On a Positive Knot without Positive Minimal Diagrams

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A. Stoimenow found an example of a positive knot which has no positive diagram with minimal crossing for the knot. In this note, we give another example inspired by Stoimenow's knot.

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1. Introduction

A *link* is a closed oriented 1-manifold embedded smoothly in the 3-sphere S^3 , and a *knot* is a link with one connected component. A *link diagram* (or *diagram*) is a generic projection of a link in S^3 onto the 2-sphere $S^2 \subset S^3$, which contains the information such that the overcrossing line is marked for each double point so that the original knot can be reconstructed. Each crossing in a link diagram has a sign, namely, positive ( ) or negative ( ). A knot is said to be *positive* if it has a diagram whose all crossings are positive. We call such a diagram a *positive diagram*. Positive knots have many interesting questions to provide us in analogy to *alternating knots*. An alternating knot is defined as a knot which has an *alternating diagram*. A diagram is said to be alternating if an overcrossing and an undercrossing appear alternately as one goes along each component. The *minimal crossing number* of a knot is the minimal number of crossings that occur in any diagram presenting the knot. We call a diagram realizing the minimal crossing number of a knot a *minimal diagram*. It is quite hard to determine the minimal crossing number of a given knot, to determine whether a given diagram is minimal or not. We have known the following famous theorem concerning a minimal diagram proved by Kauffman, Murasugi and Thistlethwaite:

Theorem 1.1 ([5, 10, 23]). *Any reduced alternating diagram is a minimal diagram. Moreover, any minimal diagram of a prime alternating knot is alternating.*

A diagram is said to be *reduced* if there exists no crossing such that the diagram is separated by splicing the crossing.

We consider Theorem 1.1 for the case of positive knots. There exists, however, a positive knot with a reduced positive diagram which is not minimal. An example of such a knot is the trefoil knot. See Fig. 1.

Fig. 1.

The latter claim of Theorem 1.1 also no longer holds for the case of positive knots, that is, there exists a knot with a positive diagram and a non-positive diagram as a minimal diagram though the positive knot is prime. An example of such a knot is a famous knot called *Perko's pair* 10_{161} and 10_{162} in the knot table in Rolfsen's book [19]. The knots 10_{161} and 10_{162} had been considered as different knots for a long time since knots had been tabulated by Little or Tait in the end of 19th century. In 1970, Perko [18] pointed out that they were equivalent to each other. The diagram of Fig. 2(a) presenting the knot 10_{161} is a non-positive diagram. Only the crossing in the dotted circle is negative. The diagram of Fig. 2(b) presenting the knot 10_{162} is a positive diagram.

So the question whose affirmative answer corresponds to Theorem 1.1 is the following:

Fig. 2.

Question 1.2. *Is a minimal diagram of any positive knot realized by a positive diagram?*

This question is affirmatively solved for torus knots [12], positive alternating knots [15] and positive knots of genus one or two [20, 21].

Throughout this paper, all manifolds are oriented in S^3 unless otherwise stated. For the definition of standard terms in Knot Theory, we refer to [1, 6, 13, 19].

2. Stoimenow's Example

Stoimenow [22] found the following example of a positive knot which gives a negative answer for Question 1.2, illustrated in Fig. 3. We call this knot *Stoimenow's knot*.

Fig. 3.

The diagram in Fig. 3(a) is a non-positive diagram with 11 crossings. This knot has a positive diagram with 12 crossings as in Fig. 3(b). (It follows Fig. 4 that these two diagrams present the same knot. The left most diagram in Fig. 4 is obtained from the diagram in Fig. 3(a) by turning it over.)

Fig. 4.

By a precise observation of this deformation, we see that the interior of the circle in Fig. 5(a) is not affected by this deformation. So we obtain a tangle where this deformation completes inside. See Fig. 5(b). We will construct a positive knot from this tangle, and show that it has no positive diagram with minimal crossings in the next section.

Fig. 5.

Fig. 6.

3. Positive Knot without Positive Minimal Diagrams

Let S_0 be Stoimenow's knot. Let S_n be the positive knot as in Fig. 6(a), obtained from S_0 by changing the crossing in the dotted circle in Fig. 5(a) into $2n + 1$ positive half-twists.

