, \log \log P / \log \log \log P, \log M / \log \log M\})$ -competitive, where C is the ratio between the highest and the lowest capital costs, P is the ratio between the highest and the lowest production costs, and M is the number of investment opportunities. Thus in the nonconvex case, the performance of any online algorithm cannot be within a constant factor to that of the optimal offline algorithm (so we have little hope to drastically improve the performance).

In the convex case, we have an online algorithm that performs within a constant factor to the optimal offline algorithm, however, we do not know whether it can or cannot be improved further. In this paper, we shall investigate the convex case (online) capital investment problem more precisely to capture the inherent nature of the problem with respect to the competitive ratio.

1.2 Main Results

As we have described above, Azar, et al. [2] showed that the algorithm CONVEX $(4 + 2\sqrt{2})$ -competitive for the convex case. To see the exact performance of the algorithm CONVEX, we show that (1) for any $\varepsilon > 0$, there ex-

ists a sequence σ such that $\text{CONVEX}(\sigma)/\text{OPT}(\sigma) \geq 4 + 2\sqrt{2} - \varepsilon$ (Theorem 3.3), i.e., the performance of the algorithm CONVEX is completely characterized. In the practical point of view, it seems natural to assume that for any machine $m = (c, p)$, the capital cost c is much higher than the production cost p . We say that a convex case capital investment is γ -restricted if for any available machine $m = (c, p)$, it satisfies that $c/p \geq \gamma$. Then we consider the performance of the algorithm CONVEX and show that for the γ -restricted case, (2) the algorithm CONVEX is $\{5 + 4/(\gamma - 4)\}$ -competitive for any $\gamma \geq 6$ (Theorem 4.3); (3) for any $\varepsilon > 0$, the competitive ratio of the algorithm CONVEX is at least $5 - \varepsilon$ (Theorem 4.4). Note that Theorem 4.3 guarantees the better performance of the algorithm CONVEX for the γ -restricted case than for general convex case, and Theorems 4.3 and 4.4 provide almost complete characterization of the performance of the algorithm CONVEX for the γ -restricted case capital investment. To capture the nature of the γ -restricted case capital investment, we finally consider a general lower bound for the competitive ratio and show that (4) for any $\varepsilon > 0$ and any online algorithm ON, there exists a sequence σ such that $\text{ON}(\sigma)/\text{OPT}(\sigma) \geq 2 - \varepsilon$ (Theorem 5.4). As we shall mention in Section 5 precisely, Theorem 5.4 captures the inherent nature of the γ -restricted case capital investment in the natural sense.

2 Preliminaries

In the online capital investment problem, an instance of the problem consists of a sequence of machines and a sequence of production requests. A machine m_i is defined by the triplet (t_i, c_i, p_i) , where t_i is the time at which the machine becomes available, c_i is its capital cost, and p_i is its production cost. We note that the capital cost is the capital investment necessary to buy the machine and the production cost is the cost of producing one unit of commodity by the machine. We assume that once a machine becomes available, then it is available forever, and we also assume that any machine can produce an unbounded number of units of commodity. Every production request is defined by its arrival time. Without loss of generality, the j th production request is assumed to appear at time j . Since any reasonable online algorithm will not buy a new machine when there are no pending production request, we can assume that $t_i = j + 1$ for any machine $m_i = (t_i, c_i, p_i)$ such that $j < t_i \leq j + 1$. So at any time j , the algorithm can buy any of the available machines m_i such that $t_i \leq j$, and then produce one unit of the commodity.

Let Σ be the set of sequences of machines and production requests. For any $\sigma \in \Sigma$, we use $\text{ON}(\sigma)$ to denote the total cost of the online algorithm ON over the sequence σ and $\text{OPT}(\sigma)$ to denote the total cost of the optimal offline algorithm that knows entire sequence σ in advance, where the total cost is the sum of the total capital cost and the total production cost. The goal of the online algorithm is to minimize the total cost and its performance is measured by the competitive ratio [6].

Definition 2.1: An online algorithm ON is ρ -competitive if $\text{ON}(\sigma)/\text{OPT}(\sigma) \leq \rho$ for any $\sigma \in \Sigma$.

Definition 2.2: The (online) capital investment problem is ρ -competitive if there exists an online algorithm ON that is ρ -competitive, i.e., $\text{ON}(\sigma)/\text{OPT}(\sigma) \leq \rho$ for any $\sigma \in \Sigma$.

3 Competitive Ratio of the Algorithm CONVEX

In this section, we investigate the convex case capital investment. We say that the capital investment is *convex* if $p_i < p_j$ implies $c_i \geq c_j$ for any pair of machines $m_i = (t_i, c_i, p_i)$ and $m_j = (t_j, c_j, p_j)$.

3.1 Upper Bound for Competitive Ratio of the Algorithm CONVEX

For the convex case capital investment, Azar, et al. [2] proposed the following (online) algorithm CONVEX where α and β are positive constants satisfying that $\alpha \geq 2$ and $1/\alpha + 2\beta \leq 1$.

The Online Algorithm CONVEX:

Input: a sequence $\sigma \in \Sigma$.

Step 1: For $j = 1$, buy the machine m_i that minimizes $c_i + p_i$ among all machines available at time $j = 1$ and then produce the first unit of the commodity.

Step 2: For $j \geq 2$, let c and p be the capital and production cost, respectively, of the currently used machine that is bought at time $j' < j$, and for each $1 \leq k < j$, let $\text{prod}[k, j - 1]$ be the total production cost incurred from time k to time $j - 1$.

Step 2-1: If $\text{prod}[j', j - 1] < \beta c$, then produce the j th unit of commodity by the current machine.

Step 2-2: Buy the machine m_i that minimizes the production cost $p_i < p$ among all available machines m' such that $c' \leq \alpha \cdot \text{prod}[1, j - 1]$, and then produce the j th unit of commodity by the machine just bought. If there exist no such machines, then produce the j th unit of commodity by the currently used machine.

For the competitive ratio of the algorithm CONVEX, Azar, et al. [2] showed the following theorem:

Theorem 3.1 [2, Theorem 3.1]: For any positive constants α and β such that $\alpha \geq 2$ and $1/\alpha + 2\beta \leq 1$, the algorithm CONVEX is $(1 + \alpha + 1/\beta)$ -competitive.

From Theorem 3.1, Azar, et al. [2] concluded that the algorithm CONVEX is $(4 + 2\sqrt{2})$ -competitive by choosing $\alpha = \alpha^* = 1 + \sqrt{2}$ and $\beta = \beta^* = 1/(2 + \sqrt{2})$. Note that $\alpha^* \geq 2$ and $1/\alpha^* + 2\beta^* = 1$.

3.2 Lower Bound for Competitive Ratio of the Algorithm CONVEX

The algorithm CONVEX is shown to be $(4 + 2\sqrt{2})$ -competitive. One may expect the algorithm CONVEX to perform much better, however, it is impossible. To show this, the following lemma is useful.

Lemma 3.2: For any $0 \leq x \leq 1/3$ and any integer $k \geq 1$, $(1 - x)(1 - x^2) \cdots (1 - x^k) \geq 1 - 3x$.

The proof of Lemma 3.2 is given in Appendix A. Then for the convex case capital investment, we can show the following theorem on the optimality for the competitive ratio of the algorithm CONVEX.

