The Space of Closed Geodesics on a Surface

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We introduce a topology to the set of closed geodesics on a hyperbolic surface in certain natural way and show that the space is not a Hausdorff space.

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1. Introduction

For a given set of mathematical objects, introducing a ‘natural’ topology to the set and studying its topological properties would be a basic, elementary subject in topology.

In this article, we consider the set $\mathcal{G}_g$ of closed geodesics on a closed, orientable surface $F$ of genus $g$ endowed with a constant curvature metric. In low-dimensional topology and Teichmüller space theory, such a set often appears, and it is worthwhile to study how it reflects the topology of the surface. Here, following [2], a closed geodesic will mean the image of a periodic geodesic map of $\mathbb{R}$ into $F$. Remark that each element of our set $\mathcal{G}_g$ is a single closed geodesic but a set of them. Thus our space $\mathcal{G}_g$ is not a subset of the space of projective geodesic currents [1].

As an instructive example, let us observe the easiest case; $F$ is the 2-sphere $S^2$. By $S^2$ we mean the unit sphere in $\mathbb{R}^3$ endowed with induced positive constant curvature metric. In this case, any geodesic line is periodic and the closed geodesics are exactly the great circles on $S^2$. For a great circle on $S^2$, there is a unique line in $\mathbb{R}^3$ which runs through the origin and is orthogonal to the plane including the great circle. The correspondence between a great circle and such a line gives a bijection of the set of great circles on $S^2$, equivalently the set $\mathcal{G}_0$, and the real projective plane $\mathbb{R}P^2$. By using this bijection, we can introduce a topology to $\mathcal{G}_0$, and then $\mathcal{G}_0$ is homeomorphic to $\mathbb{R}P^2$.

This method of topologizing is very natural, but it unfortunately seems that there exists no such bijection between any well-known topological space and $\mathcal{G}_g$ for $g > 0$ as in the case of $S^2$.

2. A Method for Topologizing $\mathcal{G}_g$

We note the following from the previous observation: Two great circles on $S^2$, equivalently two points of $\mathcal{G}_0$, are ‘close’ together if and only if one of the circle lies in a ‘neighborhood’ of the other one.

Based on this, let us review the topology of $\mathcal{G}_g$. Let $c$ be a great circle on $S^2$, or equivalently an element of $\mathcal{G}_0$, and let $N_\varepsilon(c)$ denote the $\varepsilon$-neighborhood of $c$ on $S^2$ for $\varepsilon > 0$. We obtain a function $\delta_{c,x} : \mathcal{G}_0 \to \mathbb{R}_+$ by defining $\delta_{c,x}(c')$ as the length of an arc appearing as $c' \cap N_\varepsilon(c)$ for $c' \in \mathcal{G}_0$. A topology of $\mathcal{G}_0$ is introduced by setting the family

$$\mathcal{U}_\varepsilon(c) = \{c' \in \mathcal{G}_0 | \delta_{c,x}(c') \geq \text{length}(c)\}$$

for $\varepsilon > 0$ as a fundamental system of neighborhoods. It is easy to see that a subset of $\mathcal{G}_0$ is open with respect to this topology if and only if it is open with respect to the topology defined in the previous section. Thus two topologies on $\mathcal{G}_0$ are equivalent.

This will be our base to introduce a topology to $\mathcal{G}_g$ for $g > 0$.

Remark. In fact, we can create a distance function $d$ of $\mathcal{G}_0$ by using $\delta_{c,x}$ as follows. Define the function $d : \mathcal{G}_0 \times \mathcal{G}_0 \to \mathbb{R}$ by setting $d(c,c')$ as

$$\min\{\varepsilon | \delta_{c,x}(c') \geq \text{length}(c)\}$$

for $c, c' \in \mathcal{G}_0$. It is easily checked that this $d$ satisfies the axiom of distance and coincides with the distance induced from the canonical one of $\mathbb{R}P^2$. This also coincides with the metric induced from the Hausdorff distance on the set.

