Computer Experiments for Circle Packings on Surfaces

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Circle packings on surfaces with complex projective structures were investigated in the previous paper “Circle packings on surfaces with projective structures”. This paper describes computer experiments performed in the process.

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1. Introduction

An orientable closed surface Σ_g of genus g admits a spherical metric ($g = 0$), a Euclidean metric ($g = 1$) or a hyperbolic metric ($g \geq 2$) according to its genus. Though the sphere admits the unique metric essentially, the space of Euclidean metrics on the torus and the space of hyperbolic metrics on a surface with genus higher than one are topologically cells of dimension 2 and $6g - 6$ respectively. In spite of these plentifulness of metrics for $g \geq 1$, the Andreev–Thurston theorem [1, 4] shows the uniqueness of the metric which admits a circle packing for each combinatorial type of a circle packing.

In the previous paper [2], we started from the Andreev–Thurston theorem, let the metric structures relax into complex projective structures, and then deformed the structure admitting packing. The main result of the paper states that the space of deformations for packings with one circle is a cell of dimension 2 for $g = 1$ and $6g - 6$ for $g \geq 2$. (Note that the dimension of the space of complex projective structures on a surface is 4 for $g = 1$ and $12g - 12$ for $g \geq 2$.)

In the process of the study, computer experiments were performed with some programs. These programs simply calculate concrete values of a parameter and draw circle packings for the values. Though the experiments were not such as to bring inconceivable results to us, they enabled us to advance with verifications of logical arguments. In this paper, we describe our computer experiments.

2. Circle Packings and Complex Projective Structures

In this section, we review the results of [2, 3]. Orientable closed surfaces admit metrics of constant curvature. The sphere admits a unique spherical metric. The torus admits Euclidean metrics. Though one metric trivially gives a family of its scalar multiplications, by identifying them, the space of marked Euclidean metrics is of dimension 2. A surface of genus $g \geq 2$ admits hyperbolic metrics. The space of hyperbolic metrics is topologically a cell of dimension $6g - 6$.

For a set of circles on a surface with a metric of constant curvature, we call it a *circle packing* if any distinct two circles have disjoint interiors, and moreover if the complementary region of the union of all circles consists of triangular interstices.

We express the combinatorial type of tangencies of a circle packing by a nerve τ . The *nerve* of a packing is defined as a graph on the surface constructed as follows. For each circle, we pick up a point in the circle and add it to the graph as a vertex. For each contact point of circles, we draw a curve connecting two points for the contacting two circles through the contact point, and then add it to the graph as an edge. From the condition about the complementary region, the nerve forms the 1-skeleton of a certain triangulation. Besides, for the universal cover and its covering projection, the preimage of the nerve by the covering projection is simple in the graph theoretical sense, since two distinct circles cannot share two contact points and a circle cannot contact with itself in the plane or the sphere.

The Andreev–Thurston theorem concerns the metrics that allow a packing, and states rigidity, that is, uniqueness of the metric for a packing. We call the unique metric structure the *Andreev–Thurston solution*.

Theorem 2.1 (Andreev [1], Thurston [4]). *For an orientable closed surface Σ_g of genus g and a combinatorial type τ of a circle packing, there exists a unique metric of constant curvature which admits a circle packing of the specified combinatorial type τ . Moreover, the packing is essentially unique.*

Let us rethink the situation from the beginning. Metrics may seem to play an important role in the study of circle packings. But, for a (G, X) -structure (see [4]), if a figure in the base space X and its image by an element of the group G are both naturally recognized as a circle in a usual sense, then by defining a circle as such a figure, we can still define a

circle and a circle packing naturally without metrics.

Therefore, in [2], we let the metric structures relax into the complex projective structures and studied the space of the structures that admit a circle packing.

Definition 2.2. A *complex projective structure* is a (G, X) -structure, where the group G is the group $\mathrm{PSL}_2(\mathbb{C})$ of complex projective transformations and the base space X is the Riemann sphere $\hat{\mathbb{C}}$.