Ozawa [17, Theorem 1.4.] showed that if a nontrivial positive knot is prime then its reduced positive diagram is prime. A diagram D on the 2-sphere S is said to be *prime* if for any simple loop l on S intersecting D in 2 points, then l bounds a disk intersecting D in a simple arc. Hence we see that the positive knot S_n is prime.

In this note, we concentrate on the positive knot S_1 [Fig. 6(b)]. We note that the positive knot S_1 has a positive diagram with 14 crossings and its minimal crossing number is 13 or less. Then we have the following:

Theorem 3.1. *The positive knot S_1 has no positive diagram with 13 crossings or less.*

Hence the positive knot S_1 is an example of a positive knot without a positive minimal diagram.

Remark 3.2. If we consider this theorem for a non-prime positive knot, then we can easily obtain a positive knot without a positive minimal diagram by the composition of several copies of Stoimenow's knot S_0 . Let K be the composite knot of n copies of S_0 and $c^+(K)$ the minimal crossing number that occur in any positive diagram presenting K . We see that the minimal crossing number $c(K)$ of K is at most $11n$. Now we can see that $c^+(K) \leq 12n$. However, if $c^+(K) < 12n$ then S_0 must have a positive diagram with 11 crossings or less since any positive diagram of a composite positive knot is non-prime by Ozawa [17, Theorem 1.4.]. This is a contradiction. Thus we have $c^+(K) = 12n$. Hence the difference between $c(K)$ and $c^+(K)$ can be arbitrarily large.

We propose the following conjecture.

Conjecture 3.3. *The positive knot S_n has no positive diagram with the minimal crossing for S_n for $n \geq 2$.*

In order to prove Theorem 3.1 in the simplest way, we first list up the positive diagrams with 13 crossings or less and show that the positive knot S_1 is not contained in the list by some invariants. It is, however, not effective since there exists a huge number of positive diagrams with 13 crossings or less. We consider some properties of S_1 in order to reduce the number of positive diagrams to be investigated.

Algebraic properties: Let $V_K(t)$ be the Jones polynomial of a knot K , which calculated by the following recursive relations.

- (1) $V_O(t) = 1$,
- (2) $t^{-1}V_{K_+}(t) - tV_{K_-}(t) = (t^{-1/2} - t^{1/2})V_{K_0}(t)$,

where O is the trivial knot and K_+ , K_- and K_0 are three links that are identical except near one point and respectively.

By an easy calculation, we have

$$V_{S_1}(t) = t^{15} - 3t^{14} + 5t^{13} - 8t^{12} + 9t^{11} - 10t^{10} + 9t^9 - 6t^8 + 4t^7 - t^6 + t^4.$$

We also have the ordinary knot signature $\sigma(S_1)$ of the positive knot S_1 is equal to -6 by an easy calculation.

There exists the following theorem concerning the relation between the degree of the Jones polynomial and the number of crossings in a diagram.

Theorem 3.4 ([11]). *For any knot K and any diagram D of K ,*

$$\max \deg V_K(t) - \min \deg V_K(t) \leq c(D),$$

$$\max \deg V_K(t) + \frac{1}{2} \sigma(K) \leq c_+(D),$$

where $c(D)$ (resp. $c_+(D)$) is the number of crossings (resp. positive crossings) in D .

Then we have

$$11 \leq c(S_1), \tag{1}$$

$$12 \leq c_+(D), \tag{2}$$

where D is a diagram of S_1 . By the inequality (1), it is enough to check the existence of positive diagrams of S_1 with 13, 12 and 11 crossings. However, it is unnecessary to check the existence of positive diagrams with 11 crossings by the inequality (2).