Theorem 3.3: For any $\varepsilon > 0$, there exists a sequence $\sigma \in \Sigma$ such that $\text{CONVEX}(\sigma)/\text{OPT}(\sigma) \geq 4 + 2\sqrt{2} - \varepsilon$, i.e., for any $\varepsilon > 0$, the competitive ratio of the algorithm CONVEX is at least $4 + 2\sqrt{2} - \varepsilon$.

Proof: Let $\delta > 0$ be an extremely small constant and let $\alpha^* = 1 + \sqrt{2}$ and $\beta^* = 1/(2 + \sqrt{2})$. Define a sequence $\sigma_s \in \Sigma$ as follows: At time $j = 1$, only the machine $m_1 = (t_1, c_1, p_1)$ is available, where $t_1 = 1$, $c_1 = c$, and $p_1 = p$. At time $j = 2$, the machine $m_2 = (t_2, c_2, p_2)$ becomes available, where $t_2 = 2$, $c_2 = c$, and $p_2 = p(1 - \delta)$. For each $3 \leq k \leq s$, let $c_k = (1 + \alpha^*\beta^*)^{k-2}c$ and $p_k = p(1 - \delta)(1 - \delta^2) \cdots (1 - \delta^{k-1})$. Let $j_1 = n_1 = \lceil c/\alpha^*p \rceil$ and for each $2 \leq l \leq s - 1$, let $j_l = \lceil \beta^*c_l/p_l \rceil$ and $n_l = j_1 + j_2 + \cdots + j_l$. For each $3 \leq k \leq s$, the machine $m_k = (t_k, c_k, p_k)$ becomes available at time $j = n_{k-2} + 2$ and the sequence $\sigma_s \in \Sigma$ ends at time $j = n_{s-1} + 1$ (i.e., the last machine m_s produces only one unit of commodity).

We analyze the behavior of CONVEX on the sequence $\sigma_s \in \Sigma$. Let $\text{CONVEX}_p(\sigma_s)$ be the total production cost of CONVEX on the sequence $\sigma_s \in \Sigma$ and $\text{CONVEX}_c(\sigma_s)$ be the total capital cost of CONVEX on the sequence $\sigma_s \in \Sigma$. From the definition of the algorithm CONVEX, it buys the machine m_1 at time $j = 1$, and for $2 \leq k \leq s$, it buys the machines m_k at time $j = n_{k-1} + 1$. So we have that

$$\text{CONVEX}_c(\sigma_s) \geq \frac{c}{\alpha^*\beta^*} \{(1 + \alpha^*\beta^*)^{s-1} - 1\}, \quad (1)$$

$$\text{CONVEX}_p(\sigma_s) \geq \frac{c}{\alpha^*} (1 + \alpha^*\beta^*)^{s-2}, \quad (2)$$

where the proof for equations (1) and (2) are given in Appendices B.1 and B.2, respectively.

Let us consider the *offline* algorithm ADV: For the sequence $\sigma_s \in \Sigma$, it buys only the machine m_1 and then produces all units of commodity by the machine m_1 . Let $\text{ADV}_p(\sigma_s)$ be the total production cost of ADV on the sequence $\sigma_s \in \Sigma$ and $\text{ADV}_c(\sigma_s)$ be the total capital cost of ADV on the sequence $\sigma_s \in \Sigma$. So from the definition of the offline algorithm ADV, we have that

$$\text{ADV}_c(\sigma_s) = c_1 = c. \quad (3)$$

$$\text{ADV}_p(\sigma_s) \leq sp + \frac{c}{(1 - 3\delta)\alpha^*} (1 + \alpha^*\beta^*)^{s-2}, \quad (4)$$

where the proof of equation (4) is given in Appendix B.3. Thus it follows that

$$\frac{\text{CONVEX}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{1 + \alpha^* + \frac{1}{\beta^*}}{\frac{\alpha^*}{(1 + \alpha^*\beta^*)^{s-2}} + \frac{\alpha^*p}{c} \frac{s}{(1 + \alpha^*\beta^*)^{s-2}} + \frac{1}{1 - 3\delta}}, \quad (5)$$

where the proof of equation (5) is given in Appendix B.4.

Recall that $\delta > 0$ is an extremely small constant. Since $\alpha^* = 1 + \sqrt{2}$ and $\beta^* = 1/(2 + \sqrt{2})$, we have that $1 + \alpha^*\beta^* \geq 1.7$ and $1 + \alpha^* + 1/\beta^* = 4 + 2\sqrt{2}$. By letting $\delta = 1/s$, it follows from equation (5) that

$$\frac{\text{CONVEX}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{1 + \alpha^* + \frac{1}{\beta^*}}{\frac{\alpha^*}{(1 + \alpha^*\beta^*)^{s-2}} + \frac{\alpha^*p}{c} \frac{s}{(1 + \alpha^*\beta^*)^{s-2}} + \frac{1}{1 - 3\delta}} \rightarrow 1 + \alpha^* + \frac{1}{\beta^*} (s \rightarrow \infty).$$

Thus for any $\varepsilon > 0$, there exists a sequence $\sigma \in \Sigma$ such that $\text{CONVEX}(\sigma)/\text{OPT} \geq 4 + 2\sqrt{2} - \varepsilon$. $\blacksquare$

4 Competitive Ratio for the γ -Restricted Convex Case

In this section, we consider the restricted convex case capital investment. For the convex case, we have assumed that for any pair of machines $m_i = (t_i, c_i, p_i)$ and $m_j = (t_j, c_j, p_j)$, $p_i < p_j$ implies $c_i \geq c_j$. We say that the convex case capital investment is γ -restricted if $c/p \geq \gamma$ for any machine $m = (t, c, p)$.

It seems reasonable to assume that γ is large for most of the natural convex case capital investment. Note that

for any $\alpha, \beta > 0$, the following lemma holds for the algorithm CONVEX.

Lemma 4.1 [2, Lemma 3.1]: *Let $\alpha \geq 2$ and $1/\alpha + 2\beta \leq 1$. Then for the convex case capital investment, the following holds: For any sequence $\sigma \in \Sigma$, $\text{CONVEX}_c(\sigma) \leq (\alpha + 1/\beta)\text{CONVEX}_p(\sigma)$.*

For the γ -restricted convex case capital investment, the following lemma also holds for the algorithm CONVEX, which improves the claim due to Azar, et al. [2, Lemma 3.2].

Lemma 4.2: *Let $\gamma \geq 6$. Then for the γ -restricted convex case capital investment, the following holds: For any $\alpha, \beta > 0$, if $(\beta\gamma + 1)/\alpha\beta\gamma \leq 1$ and $1/\alpha + \beta^2\gamma/(\beta\gamma - 1) \leq 1$, then $\text{CONVEX}_p(\sigma) \leq \text{OPT}(\sigma)$ for any sequence $\sigma \in \Sigma$.*

The proof of Lemma 4.2 is given in Appendix C. Combining Lemmas 4.1 and 4.2, we can show the following theorem for the γ -restricted convex case capital investment.