The space $\mathcal{P}_g$ of complex projective structures of the surface Σ_g is a fiber bundle with the uniformization map as the bundle projection over the Teichmüller space $\mathcal{T}_g$, which is the space of conformally equivalent classes of conformal structures. The Teichmüller space and the fiber are both topologically a cell of dimension 0 for $g = 0$, 2 for $g = 1$ and $6g - 6$ for $g \geq 2$, and hence, the space $\mathcal{P}_g$ of complex projective structures is also a cell of dimension 0 for $g = 0$, 4 for $g = 1$ and $12g - 12$ for $g \geq 2$. The space of metrics of constant curvature appears as the zero section of this bundle structure.

Here we confirm the definition of a circle on the surface Σ_g with a complex projective structure.

Definition 2.3. An *open disk* on a surface Σ_g with a complex projective structure is a point set which is topologically an open disk and each lift of which is homeomorphically mapped to a Euclidean disk in $\hat{\mathbb{C}}$ by a developing map, where a *developing map* is a map from the universal cover of Σ_g to $\hat{\mathbb{C}}$ obtained by continuing local charts along paths on Σ_g .

The space $\mathrm{CPDef}_{g,\tau}$ of complex projective structures on Σ_g which admit a circle packing of a combinatorial type τ intersects with the zero section at one point, as the Andreiev–Thurston theorem states.

Though the result was described in terms of complex affine structures in [3], we state the result in terms of complex projective structures:

Theorem 2.4 (Mizushima). *Let τ be the unique combinatorial type of circle packings by one circle on the torus. The space $\mathrm{CPDef}_{1,\tau}$ of complex projective structures on the torus which admit a circle packing of the combinatorial type τ is topologically a cell of dimension 2, and is homeomorphically projected down to the Teichmüller space $\mathcal{T}_1$ by the uniformization map. Each complex projective structure in $\mathrm{CPDef}_{1,\tau}$ admits the unique circle packing by one circle up to “sliding moves”.*

This gives other uniformization of conformally equivalent classes that maps each conformal structure to the unique complex projective structure admitting a circle packing by one circle. The result shows the neat correspondence between the $\mathrm{CPDef}_{1,\tau}$ and the Teichmüller space $\mathcal{T}_1$.

But the proof depends on a parameter, that is not applicable for $g \geq 2$. Hence, we introduced a new parameter, which is described in the next section, and studied for the higher genus case. Then, we obtained the following.

Theorem 2.5 (Kojima, Mizushima and Tan). *Let Σ_g denote an orientable surface of fixed genus $g \geq 2$. Let τ be a combinatorial type of a circle packing by one circle on Σ_g . The space $\mathrm{CPDef}_{g,\tau}$ of complex projective structures on Σ_g which admit a circle packing of type τ is topologically a cell of dimension $6g - 6$. Each complex projective structure in $\mathrm{CPDef}_{g,\tau}$ admits the unique circle packing by one circle.*

In the paper, we also studied $\mathrm{CPDef}_{g,\tau}$ for an arbitrary combinatorial type τ , and then obtained the following result about a local situation around the Andreiev–Thurston solution.

Theorem 2.6 (Kojima, Mizushima and Tan). *For an arbitrary $g \geq 1$ and τ , the local dimension of $\mathrm{CPDef}_{g,\tau}$ around the Andreiev–Thurston solution matches with the dimension of the Teichmüller space $\mathcal{T}_g$.*

From these results, the authors expect that the result similar to Theorem 2.4 holds for arbitrary g and τ .

3. Cross Ratio Parameters and Space of Circle Packings

In this section, we introduce a cross ratio parameter which was used in the paper [2] and computer experiments, and then describe the space of circle packings.