Next we proceed the case of genus one: $F$ is the torus $T^2$ endowed with a fixed flat metric. Our method above unfortunately does not yield a natural topology on $\mathcal{G}_1$ for the following reason. Assume for example there are closed geodesics $m$ and $l$ on $T^2$ which intersect at single point perpendicularly. Let $c_n$ be a closed geodesic meeting $m$ once and $l$ at $n$ points for $n = 1, 2, 3, \ldots$. The angle between $l$ and $c_n$ become bigger as $n$ is larger, and so $c_n$ should become...
'further' from $l$ as $n$ is larger. However, with the same definition of $\delta_{l,c}$ as before, the set
\[
\min \{ \varepsilon \mid \delta_{l,c}(c_n) \geq \text{length}(l) \}
\]
is bounded by half of the length of $m$ above.

To avoid this phenomenon, we consider the universal cover $\mathbb{E}^2$ of $T^2$, and modify the definition of $\delta$. For $c \in \mathcal{G}_1$, equivalently for a closed geodesic $c$ on $T^2$, fix a lift $\tilde{c}$ in $\mathbb{E}^2$ (i.e., $\tilde{c}$ is a connected component of the preimage of $c$). With a constant $\varepsilon > 0$, let $N_\varepsilon(\tilde{c})$ denote the $\varepsilon$-neighborhood of $\tilde{c}$ in $\mathbb{E}^2$. We are interested in the function $\delta_{l,c} : \mathcal{G}_1 \to \mathbb{R}_+ \cup \{ \infty \}$ by defining $\delta_{l,c} : \mathcal{G}_1 \to \mathbb{R}_+ \cup \{ \infty \}$ by defining $\delta_{l,c}(\tilde{c}) = \infty$ if some lift $\tilde{c}'$ in $\mathbb{E}^2$ of $c' \in \mathcal{G}_1$ is included in $N_\varepsilon(\tilde{c})$, or otherwise, defining $\delta_{l,c}(\tilde{c})$ as the length of an arc appearing as $\tilde{c}' \cap N_\varepsilon(\tilde{c})$ for a lift $\tilde{c}'$ in $\mathbb{E}^2$ of $c' \in \mathcal{G}_1$. Obviously this definition does not depend on the choice of $\tilde{c}$. In the same way as before, consider the family
\[
\mathcal{U}_\varepsilon(c) = \{ c' \in \mathcal{G}_1 \mid \delta_{l,c}(c') \geq \text{length}(c) \}.
\]

By setting $\{ \mathcal{U}_\varepsilon(c) \}_{\varepsilon > 0}$ as a fundamental system of neighborhoods, we introduce a topology to $\mathcal{G}_1$.

About $\mathcal{G}_1$ with this topology, we easily verify that this is a Hausdorff space. Also, under this topology, $c_n$ is 'further' from $l$ as $n$ is larger in the example above.

### 3. Topology of $\mathcal{G}_g$ for $g > 1$

The topology of $\mathcal{G}_g$ for $g > 1$ is also introduced in the same way as for the case $g = 1$. Throughout the following, we assume $g > 1$, fix a hyperbolic metric on $F$ and identify the universal cover of $F$ with the hyperbolic plane $\mathbb{H}^2$ as usual.

For $c \in \mathcal{G}_g$, the preimage of $c$ in $\mathbb{H}^2$ consists of geodesics in $\mathbb{H}^2$. Note that they are mutually disjoint if and only if $c$ is simple. We call each one of them a lift of $c$. With a constant $\varepsilon > 0$, let $N_\varepsilon(\tilde{c})$ denote the $\varepsilon$-neighborhood of a lift $\tilde{c}$ of $c$ in $\mathbb{H}^2$. We obtain a function $\delta_{l,c} : \mathcal{G}_g \to \mathbb{R}_+ \cup \{ \infty \}$ by defining $\delta_{l,c}(\tilde{c})$ as the maximum of the length of an arc appearing as $\tilde{c}' \cap N_\varepsilon(\tilde{c})$ for a lift $\tilde{c}'$ in $\mathbb{H}^2$ of $c' \in \mathcal{G}_g$. Again this definition does not depend on the choice of $\tilde{c}$. In the same way as before, by using
\[
\mathcal{U}_\varepsilon(c) = \{ c' \in \mathcal{G}_g \mid \delta_{l,c}(c') \geq \text{length}(c) \}
\]
we introduce a topology to $\mathcal{G}_g$.

This topology is natural in the sense that it is equivalent to the one obtained as follows.