Theorem 4.3: *For γ -restricted convex case capital investment, the algorithm CONVEX is $\{5 + 4/(\gamma - 4)\}$ -competitive, i.e., $\text{CONVEX}(\sigma)/\text{OPT}(\sigma) \leq 5 + 4/(\gamma - 4)$ for any sequence $\sigma \in \Sigma$.*

Proof: From Lemmas 4.1 and 4.2, it follows that the algorithm CONVEX is $(1 + \alpha + 1/\beta)$ -competitive for any $\alpha, \beta > 0$ if $(\beta\gamma + 1)/\alpha\beta\gamma \leq 1$ and $1/\alpha + \beta^2\gamma/(\beta\gamma - 1) \leq 1$. So our goal is as follows:

$$\text{minimize } \left(1 + \alpha + \frac{1}{\beta}\right) \text{ subject to } \frac{\beta\gamma + 1}{\alpha\beta\gamma} \leq 1 \quad \text{and} \quad \frac{1}{\alpha} + \frac{\beta^2\gamma}{\beta\gamma - 1} \leq 1. \quad (6)$$

It seems difficult to derive the closed form for the minimum of the problem (6). To overcome this difficulty, we let $\beta = 1/2$. Then the problem (6) can be simplified as follows:

$$\text{minimize } (3 + \alpha) \text{ subject to } \alpha \geq \frac{\gamma + 2}{\gamma} \quad \text{and} \quad \frac{1}{\alpha} + \frac{\gamma}{2\gamma - 4} \leq 1. \quad (7)$$

This can be easily solved. In fact, the minimum of the problem (7) is given by $5 + 4/(\gamma - 4)$, which is achieved at $\alpha' = (2\gamma - 4)/(\gamma - 4)$ and $\beta' = 1/2$. ■

If $\gamma \gg 1$, then the algorithm CONVEX performs better for the γ -restricted case than for the convex case. One may expect CONVEX to perform much better, however, it is almost impossible.

Theorem 4.4: *For the γ -restricted convex case capital investment, the following holds: For any $\gamma \geq 6$ and any $\varepsilon > 0$, there exists a sequence $\sigma \in \Sigma$ such that $\text{CONVEX}(\sigma)/\text{OPT}(\sigma) \geq 5 - \varepsilon$.*

Proof: As in the proof of Theorem 4.3, the algorithm CONVEX is $(1 + \alpha^* + 1/\beta^*)$ -competitive, where α^*, β^* are the values which achieve the minimum of the problem (6). Since $1/\alpha < (\beta\gamma + 1)/\alpha\beta\gamma \leq 1$ and $1/\alpha + \beta < 1/\alpha + \beta^2\gamma/(\beta\gamma - 1) \leq 1$, we have the following the minimization problem:

$$\text{minimize } \left(1 + \alpha + \frac{1}{\beta}\right) \text{ subject to } \frac{1}{\alpha} \leq 1 \quad \text{and} \quad \frac{1}{\alpha} + \beta \leq 1. \quad (8)$$

The minimum of the problem (8) is 5. Since the problem (8) is the relaxation of the problem (6), it is obvious that $5 \leq 1 + \alpha^* + 1/\beta^*$. Then in a way similar to the proof of Theorem 3.3, we can show that for any $\varepsilon > 0$, there exists a sequence $\sigma \in \Sigma$ such that $\text{CONVEX}(\sigma)/\text{OPT}(\sigma) \geq 5 - \varepsilon$. ■

5 General Lower Bound for the γ -Restricted Convex Case

In Section 4, we have shown a lower bound for the specific algorithm CONVEX (Theorem 4.4). In this section, we shall show a general lower bound for the γ -restricted convex case capital investment.

The *ski rental problem* [4] is the practical online problem and can be stated as follows: You do not know in advance how many times you will go skiing; renting a pair of skis costs r ; buying your own pair of skis costs c . When do you buy? It is not hard to see that the ski rental problem is 2-competitive and for any $\varepsilon > 0$, there exist no online algorithm for the ski rental problem that is $(2 - \varepsilon)$ -competitive.

Assume that machines $m_1 = (c_1, p_1)$ and $m_2 = (c_2, p_2)$ are available at time 0, where $c_1 = 0, p_1 = r$ and $c_2 = c, p_2 = 0$. So the ski rental problem can be regarded as a special case of the convex case capital investment, which implies that for the convex case capital investment, the lower bound for the competitive ratio is 2. In this section, we derive the lower bound for the competitive ratio of the γ -restricted convex case capital investment to capture the inherent nature of the γ -restricted convex case capital investment.

For any online algorithm ON, define the sequence $\sigma \in \Sigma$ as follows: Let $j_0 = 0$ and $\kappa \gg 1$ be an integer. At time $j = 1$, only the machine $m_1 = (t_1, c_1, p_1)$ is available, where $c_1 = c, p_1 = p$, and $c_1/p_1 \geq \gamma$. At time $j = 2$, the machine $m_2 = (t_2, c_2, p_2)$ becomes available, where $c_2 = \kappa c$ and $p_2 = p/\kappa$. For each $l \geq 3$, if the algorithm ON buys the machine m_{l-1} at time $j = j_{l-2} + 1$, then the machine $m_l = (t_l, c_l, p_l)$ becomes available at time $j = j_{l-2} + 2$, where $c_l = \kappa^{l/2}c$ and $p_l = p/\kappa^{l-1}$. Since $c_l/p_l \geq \gamma$ and $\kappa \gg 1$, all these machines $m_l = (t_l, c_l, p_l)$ satisfy the γ -restricted condition, i.e., $c_l/p_l \geq \gamma$ for each $l \geq 1$. For any $\kappa \gg 1$ and any algorithm ON, the following cases are possible: (Case 1) The algorithm ON buys only the machine m_1 at time $j = j_0 + 1$; (Case 2) The algorithm ON buys at least $r \geq 2$ machines $m_1, m_2, \dots, m_r$ at time $j = j_0 + 1, j = j_1 + 1, \dots, j = j_{r-1} + 1$, respectively.

Lemma 5.1: For any $\gamma > 0$, any $\varepsilon > 0$, and any online algorithm ON in (Case 1), there exists a sequence $\sigma \in \Sigma$ such that $\text{ON}(\sigma)/\text{OPT}(\sigma) \geq 2 - \varepsilon$.

Proof: Since the algorithm ON buys only the machine m_1 at time $j_0 + 1 = 1$, we define the sequence $\sigma_n \in \Sigma$ that ends at time n for sufficiently large $n > 0$. It is obvious that for any $n \geq 1$,

$$\text{ON}(\sigma_n) = c_1 + p_1 n. \quad (9)$$

Let us consider the offline algorithm ADV: For the sequence $\sigma_n \in \Sigma$, it buys the machines m_1 and m_2 at time 1 and 2, respectively. It is immediate that for any $n \geq 2$,

$$\text{ADV}(\sigma_n) = c_1 + p_1 + c_2 + p_2(n - 1). \quad (10)$$

From equations (9) and (10) and the fact that $\text{OPT}(\sigma_n) \leq \text{ADV}(\sigma_n)$ for any $n \geq 2$, it follows that

$$\frac{\text{ON}(\sigma_n)}{\text{OPT}(\sigma_n)} \geq \frac{\text{ON}(\sigma_n)}{\text{ADV}(\sigma_n)} = \frac{c_1 + p_1 n}{c_1 + p_1 + c_2 + p_2(n - 1)} \rightarrow \frac{p_1}{p_2} = \kappa (n \rightarrow \infty).$$