We fix a surface Σ_g of genus g and a combinatorial type τ of a packing on the surface. We start from introducing the set of pairs (μ, P) where $\mu \in \mathcal{P}_g$ is a complex projective structure admitting a circle packing P of the combinatorial type τ . We call it the *circle packing space* and denote it by $\mathrm{CPS}_{g,\tau}$. Note that, at least at the beginning, it is unknown whether a projective structure admits more than one circle packing, or $\mathrm{CPS}_{g,\tau}$ is bijectively projected down to $\mathrm{CPDef}_{g,\tau}$ by forgetting the packing P of (μ, P) .

Though the parameter used in [3] is roughly of $\mathcal{P}_1$, the cross ratio parameter is a parameter for $\mathrm{CPS}_{g,\tau}$ rather than $\mathrm{CPDef}_{g,\tau}$ or $\mathcal{P}_g$. A *cross ratio parameter* is defined as follows.

We fix a developed image of a circle packing of type τ . For an edge e_i of the nerve, we watch two triangles sharing the edge and four circles corresponding to four vertices of the two triangles, then calculate the cross ratio q of the four

contact points p_2, p_3, p_4 and p_1 (Figure 1a);

$$q(p_2, p_3, p_4, p_1) = \frac{(p_2 - p_3)(p_4 - p_1)}{(p_3 - p_4)(p_1 - p_2)}.$$

Let f be the complex projective transformation sending p_1, p_2, p_3 to $0, 1, \infty$. A rectangle appears when we normalize the configuration of four circles by the transformation f as in Fig. 1b. Since $q(p_2, p_3, p_4, p_1)$ is equal to $f(p_4)$, the value of the cross ratio is a purely imaginary number $x_i\sqrt{-1}$ with a positive number x_i . For the edge e_i , we assign the positive real number x_i . Even if we choose another set of contact points $\{p_5, p_4, p_3, p_2\}$, a rectangle in its normalized picture is just a rotated picture of the previous one and hence the positive real number x'_i calculated in the same way coincides with the value x_i . Therefore, the value x_i does not depend on the direction of the edge. Of course, since a cross ratio is invariant under complex projective transformations, the value x_i does not depend on any developing map and any choice of a developed image of the four circle configuration, and hence is well-defined. By calculating the value x_i for all edges e_i in this process, we define the cross ratio parameter $\mathbf{c} : E(\tau) \rightarrow \mathbb{R}_+$ for a packing, where $E(\tau)$ is the set of edges of the nerve τ . We call each x_i a *variable*. In the normalized picture, there exists a one-parameter family of rectangles, which corresponds to the family of configurations up to complex projective transformations. The value x_i parameterizes the family.

Fig. 1. Configuration and its normalized picture

This cross ratio parameter must satisfy two kinds of conditions in order to realize a packing. When we normalize the four circle configuration around an edge e_i with $\mathbf{c}(e_i) = x_i$ by the complex projective transformation sending p_1, p_2 and p_3 to $\infty, \sqrt{-1}$ and 0 , the complex projective transformation which maps the interstice $\mathcal{I}_1$ to $\mathcal{I}_2$ and fixes the real axis as a set is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & x_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

(see Fig. 2). We call this an *associated matrix* of the edge e_i .

Fig. 2. Another normalized picture

For each vertex v , through a sequence of interstice-sliding transformations represented by A_i 's, an interstice at last comes back to the starting position. If we read off the edges sharing v as an end in clockwise order and denote them $e_{i_1}, e_{i_2}, \dots, e_{i_n}$, the following equation must hold.

$$(1) \quad A_{i_1} A_{i_2} \dots A_{i_n} = -E.$$

Here, E denotes the identity matrix of dimension 2.

Fig. 3. Flower and edges around a vertex

Since the equation allows circles to surround the central circle twice or more as well as once, we need to exclude the case other than once. For a sequence of matrices which appears on the left side of Eq. (1), when we watch the product of each subsequence, an additional condition is represented as positivity or negativity of the entries of the product.