The preimage in $\mathbb{H}^2$ of a geodesic on $F$ is a $\Gamma$-invariant set of geodesics in $\mathbb{H}^2$, where $\Gamma$ denotes the covering transformation group. The set of endpoints (on the sphere at infinity $S^1_{\infty}$ of $\mathbb{H}^2$) of such geodesics gives a $\Gamma$-invariant set of points in $M = (S^1_{\infty} \times S^1_{\infty} - \Delta)/\mathbb{Z}_2$, where $\Delta$ denotes the diagonal set and $\mathbb{Z}_2$ acts on $S^1_{\infty} \times S^1_{\infty}$ by exchanging the two factors. By considering the conformal disk model of $\mathbb{H}^2$, the sphere at infinity $S^1_{\infty}$ is identified with the unit circle in $\mathbb{H}^2$. Thus the set $M$ has a natural topology induced from that of $S^1_{\infty}$, and then $M$ is homeomorphic to an open Möbius strip. Conversely a $\Gamma$-invariant set of points in $M$ gives a $\Gamma$-invariant set of geodesics in $\mathbb{H}^2$, and so it gives a geodesic on $F$. As a result, the set $\mathcal{G}_g$ of (not necessarily closed) geodesics on $F$ is identified with the set $M/\Gamma$, from which a topology on $\mathcal{G}_g$ is induced.

**Proposition.** The relative topology of $\mathcal{G}_g$ induced from $\mathcal{G}_0$ is equivalent to that defined by the fundamental system of neighborhoods $\{ \mathcal{U}_\varepsilon(c) \}_{\varepsilon > 0}$.

**Proof.** It suffices to show:

- for any neighborhood $V \subset \mathcal{G}_g$ of $c \in \mathcal{G}_g$, there exists $U(c) \subset \mathcal{G}_g$ such that $U(c) \subset V$,
- for any $c \in \mathcal{G}_g$ and $\mathcal{U}_\varepsilon(c) \subset \mathcal{G}_g$, there exists a neighborhood $V$ of $c$ in $\mathcal{G}_g$ such that $V \cap \mathcal{G}_g \subset \mathcal{U}_\varepsilon(c)$.

Let $p : M \to M/\Gamma = \mathcal{G}_g$ be the natural projection.

Suppose that a neighborhood $V \subset \mathcal{G}_g$ of $c \in \mathcal{G}_g$ is given. The preimage $p^{-1}(V)$ of $V$ is written as $\bigcup_{v \in V} \gamma V$ with some component $\gamma$ fixed. This $\gamma$ is regarded as a set of geodesics in $\mathbb{H}^2$ with endpoints sufficiently close to those of some lift $\tilde{c}$ of $c$. Without loss of generality, we can suppose that $\tilde{c}$ is the nearest one to the origin $O$ of the conformal disk model of $\mathbb{H}^2$. Let $c'$ be an element of $\mathcal{U}_\varepsilon(c)$ for some $\varepsilon$ and take the lift $\tilde{c}'$ of $c'$ in $\mathbb{H}^2$ such that the nearest point $x$ of $\tilde{c}'$ to $\tilde{c}$ is nearest to $O$. Note that the distance between $x$ and $O$ is bounded above as $x$ must be contained in the finite fundamental region of $F$ containing $O$. Then, by a direct calculation, we see that, if $\varepsilon$ is sufficiently small, then $\tilde{c}'$ has the endpoints which close to those of $\tilde{c}$ so that $\tilde{c}'$ is contained in $\tilde{V}$. This means that there exists a connected component of $p^{-1}(\mathcal{U}_\varepsilon(c)) \subset \tilde{V}$, and thus $\mathcal{U}_\varepsilon(c) \subset V$ for this $\varepsilon$.

On the other hand, for a lift $\tilde{c}$ of a $c \in \mathcal{G}_g$, any geodesic $\tilde{c}'$ whose endpoints are sufficiently close to those of $\tilde{c}$ satisfies that length($\tilde{c}' \cap N_\varepsilon(\tilde{c})$) $\geq$ length($\tilde{c}$) for arbitrary $\varepsilon > 0$. This means that there exists a neighborhood $V$ of $\tilde{c}$ in $\mathbb{H}^2$ such that $\tilde{V} \cap p^{-1}(\mathcal{G}_g)$ is included in a component of $p^{-1}(\mathcal{U}_\varepsilon(c))$. Thus we find $V = p(\tilde{V})$ in $\mathcal{G}_g$ such that $V \cap \mathcal{G}_g \subset \mathcal{U}_\varepsilon(c)$ for any $c \in \mathcal{G}_g$ and $\varepsilon > 0$.