Thus for any $\varepsilon > 0$, there exists a sequence $\sigma_n \in \Sigma$ such that $\text{ON}(\sigma_n)/\text{OPT}(\sigma_n) \geq \kappa - \varepsilon \geq 2 - \varepsilon$. $\blacksquare$

Since the algorithm ON buys at least $r \geq 2$ machines $m_1, m_2, \dots, m_r$ at time $j_0 + 1, j_1 + 1, \dots, j_{r-1} + 1$ in (Case 2), respectively, we consider the first $2s$ machines $m_1, m_2, \dots, m_{2s}$ (in fact, we can take any integer s such that $1 \leq s < \infty$ and $2s \leq r$). We define the sequence $\sigma_s \in \Sigma$ that ends at time $j_{2s-1} + 1$, i.e., it ends just after the algorithm ON buys the machine m_{2s} and produces only one unit of commodity. Since $c_l = \kappa^{\lfloor l/2 \rfloor} c$ for each $1 \leq l \leq 2s$, we have that

$$\text{ON}_c(\sigma_s) = \sum_{i=1}^{2s} c_i = c + 2\kappa c + 2\kappa^2 c + \dots + 2\kappa^{s-1} c + \kappa^s c = c(\kappa + 1) \frac{\kappa^s - 1}{\kappa - 1}; \quad (11)$$

$$\text{ON}_p(\sigma_s) = \sum_{i=1}^{2s-1} p_i(j_i - j_{i-1}) + p_{2s} = \sum_{i=1}^{2s-1} p\kappa^{1-i}(j_i - j_{i-1}) + p_{2s}. \quad (12)$$

Then the following subcases are possible: (Case 2.1) $\text{ON}_c(\sigma_s) < \text{ON}_p(\sigma_s)$; (Case 2.2) $\text{ON}_c(\sigma_s) \geq \text{ON}_p(\sigma_s)$.

Lemma 5.2: For any $\gamma > 0$, any $\varepsilon > 0$, and any online algorithm ON in (Case 2.1), there exists a sequence $\sigma \in \Sigma$ such that $\text{ON}(\sigma)/\text{OPT}(\sigma) \geq 2 - \varepsilon$.

Proof: Let us consider the offline algorithm ADV^1 : For the sequence $\sigma_s \in \Sigma$, it buys the machine m_1 at time 1 and produces one unit of commodity by the machine m_1 . For $2 \leq l \leq 2s$, the algorithm ADV^1 buys the machine m_l at time $j_{l-2} + 2$. So have that

$$\text{ADV}_c^1(\sigma_s) = \sum_{i=1}^{2s} c_i = \text{ON}_c(\sigma_s); \quad (13)$$

$$\text{ADV}_p^1(\sigma_s) \leq \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} \text{ON}_c(\sigma_s) + \frac{1}{\kappa} \text{ON}_p(\sigma_s), \quad (14)$$

where the proof of equation (14) is given in Appendix D.1. Thus it follows that

$$\frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{2}{1 + \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} + \frac{1}{\kappa}}, \quad (15)$$

where the proof of equation (15) is given in Appendix D.2. So we have from equation (15) that

$$\frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{2}{1 + \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} + \frac{1}{\kappa}} \rightarrow 2 (\kappa \rightarrow \infty).$$

Thus for any $\varepsilon < 0$, there exists a sequence $\sigma_s \in \Sigma$ such that $\text{ON}(\sigma_s)/\text{OPT}(\sigma_s) \geq 2 - \varepsilon$. $\blacksquare$

Lemma 5.3: For any $\gamma > 0$, any $\varepsilon > 0$, and any online algorithm ON in (Case 2.2), there exists a sequence $\sigma \in \Sigma$ such that $\text{ON}(\sigma)/\text{OPT}(\sigma) \geq 2 - \varepsilon$.

Proof: For $1 \leq i \leq 2s - 2$, we use $I_i = [j_{i-1} + 1, j_i]$ to denote the i th interval for the sequence $\sigma_s \in \Sigma$, and $I_{2s-1} = [j_{2s-2} + 1, j_{2s-1} + 1]$ to denote the $(2s - 1)$ th interval for the sequence $\sigma_s \in \Sigma$. Assume that I_{2s} is empty. For each $1 \leq l \leq s$, let $J_l = [I_{2l-1}, I_{2l}]$ be the l th phase for the sequence σ_s and $J_l = [I_{2l-1}, I_{2l}]$ is said to be *impatient* if $|I_{2l}| \leq \kappa$; *patient* if $|I_{2l}| \geq \kappa + 1$. Let us consider the offline algorithm ADV^2 : For the sequence $\sigma_s \in \Sigma$, the algorithm ADV^2 buys the machine m_1 at time 1. For each $1 \leq l \leq s - 1$, the algorithm ADV^2 buys the machine m_{2l+1} at time $j_{2l} + 1 \geq 2j_{2l-1} + 2$ if the phase J_l is impatient; the algorithm ADV^2 buys the machine m_{2l+1} at time $j_{2l-1} + \lceil |I_{2l}| / (\kappa + 1) \rceil + 1 \geq j_{2l-1} + 2$ otherwise. Note that for the phase $J_l = [I_{2l-1}, I_{2l}]$, the algorithm ADV^2 always can buy the machine m_{2l+1} , because the machine m_{2l+1} has become available at time $j_{2l-1} + 2 \in J_l$. For the intuition on the behavior of the algorithm ADV^2 , see Appendix E. So from equation (11), we have that

$$\text{ADV}_c^2(\sigma_s) = \sum_{l=1}^s c_{2l-1} = c + \kappa c + \kappa^2 c + \cdots + \kappa^{s-1} c = c \frac{\kappa^s - 1}{\kappa - 1} = \frac{\text{ON}_c(\sigma_s)}{\kappa + 1}. \quad (16)$$

Let $\text{IP} = \{1 \leq l \leq s : J_l \text{ is impatient}\}$ and $\text{P} = \{1 \leq l \leq s : J_l \text{ is patient}\}$. For each $l \in \text{IP}$, let $\text{ADV}_p^2(J_l^{\text{IP}})$ be the production cost of the algorithm ADV^2 in the impatient phase J_l^{IP} , and for each $l \in \text{P}$, let $\text{ADV}_p^2(J_l^{\text{P}})$ be the production cost of the algorithm ADV^2 in the patient phase J_l^{P} . So we have that for each $1 \leq l \leq s$,

$$\text{ADV}_p^2(J_l^{\text{IP}}) \leq p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| + \frac{\kappa(\kappa - 1)}{\gamma} \cdot \frac{c_{2l}}{\kappa^{3l-1}}, \quad (17)$$

$$\text{ADV}_p^2(J_l^{\text{P}}) \leq p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}|, \quad (18)$$

where the proof of equations (17) and (18) are given in Appendices F.1 and F.2, respectively. Thus

$$\text{ADV}_p^2(\sigma_s) \leq \text{ON}_p(\sigma_s) + \frac{\kappa - 1}{\kappa^s - 1} \cdot \frac{\text{ON}_c(\sigma_s)}{\gamma}, \quad (19)$$

where the proof of equation (19) is given in Appendix F.3. So we have that

$$\frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{2}{\frac{1}{\kappa + 1} + \frac{\kappa - 1}{(\kappa^s - 1)\gamma} + 1}, \quad (20)$$

where the proof of equation (20) is given in Appendix F.4. Therefore we have from equation (20) that

$$\frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{2}{\frac{1}{\kappa + 1} + \frac{\kappa - 1}{(\kappa^s - 1)\gamma} + 1} \rightarrow 2(\kappa \rightarrow \infty). \quad (20)$$