Thus the cross ratio parameter must satisfy Eq. (1) and these inequality conditions. Hence, the space of valid cross ratio parameters is a semi-algebraic set. We call it the *cross ratio parameter space* and denote it by $\mathcal{C}_{g,\tau}$. Since the correspondence between the circle packing space and the cross ratio parameter space turns out to be bijective, the circle packing space $\text{CPS}_{g,\tau}$ is explicitly represented by the cross ratio parameters.

By Gauss–Bonnet theorem, the number of edges is $6g - 6 + 3|V(\tau)|$ for $|V(\tau)|$ as the number of vertices. For each vertex, the condition (1) is a real $(2, 2)$ -matrix equation. Since an associated matrix and the product of associated matrices have determinant 1, the formal dimension of the set $\mathcal{C}_{g,\tau}$ is $6g - 6$.

4. Computer Experiments

When we wanted to know how circle packings vary, it seemed impossible to imagine the behavior of circle packings only with the parameters.

In contrast with translations, rotations, magnifications and reductions in the Euclidean transformations which preserve the shapes of figures, non-Euclidean transformations in the complex projective transformations deform the configuration of circles in a circle packings. Therefore a circle packing has different developed images depending on the choice of developing maps and developed images.

Those were the reasons that let us start computer experiments to see what happens in the space of circle packings. We produced programs that solve the equation numerically, draw the circle packing for the solution, and show various types of pictures of one circle packing.

In the process of this study, we mainly assumed that a combinatorial type τ has only one vertex as a graph, or a circle packing consists of one circle. Here we describe the case where τ corresponds to a packing with one circle on a surface of genus two. In this case, the cross ratio parameter has 9 variables corresponding to the 9 edges of the nerve. By using Eq. (1), the formal dimension of $\text{CPS}_{2,\tau}$ is shown to be 6, which was actually proven later.

Note that even in this one circle case, there are essentially different eight combinatorial types in contrast with the torus case which has the unique combinatorial type with one vertex. Therefore, the matrix condition depends on the type τ . For example, the following equation is the matrix equation (1) for the type illustrated in Fig. 4.

$$A_1 A_2 A_3 A_4 A_2 A_5 A_3 A_4 A_5 A_1 A_6 A_7 A_8 A_6 A_9 A_7 A_8 A_9 = -E.$$

For every edge e_i , the associated matrix A_i with the variable x_i in an entry appears exactly twice in this equation.

We produced a computer program. The program first divides variables into two classes — 6 free variables and 3 dependent variables, then solves the equation (1) with appropriate values of free variables given, and obtains concrete values of dependent variables.

Fig. 4. An example of a type

According to the concrete solutions with respect to the cross ratio parameter, the program draws a developed image of the corresponding circle packing.

By repeatedly changing values of free variables, solving the equation and drawing a circle packing, the program produces animations of circle packings. Some kind of animation is available by changing one variable in a back-and-forth motion, changing all variables synchronously, or changing variables with random variation.

Here, results of the experiments are described with some illustrations. The older version of the program sometimes yields an output such as Fig. 5. In the figure, 18 circles surround the center circle three times. This figure is drawn because it satisfies Eq. (1). But it does not satisfy the inequality conditions. This shows the necessity of the inequality conditions.

With respect to Eq. (1), there was a possibility of the free dimension to be more or less than the formal dimension 6. But, generally a solution was correctly obtained for any assignment of free variables, and moreover a figure seemed to be continuously transformed as values of free variables are continuously changed. From this experiments, we suspected that the formal dimension coincides with the actual free dimension, which is affirmatively confirmed in [2] later.

For a solution near the boundary of $\text{CPS}_{2,\tau}$, some edges of a polygonal region circumscribing a circle seems to be degenerated as in Fig. 6. This may imply that the limit case corresponds to the end of the Teichmüller space.