However, we have:

**Theorem.** This topological space $\mathcal{G}_g$ is not a Hausdorff space.
Proof. Let $c_1$ and $c_2$ be two closed geodesics on $F$, equivalently two elements of $\mathcal{G}_g$. It suffices to show that $\mathcal{W}_\varepsilon(c_1) \cap \mathcal{W}_\varepsilon(c_2) \neq \emptyset$ in $\mathcal{G}_g$ for any $\varepsilon > 0$.

Take a lift $\tilde{c}_i$ of $c_i$, and let $\tilde{N}_i$ denote the $\varepsilon$-neighborhood of $\tilde{c}_i$ and $N_i$ the image of $\tilde{N}_i$ by the covering projection, for $i = 1, 2$. Without loss of generality, we can assume that $\varepsilon > 0$ is sufficiently small so that $N_1 \cup N_2$ is a regular neighborhood of $c_1 \cup c_2$ on $F$. For a positive integer $m$, let $\gamma^m$ denote the $m$-th power of the covering transformation corresponding to $c_i$ with suitable orientation for $i = 1, 2$.

In the following, we construct a piece-wise geodesic, closed curve $p_m$ on $F$ sufficiently ‘winding’ $c_1$ and $c_2$. First assume that $c_1 \cap c_2 \neq \emptyset$. Let $z$ and $z'$ be two of four intersection points $\partial\tilde{N}_1 \cap \partial\tilde{N}_2$ point symmetric with respect to the intersection $\tilde{c}_1 \cap \tilde{c}_2$. Let $a_i$ be the geodesic segment connecting $z$ and $\gamma^m(z')$ for $i = 1, 2$ and for a positive integer $m$. The image of $a_1 \cup a_2$ by the covering projection gives a piece-wise geodesic, closed curve on $F$, which we denote $p_m$.

Next assume that $c_1 \cap c_2 = \emptyset$. Let $q$ be the shortest path connecting $\tilde{c}_1$ and $\tilde{c}_2$, $\tilde{q}$ the geodesic in $\mathbb{H}^2$ containing $q$ as a subset and $\tilde{N}_q$ the $\varepsilon$-neighborhood of $\tilde{q}$. Then let $z_i$ and $z'_i$ be two of four intersection points $\partial\tilde{N}_i \cap \partial\tilde{N}_q$ point symmetric with respect to the intersection $\tilde{c}_i \cap \tilde{q}$ for $i = 1, 2$. Let $a_i$ be the geodesic segment connecting $z_i$ and $\gamma^m(z'_i)$ for $i = 1, 2$ and for a positive integer $m$. The image of $a_1 \cup a_2 \cup \tilde{b} \cup \tilde{b}'$ by the covering projection gives a piece-wise geodesic, closed curve $p_m$ on $F$, where $\tilde{b}, \tilde{b}'$ denote the geodesic path connecting $z_1$ and $z_2$, $z'_1$ and $z'_2$, respectively.

Now let $\tilde{p}_m$ be the lift of $p_m$ including $a_1$ and $a_2$. Then we see that one of $\partial\tilde{p}_m$ converges to one of $\partial\tilde{c}_1$ and the other converges to one of $\partial\tilde{c}_2$ on $S^1_{\infty}$ as $m \to \infty$. Since the closed geodesic freely homotopic to $p_m$ has the lift with the same endpoints as $p_m$, and then, by direct calculations, it is shown to be included in $\mathcal{W}_\varepsilon(c_1) \cap \mathcal{W}_\varepsilon(c_2)$ for sufficiently large $m$.

As an immediate corollary, we have:

**Corollary.** The space $\mathcal{G}_g^p = \{(S^1_\infty \times S^1_{\infty} - \Delta)/\Gamma\}$ of geodesics on $F$ is not a Hausdorff space for $g \geq 1$.

**REFERENCES**