Thus for any $\varepsilon > 0$, there exists a sequence $\sigma_s \in \Sigma$ such that $\text{ON}(\sigma_s)/\text{OPT}(\sigma_s) \geq 2 - \varepsilon$. $\blacksquare$

From Lemmas 5.1, 5.2, and 5.3, we have the following theorem:

Theorem 5.4: *For the γ -restricted convex case capital investment, the following holds: For any $\gamma > 0$, any $\varepsilon > 0$, and any online algorithm ON , there exists a sequence $\sigma \in \Sigma$ such that $\text{ON}(\sigma)/\text{OPT}(\sigma) \geq 2 - \varepsilon$.*

6 Concluding Remarks

In this paper, we have investigated the competitive ratio for the convex case capital investment and have introduced a new (natural) notion of “ γ -restricted” to the convex case capital investment.

For the (γ -restricted) convex case capital investment, the following are interesting open problems to be studied. From Theorem 3.3, we already know that the algorithm CONVEX is $(4 + 2\sqrt{2})$ -competitive in the strict sense. Then as the open problem for the convex case capital investment,

- (1) construct an efficient online algorithm ON that is ρ -competitive for some $\rho < 4 + 2\sqrt{2}$.

As we have shown in Theorems 4.3 and 4.4, the algorithm CONVEX is $\{5 + 4/(\gamma - 4)\}$ -competitive for the γ -restricted convex case capital investment, which is almost optimal for sufficiently large $\gamma > 0$. Then as the open problem for the γ -restricted convex case capital investment,

- (2) construct an efficient online algorithm ON that is ρ -competitive for some $\rho < 5$.

For the convex case capital investment, we have noticed in Section 5 that the lower bound of the competitive ratio is 2, however, the best known online algorithm is the algorithm CONVEX that is $(4 + 2\sqrt{2})$ -competitive. Then as the open problem for the convex case capital investment,

- (3) improve the best known lower bound for the competitive ratio.

For the γ -restricted convex case capital investment, we have shown in Theorem 5.4 that the lower bound of the competitive ratio is 2, however, the best known online algorithm is the algorithm CONVEX that is $(5 + 4/(\gamma - 4))$ -competitive. Then as the open problem for the γ -restricted convex case capital investment,

- (4) improve the best known lower bound for the competitive ratio.

REFERENCES

- [1] al-Binali, S., “The Competitive Analysis of Risk Taking with Application to Online Trading,” *Proc. of the 38th IEEE Annual Symposium on Foundations of Computer Science*, pp. 336–344 (1997).
- [2] Azar, Y., Bartal, Y., Feuerstein, E., Fiat, A., Leonardi, S. and Rosén, A., “On Capital Investment,” *Algorithmica*, Vol. 25, pp. 22–36 (1999).
- [3] El-Yaniv, E., Fiat, A., Karp, R. and Turpin, G., “Competitive Analysis of Financial Games,” *Proc. of the 33rd IEEE Annual Symposium on Foundations of Computer Science*, pp. 327–333 (1992).
- [4] Karp, R. M., “On-Line Algorithms Versus Off-Line Algorithms: How Much Is It Worth to Know the Future?” *Proc. of the 12th World Computer Congress*, pp. 416–429 (1992).

- [5] Raghavan, P., “A Statistical Adversary for Online Algorithms,” *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, Vol. 7, pp. 79–83 (1992).
- [6] Sleator, D. D. and Tarjan, R. E., “Amortized Efficiency of List Update and Paging Rules,” *Communications of the ACM*, Vol. 28, pp. 202–208 (1985).

A Proof of Lemma 3.2

Note that $1 - x \geq e^{-2x}$ for any $0 \leq x \leq 1/3$. Thus it follows that

$$\prod_{i=1}^k (1 - x^i) \geq \prod_{i=1}^{\infty} (1 - x^i) \geq \prod_{i=1}^{\infty} e^{-2x^i} = e^{-2\sum_{i=1}^{\infty} x^i} = e^{2x/(x-1)},$$

for any $0 \leq x \leq 1/3$ and any integer $k \geq 1$ (it is obvious that $\prod_{i=1}^{\infty} (1 - x^i) < \infty$ for any $0 < x \leq 1$). Since $2x/(x-1) \geq -3x$ and $e^{-ax} \geq 1 - ax$ for any $a > 0$ and any $0 \leq x \leq 1/3$, we have that $(1 - x)(1 - x^2) \cdots (1 - x^k) \geq e^{-3x} \geq 1 - 3x$.

B Proof of Theorem 3.3

B.1 Proof of Equation (1)

From the definition of the sequence $\sigma_s \in \Sigma$, we have that

$$\begin{aligned} \text{CONVEX}_c(\sigma_s) &= \sum_{i=1}^s c_i = c + c \sum_{i=0}^{s-2} (1 + \alpha^* \beta^*)^i \\ &= c + \frac{c}{\alpha^* \beta^*} \{(1 + \alpha^* \beta^*)^{s-1} - 1\} \geq \frac{c}{\alpha^* \beta^*} \{(1 + \alpha^* \beta^*)^{s-1} - 1\}, \end{aligned}$$

where the second equality follows from the definition of the capital cost c_i .

B.2 Proof of Equation (2)

From the definition of the sequence $\sigma_s \in \Sigma$, we have that

$$\begin{aligned} \text{CONVEX}_p(\sigma_s) &= p_1 j_1 + p_2 j_2 + \cdots + p_{s-1} j_{s-1} + p_s \geq p \frac{c}{\alpha^* p} + \beta^* \sum_{i=2}^{s-1} c_i + p \prod_{i=1}^{s-1} (1 - \delta^i) \\ &\geq \frac{c}{\alpha^*} + p(1 - 3\delta) + \beta^* \sum_{i=2}^{s-1} (1 + \alpha^* \beta^*)^i c \\ &= \frac{c}{\alpha^*} + p(1 - 3\delta) + \frac{c}{\alpha^*} \{(1 + \alpha^* \beta^*)^{s-2} - 1\} \geq \frac{c}{\alpha^*} (1 + \alpha^* \beta^*)^{s-2}, \end{aligned}$$

where the first inequality follows from the definition of j_i , the property of the function $\lceil \cdot \rceil$ and the second inequality follows from Lemma 3.2 and the definition of the capital cost c_i , and the last inequality follows from the assumption that $\delta > 0$ is an extremely small constant.

B.3 Proof of Equation (4)

From the definitions of j_i , c_i , and p_i , we have that

$$\begin{aligned} \text{ADV}_p(\sigma_s) &= p j_1 + p j_2 + \cdots + p j_{s-1} + p \\ &= p + p \left\lceil \frac{c}{\alpha^* p} \right\rceil + \sum_{i=2}^{s-1} p \left\lceil \frac{\beta^* c_i}{p_i} \right\rceil \leq p + p \left(\frac{c}{\alpha^* p} + 1 \right) + \sum_{i=2}^{s-1} \left(\frac{\beta^* c_i}{1 - 3\delta} + p \right) \\ &= sp + \frac{c}{\alpha^*} + \frac{\beta^*}{1 - 3\delta} \sum_{i=2}^{s-1} c_i \\ &= sp + \frac{c}{\alpha^*} + \frac{c}{(1 - 3\delta)\alpha^*} \{(1 + \alpha^* \beta^*)^{s-2} - 1\} \geq sp + \frac{c}{(1 - 3\delta)\alpha^*} (1 + \alpha^* \beta^*)^{s-2}, \end{aligned}$$

where the first inequality follows from Lemma 3.2 and the property of the function $\lceil \cdot \rceil$ and the second inequality follows from the assumption that $\delta > 0$ is an extremely small constant.