Fig. 5. Bad solution

Fig. 6. Near the limit

With the program, we can also observe phenomena well-known in the Kleinian group theory. As for the Andreev–Thurston solution which in particular corresponds to a hyperbolic structure and whose holonomy is Fuchsian, the surface Σ_2 is developed onto an open disk together with the circle packing (Fig. 7). For a point in the deformation space near the Andreev–Thurston solution, its holonomy group is quasi-Fuchsian and the surface is developed onto a quasi disk as in Fig. 8. A developing map overlaps for the other case as in Fig. 9. Since the holonomy acts on a developed image, occasionally the self-Möbius transformation appears clearly (see Fig. 10).

In the case of the torus, the situation is simpler. There is the unique combinatorial type τ for a packing by one circle. The space of circle packings is described by three variables and conditions

$$(2) \quad c_1 c_2 c_3 = c_1 + c_2 + c_3, c_1 > 0, c_2 > 0, c_3 > 0.$$

Thus, for values of free variables, we can calculate algebraically the value of the dependent variable which satisfies the condition.

Fig. 7. Fuchsian

Fig. 8. Quasi-Fuchsian

Fig. 9. Overlapping

Fig. 10. Self-Möbius

Fig. 11. Hexagonal packing

Fig. 12. A doyle spiral

Figure 11 illustrates a developed image of the circle packing corresponding to the Andreev–Thurston solution. In this flat structure case, the torus with the packing is developed onto the plane. A generic developed image of a packing overlaps as in Fig. 12 and is well-known as a Doyle spiral.

5. Concluding Remarks

The programs show explicitly what happens in the deformation, and helped the authors of [2] in the study. Computer experiments will be useful also for further studies, that may deal with the case of arbitrary g and τ .

Although only a slight modification of the program for genus 2 will give similar programs for genus $g \geq 3$, the phenomena which will be observed with the programs for genus $g \geq 3$ may not change so much. Rather, the similar program for arbitrary combinatorial types will be useful for the successive studies.

Another interesting thing to experiment is the behavior of a developing map. If we can understand deeply the detail of a developing map through analysis of a boundary of a developed image, the uniformization which relates the deformation space and the Teichmüller space may be grasped, and hence the result may make some contributions in extending the uniformization part of Theorem 2.4 for general cases.

Appendix: Details of the Programs

Here some details of the programs are mentioned.

Formerly, the program for Σ_2 solves the equation with Newton’s method, in which a solution is calculated through repetition of making the linearization of the equation at a tentative solution, solving the linear equation and obtaining the next tentative solution. If two or more solutions exist, it is not definite that a sequence of tentative solutions converges to the “right” solution. Hence, the method sometimes replies an unsuitable answer against the inequality condition. At the present, according to the proof in [2], we can solve the equation by a method similar to the bisection method.

Important mathematical objects handled inside the programs are complex projective transformations and circles. It is natural to describe a complex projective transformation by a matrix in $SL_2(\mathbb{C})$. Although a projective transformation sends a circle to a circle, the center of the first circle does not correspond to that of the second generally. Hence, the method of describing a circle by its center and radius seems not so suitable for the complex projective geometry. Moreover, as a circle on a surface with a projective structure is developed to a circle on the Riemann sphere, and further

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -\bar{c} \\ -c & |c|^2 - r^2 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} = 0.$$
$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & a - bi \\ a + bi & 2c \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} = 0.$$

The program is written in C/C++, and is based on OpenGL, GLUT, GLUI libraries, which are convenient in animating, windowing and interfacing. A screen shot is illustrated as Fig. 13. The picture displayed on the monitor can be saved as a file. This helps a lot in putting figures into articles.

Fig. 13. Screen shot

In a picture, each circle of a circle packing is drawn through recursive calls. At present, recursive calls finish according to its depth. Some mechanism should be introduced in order to stop drawing according to its spent time. This enables us to keep the number of frames of animation movies and to avoid freezing on an excessive depth of recursive calls.

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