B.4 Proof of Equation (5)

From equations (1), (2), (3), and (4), it follows that

$$\frac{\text{CONVEX}(\sigma_s)}{\text{OPT}(\sigma_s)} \geq \frac{\text{CONVEX}(\sigma_s)}{\text{ADV}(\sigma_s)}$$

$$\begin{aligned}
&= \frac{\text{CONVEX}_c(\sigma_s) + \text{CONVEX}_p(\sigma_s)}{\text{ADV}_c(\sigma_s) + \text{ADV}_p(\sigma_s)} \geq \frac{\frac{c}{\alpha^* \beta^*} \{(1 + \alpha^* \beta^*)^{s-2} - 1\} + \frac{c}{\alpha^*} (1 + \alpha^* \beta^*)^{s-2}}{c + sp + \frac{c}{(1 - 3\delta)\alpha^*} (1 + \alpha^* \beta^*)^{s-2}} \\
&= \frac{1 + \alpha^* + \frac{1}{\beta^*}}{\frac{\alpha^*}{(1 + \alpha^* \beta^*)^{s-2}} + \frac{\alpha^* p}{c} \frac{s}{(1 + \alpha^* \beta^*)^{s-2}} + \frac{1}{1 - 3\delta}},
\end{aligned}$$

where the first inequality follows from the fact that $\text{OPT}(\sigma_s) \leq \text{ADV}(\sigma_s)$ for the sequence $\sigma_s \in \Sigma$.

C Proof of Lemma 4.2

For any sequence $\sigma \in \Sigma$ and any integer $s \geq 1$, we assume that the algorithm CONVEX buys s machines $m_1, m_2, \dots, m_s$. Let $j_0 = 0$. From the description of the algorithm CONVEX in Section 3, it is obvious that it buys the machine $m_1 = (t_1, c_1, p_1)$ at time $j = 1$. For each $1 \leq l \leq s$, we assume that the algorithm CONVEX buys the machine $m_l = (t_l, c_l, p_l)$ at time $j = j_{l-1} + 1$. For each $1 \leq l \leq s - 1$, we use $I_l = [j_{l-1} + 1, j_l]$ to denote the l th interval for the sequence $\sigma \in \Sigma$, and let $\tau_l \in I_l$ be the maximum such that $\text{prod}[j_{l-1} + 1, j_{l-1} + \tau_l] \leq \beta c_l$ in Step 2-1. For each $1 \leq l \leq s - 1$, we have that

$$\tau_l = \left\lceil \frac{c_l \beta}{p_l} \right\rceil - 1 \geq \beta \gamma - 1. \quad (21)$$

For each $1 \leq l \leq s - 1$, partition I_l into two subintervals, and let $\text{Idle}_l = [j_{l-1} + 1, j_{l-1} + \tau_l]$ be an *idle* subinterval of I_l and $\text{Active}_l = \text{prod}[j_{l-1} + \tau_l + 1, j_l]$ be an *active* subinterval of I_l . For any sequence $\sigma \in \Sigma$ and any $j \geq 1$, we use $\sigma_j \in \Sigma$ to denote the subsequence of $\sigma \in \Sigma$ up to time j . From the description of the algorithm CONVEX, it follows that $c_1 \leq c_2 \leq \dots \leq c_s$ and $p_1 > p_2 > \dots > p_s$ for all of the machines $m_1, m_2, \dots, m_s$ bought by the algorithm CONVEX. We show the lemma by induction on time $j \geq 1$.

In Step 1, the algorithm CONVEX buys the machine $m_1 = (t_1, c_1, p_1)$ that minimizes $m + p$ among all machines available at time $j = 1$. Thus we have that $\text{CONVEX}_p(\sigma_1) = p_1 \leq m_1 + p_1 = \text{OPT}(\sigma_1)$.

At time $j \geq 2$, assume that $\text{CONVEX}_p(\sigma_k) \leq \text{OPT}(\sigma_k)$ for any $1 \leq k \leq j - 1$. Let $m_l = (t_l, c_l, p_l)$ be the machine used by the algorithm CONVEX at time j and $m' = (t', c', p')$ be the machine used by the optimal offline algorithm OPT at time j , where c' is its capital cost and p' is its production cost. If $p' \geq p_l$, then we have that $\text{CONVEX}_p(\sigma_j) = \text{CONVEX}_p(\sigma_{j-1}) + p_l \leq \text{OPT}(\sigma_{j-1}) + p' \leq \text{OPT}(\sigma_j)$.

If $p' < p_l$, then the algorithm CONVEX did not buy the machine m' up to time $j - 1$. From the definition of the convex case, $p' < p_l$ implies $c' \geq c_l$. Then we have the following two cases: (Case 1) $j - 1 \in \text{Active}_l = [j_{l-1} + \tau_l + 1, j_l]$; (Case 2) $j - 1 \in \text{Idle}_l = [j_{l-1} + 1, j_{l-1} + \tau_l]$.

For Case 1, we have that $\alpha \cdot \text{CONVEX}_p(\sigma_{j-1}) < c'$, i.e., $\text{CONVEX}_p(\sigma_{j-1}) < c'/\alpha$. From equation (21), it follows that $j - 1 \geq j_{l-1} + \tau_l + 1 \geq j_0 + \tau_1 + 1 = \tau_1 + 1 = \beta \gamma$. Thus

$$\begin{aligned}
\text{CONVEX}_p(\sigma_j) &= \text{CONVEX}_p(\sigma_{j-1}) + p_l \leq \left(1 + \frac{1}{j-1}\right) \text{CONVEX}_p(\sigma_{j-1}) \\
&\leq \frac{j}{j-1} \frac{c'}{\alpha} \leq \frac{\beta \gamma + 1}{\alpha \beta \gamma} \text{OPT}(\sigma_j) \leq \text{OPT}(\sigma_j),
\end{aligned} \quad (22)$$

where the first inequality follows from the monotonicity that $p_1 > p_2 > \dots > p_s$ and the last inequality follows from the assumption that $(\beta \gamma + 1)/\alpha \beta \gamma \leq 1$. For Case 2, we have that

$$\begin{aligned}
\text{CONVEX}_p(\sigma_j) &= \text{CONVEX}_p(\sigma_{j_{l-1}}) + \sum_{i=j_{l-1}+1}^{j-1} p_l + p_l \leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \sum_{i=j_{l-1}+1}^{j_{l-1}+\tau_l} p_l + p_l \\
&= \text{CONVEX}_p(\sigma_{j_{l-1}}) + \left(1 + \frac{1}{\tau_l}\right) \sum_{i=j_{l-1}+1}^{j_{l-1}+\tau_l} p_l \leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \left(1 + \frac{1}{\tau_l}\right) \beta c_l \\
&\leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \frac{\beta^2 \gamma}{\beta \gamma - 1} c_l \leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \frac{\beta^2 \gamma}{\beta \gamma - 1} c',
\end{aligned} \quad (23)$$

where the second inequality follows from the definition of the subinterval $\text{Idle}_l = [j_{l-1} + 1, j_{l-1} + \tau_l]$, i.e., $\text{prod}[j_{l-1} + 1, \tau_l] < \beta c_l$, and the third inequality follows from equation (21).

Then we consider the following two subcases: (Case 2.1) $t' \geq j_{l-1} + 1$; (Case 2.2) $t' \leq j_{l-1}$, where t' is the time at which the machine m' becomes available. For Case 2.1, we have from equation (23) that

$$\text{CONVEX}_p(\sigma_j) \leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \frac{\beta^2\gamma}{\beta\gamma - 1} c' \leq \text{OPT}_p(\sigma_{j_{l-1}}) + c' \leq \text{OPT}(\sigma_j), \quad (24)$$

where the second inequality follows from the assumption that $1/\alpha + \beta^2\gamma/(\beta\gamma - 1) \leq 1$. For Case 2.2, we have that $c' > \alpha \cdot \text{CONVEX}_p(\sigma_{j_{l-1}})$, since if $c' \leq \alpha \cdot \text{CONVEX}_p(\sigma_{j_{l-1}})$, then the algorithm CONVEX would have bought the machine m' , which contradicts the assumption that $p' < p_l$. So from equation (23),

$$\begin{aligned} \text{CONVEX}_p(\sigma_j) &\leq \text{CONVEX}_p(\sigma_{j_{l-1}}) + \frac{\beta^2\gamma}{\beta\gamma - 1} c' \leq \frac{c'}{\alpha} + \frac{\beta^2\gamma}{\beta\gamma - 1} c' \\ &= \left(\frac{1}{\alpha} + \frac{\beta^2\gamma}{\beta\gamma - 1} \right) c' \\ &\leq \left(\frac{1}{\alpha} + \frac{\beta^2\gamma}{\beta\gamma - 1} \right) \text{OPT}(\sigma_j) \\ &\leq \text{OPT}(\sigma_j) \end{aligned} \quad (25)$$

where the last inequality follows from the assumption that $1/\alpha + \beta^2\gamma/(\beta\gamma - 1) \leq 1$. Thus from equations (22), (24), and (25), it follows that $\text{CONVEX}_p(\sigma) \leq \text{OPT}(\sigma)$ for any sequence $\sigma \in \Sigma$.

D Proof of Lemma 5.2

D.1 Proof of Equation (14)

From equations (11) and (12), it follows that

$$\begin{aligned} \text{ADV}_p^1(\sigma_s) &= p_1 + \sum_{i=2}^{2s} p_i(j_{i-1} - j_{i-2}) \\ &= p_1 + \frac{1}{\kappa} \sum_{i=1}^{2s-1} p_i(j_i - j_{i-1}) \\ &= p_1 + \frac{1}{\kappa} \{ \text{ON}_p(\sigma_s) - p_{2s} \} \\ &\leq p_1 + \frac{1}{\kappa} \text{ON}_p(\sigma_s) \\ &\leq \frac{c}{\gamma} + \frac{1}{\kappa} \text{ON}_p(\sigma_s) \\ &= \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} \text{ON}_c(\sigma_s) + \frac{1}{\kappa} \text{ON}_p(\sigma_s). \end{aligned}$$

D.2 Proof of Equation (15)

From equations (11), (12), (13), and (14), it follows that

$$\begin{aligned} \frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} &\geq \frac{\text{ON}(\sigma_s)}{\text{ADV}_c^1(\sigma_s)} = \frac{\text{ON}_c(\sigma_s) + \text{ON}_p(\sigma_s)}{\text{ADV}_c^1(\sigma_s) + \text{ADV}_p^1(\sigma_s)} \\ &\geq \frac{\text{ON}_c(\sigma_s) + \text{ON}_p(\sigma_s)}{\text{ON}_c(\sigma_s) + \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} \text{ON}_c(\sigma_s) + \frac{1}{\kappa} \text{ON}_p(\sigma_s)} \\ &= \frac{\frac{\text{ON}_c(\sigma_s)}{\text{ON}_p(\sigma_s)} + 1}{\left\{ 1 + \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} \right\} \frac{\text{ON}_c(\sigma_s)}{\text{ON}_p(\sigma_s)} + \frac{1}{\kappa}} \\ &\geq \frac{2}{1 + \frac{\kappa - 1}{(\kappa + 1)(\kappa^s - 1)\gamma} + \frac{1}{\kappa}}, \end{aligned}$$

where the last inequality follows from the fact that $s \geq 1$ and $\text{ON}_c(\sigma_s)/\text{ON}_p(\sigma_s) < 1$.

E Behavior of the Offline Algorithm ADV^2

E.1 Behavior of the Offline Algorithm ADV^2 in the Impatient Phase

The following figure describes the behavior of the algorithm ON in the impatient phase $J_l = [I_{2l-1}, I_{2l}]$, where $I_{2l-1} = [j_{2l-2} + 1, j_{2l-1}]$ and $I_{2l} = [j_{2l-1} + 1, j_{2l}]$. Note that the algorithm ON buys the machines m_{2l-1} , m_{2l} , and m_{2l+1} at time $j = j_{2l-2} + 1$, $j = j_{2l-1} + 1$, and $j = j_{2l} + 1$, respectively (in Figure A.1, these are given by “●”). We also note that the machines m_{2l} and m_{2l+1} become available at time $j = j_{2l-2} + 2$ and $j = j_{2l-1} + 2$, respectively (in Figure A.1, these are given by “○”). We finally note that in Figure A.1, the dashed lines denote the production cost incurred by the algorithm ON.

In Figure A.2, we describe the behavior of the algorithm ADV in the impatient phase J_l . The algorithm ADV buys the machines m_{2l-1} and m_{2l+1} at time $j = j_{2l-2} + 1$ and $j = j_{2l} + 1$, respectively, i.e., the algorithm ADV does not buy the machine m_{2l} at any time j in the impatient phase J_l and it produces all units of commodity by the machine m_{2l-1} throughout the impatient phase J_l . In Figure A.2, the solid lines denote the production cost incurred by the algorithm ADV.

E.2 Behavior of the Offline Algorithm ADV^2 in the Patient Phase

The following figure shows the behavior of the algorithm ON in the patient phase $J_l = [I_{2l-1}, I_{2l}]$, where $I_{2l-1} = [j_{2l-2} + 1, j_{2l-1}]$ and $I_{2l} = [j_{2l-1} + 1, j_{2l}]$. As in the impatient phase J_l , the algorithm ON buys the machines m_{2l-1} , m_{2l} , and m_{2l+1} at time $j = j_{2l-2} + 1$, $j = j_{2l-1} + 1$, and $j = j_{2l} + 1$, respectively (in Figure A.3, these are given by “●”). We note that the machines m_{2l} and m_{2l+1} become available at time $j = j_{2l-2} + 2$ and $j = j_{2l-1} + 2$

Figure A.1: Behavior of the Algorithm ON in the Impatient Phase J_l

Figure A.2: Behavior of the Algorithm ADV in the Impatient Phase J_l

2, respectively (in Figure A.3, these are given by “○”). We also note that in Figure A.3, the dashed lines denote the production cost incurred by the algorithm ON.

In Figure A.4, we describe the behavior of the algorithm ADV in the patient phase J_l . The algorithm ADV buys the machines m_{2l-1} and m_{2l+1} at time $j = j_{2l-2} + 1$ and $j = j_{2l-1} + \lfloor |I_{2l}| / (\kappa + 1) \rfloor + 1$, respectively, i.e., the algorithm ADV² does not buy the machine m_{2l} at any time j in the patient phase J_l and it produces units of commodity by the machine m_{2l-1} in $[j_{2l-2} + 1, j_{2l-1} + \lfloor |I_{2l}| / (\kappa + 1) \rfloor] \subseteq J_l$ and by the machine m_{2l+1} in $[j_{2l-1} + \lfloor |I_{2l}| / (\kappa + 1) \rfloor + 1, j_{2l}] \subseteq J_l$.

The algorithm ADV² spends more production cost than the algorithm ON at each time $j \in [j_{2l-1} + 1, j_{2l-1} + \lfloor |I_{2l}| / (\kappa + 1) \rfloor]$, while the algorithm ADV² spends less production cost than the algorithm ON at each time $j \in [j_{2l-1} + \lfloor |I_{2l}| / (\kappa + 1) \rfloor + 1, j_{2l}]$. Since the phase J_l is patient, the loss of the production cost incurred by the algorithm ADV² (the rectangle (L) in Figure A.4) is less than the gain of the production cost incurred by the algorithm ADV² (the rectangle (G) in Figure A.4).

F Proof of Lemma 5.3

F.1 Proof of Equation (17)

From the definitions of c_i and p_i , it follows that for any $l \in \text{IP}$,

$$\text{ADV}_p^2(J_l^{\text{IP}}) = p_{2l-1}(|I_{2l-1}| + |I_{2l}|)$$

Figure A.3: Behavior of the Algorithm ON in the Patient Phase J_l

Figure A.4: Behavior of the Algorithm Adv² in the Patient Phase J_l

$$\begin{aligned}
&= p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| + |I_{2l}|(p_{2l-1} - p_{2l}) \\
&= p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| + |I_{2l}|(\kappa p_{2l} - p_{2l}) \\
&\leq p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| + \kappa(\kappa - 1)p_{2l} \\
&= p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| + \frac{\kappa(\kappa - 1)}{\gamma} \cdot \frac{c_{2l}}{\kappa^{3l-1}}.
\end{aligned}$$

F.2 Proof of Equation (17)

From the definition of p_i and the property of the function $\lceil \cdot \rceil$, it follows that for any $l \in \mathbb{P}$,

$$\begin{aligned}
\text{ADV}_p^2(J_l^{\mathbb{P}}) &= p_{2l-1} \left(|I_{2l-1}| + \left\lceil \frac{|I_{2l}|}{\kappa + 1} \right\rceil \right) + p_{2l+1} \left(|I_{2l}| - \left\lceil \frac{|I_{2l}|}{\kappa + 1} \right\rceil \right) \\
&= p_{2l-1}|I_{2l-1}| + p_{2l+1}|I_{2l}| + \left\lceil \frac{|I_{2l}|}{\kappa + 1} \right\rceil (p_{2l-1} - p_{2l+1}) \\
&\leq p_{2l-1}|I_{2l-1}| + \frac{p_{2l}}{\kappa} |I_{2l}| + \frac{|I_{2l}|}{\kappa + 1} \left(\kappa p_{2l} - \frac{p_{2l}}{\kappa} \right) \\
&\leq p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}| \left\{ \frac{1}{\kappa} + \frac{\kappa}{\kappa + 1} - \frac{1}{\kappa(\kappa + 1)} \right\} \\
&= p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}|.
\end{aligned}$$

F.3 Proof of Equation (19)

From equations (17) and (18), it follows that

$$\begin{aligned}
\text{ADV}_p^2(\sigma_s) &= \sum_{l \in \mathbb{IP}} \text{ADV}_p^2(J_l^{\mathbb{IP}}) + \sum_{l \in \mathbb{P}} \text{ADV}_p^2(J_l^{\mathbb{P}}) \\
&\leq \sum_{l=1}^s (p_{2l-1}|I_{2l-1}| + p_{2l}|I_{2l}|) + \frac{\kappa(\kappa - 1)}{\gamma} \sum_{l \in \mathbb{IP}} \frac{c_{2l}}{\kappa^{3l-1}} \\
&= \text{ON}_p(\sigma_s) - p_{2s} + \frac{\kappa(\kappa - 1)}{\gamma} \sum_{l=1}^s \frac{c_{2l}}{\kappa^{3l-1}} \\
&\leq \text{ON}_p(\sigma_s) + \frac{c}{\gamma} \cdot \frac{\kappa(\kappa - 1)}{\kappa} \cdot \frac{1 - 1/\kappa^{2s}}{1 - 1/\kappa^2} \\
&\leq \text{ON}_p(\sigma_s) + \frac{c}{\gamma} \cdot \frac{\kappa - 1}{1 - 1/\kappa^2} = \text{ON}_p(\sigma_s) + \frac{c}{\gamma} \cdot \frac{\kappa^2}{\kappa + 1} \\
&= \text{ON}_p(\sigma_s) + \frac{\kappa^2(\kappa - 1)}{(\kappa + 1)(\kappa^s - 1)} \cdot \frac{\text{ON}_c(\sigma_s)}{\gamma} \\
&\leq \text{ON}_p(\sigma_s) + \frac{\kappa - 1}{\kappa^s - 1} \cdot \frac{\text{ON}_c(\sigma_s)}{\gamma}.
\end{aligned}$$

F.4 Proof of Equation (20)

From equations (11), (12), (16), and (19), it follows that

$$\begin{aligned}
\frac{\text{ON}(\sigma_s)}{\text{OPT}(\sigma_s)} &\geq \frac{\text{ON}(\sigma_s)}{\text{ADV}^2(\sigma_s)} \\
&= \frac{\text{ON}_c(\sigma_s) + \text{ON}_p(\sigma_s)}{\text{ADV}_c^2(\sigma_s) + \text{ADV}_p^2(\sigma_s)} \\
&\geq \frac{\text{ON}_c(\sigma_s) + \text{ON}_p(\sigma_s)}{\frac{\text{ON}_c(\sigma_s)}{\kappa + 1} + \text{ON}_p(\sigma_s) + \frac{\kappa - 1}{\kappa^s - 1} \cdot \frac{\text{ON}_c(\sigma_s)}{\gamma}} \\
&= \frac{1 + \frac{\text{ON}_p(\sigma_s)}{\text{ON}_c(\sigma_s)}}{\frac{1}{\kappa + 1} + \frac{\kappa - 1}{(\kappa^s - 1)\gamma} + \frac{\text{ON}_p(\sigma_s)}{\text{ON}_c(\sigma_s)}} \\
&\geq \frac{2}{\frac{1}{\kappa + 1} + \frac{\kappa - 1}{(\kappa^s - 1)\gamma} + 1},
\end{aligned}$$

where the last inequality follows from the fact that $s \geq 1$ and $\text{ON}_p(\sigma_s)/\text{ON}_c(\sigma_s) \leq 1$.