Topological properties: A knot is said to be *fibered* if the complement of the knot is a surface bundle over a circle, whose fiber is a Seifert surface of the knot. It is known that the Seifert surface as a fiber of a fibered knot is unique up to ambient isotopy and gives the minimal genus of the knot (for example, see [8] and [7]). It is known that a Seifert surface obtained from a positive diagram by applying Seifert's algorithm gives the minimal genus of the knot ([2, 14]). We can see that a Seifert surface of the positive knot S_1 obtained from the positive diagram in Fig. 6(b) is the Hopf plumbing, namely, it is constructed from several Hopf bands by plumbing. The Hopf band is the fiber surface of the Hopf link. Plumbing is a special case of an operation ‘‘Murasugi sum’’. The Murasugi sum is a way to create a new Seifert surface by pasting two Seifert surfaces along a disk (more precisely $2n$ -gon). We call such a disk a *summing disk*. Gabai [4] showed that the resulting Seifert surface by the Murasugi sum is a fiber surface if and only if the summands are fiber surfaces. Plumbing is pasting two Seifert surfaces along 4-gon. So S_1 is a fibered knot of genus 4 (we can also see that S_n is a fibered knot of genus $n + 3$).

The spine of a Seifert surface F obtained from a diagram by applying Seifert's algorithm can be regarded as a graph G . The vertices of G correspond to the Seifert disks of F . The edges of G correspond to the twisted rectangles of F , hence to the crossings in the diagram. We call this graph a *Seifert graph*. We note that a Seifert graph is a planar bipartite graph. We also note that a *cut-vertex* (i.e., a vertex which divides the graph into several graphs by deleting itself) of a Seifert graph corresponds to a summing disk of a Seifert surface.

Proposition 3.5. *Let F be the fiber surface of a fibered positive knot obtained from a positive diagram D and G the Seifert graph from F . Then G has no cycle bounding a disk-face with the length greater than 2 when G is embedded in the 2-sphere S^2 .*

Proof. We may assume that D is a *special* diagram, i.e., any Seifert disk of F is unnnested. If not so, we can perform a reversing operation of the Murasugi sum at the nested Seifert disks. This preserves the fiberedness of surfaces. We note a special positive diagram is alternating (for example see [15]). We also assume that D is prime. Hence we may assume that G has no cut-vertex. By [9]: *A knot or link with a special alternating diagram is fibered if and only if it is the connected sum of torus knots or links of type $(2, n)$, G must consist of 2 vertices and several edges connecting them.* $\square$

So the cycles are derived from only multiedges. Since S_1 is a fibered knot of genus 4 and the number of crossings of a positive diagram which we will check its existence in Sects. 4 and 5 is 12 or 13, its Seifert graph consists of 5 (resp. 6) vertices and 12 (resp. 13) edges. All such graphs are illustrated in Fig. 7 as spanning trees. We assign a positive integer to each edge of graphs $G_1, \dots, G_9$. If an edge e joining two vertices v_1 and v_2 is assigned an integer n , e presents n edges joining v_1 and v_2 , namely, a multiedge of multiplicity n . Hence each integer is greater than 1 and the sum of integers in each graph of $G_1, \dots, G_3$ (resp. graphs $G_4, \dots, G_9$) is equal to 12 (resp. to 13). There exist five types of assignments for graphs $G_1, \dots, G_3$ such as $\{6, 2, 2, 2\}$, $\{5, 3, 2, 2\}$, $\{4, 4, 2, 2\}$, $\{4, 3, 3, 2\}$ and $\{3, 3, 3, 3\}$. There exist three

Fig. 7.

types of assignments for graphs $G_4, \dots, G_9$ such as $\{5, 2, 2, 2, 2\}$, $\{4, 3, 2, 2, 2\}$ and $\{3, 3, 3, 2, 2\}$.

4. DE-Graph

For each graph in Fig. 7, we reconstruct a positive diagram and calculate polynomial invariants. However we can reduce the number of graphs to check before calculating polynomial invariants. Let $P_K(v, z) = \sum P_{2i}(v)z^{2i}$ be the HOMFLY polynomial of a knot, which calculated by the following recursive relations.

$$(1) P_O(v, z) = 1,$$

$$(2) v^{-1}P_{K_+}(v, z) - vP_{K_-}(v, z) = zP_{K_0}(v, z),$$

where O , K_+ , K_- and K_0 are four links as in the definition of the Jones polynomial. We call a polynomial $P_{2i}(v)$ the i -th coefficient polynomial of $P_K(v, z)$.

We can calculate the HOMFLY polynomial for a given link by using only the above recursive relations (1) and (2). Then it is convenient to use a binary tree, called a *resolution tree*, for calculating the polynomial from a diagram. Let D be a diagram for a link L . Choose a crossing c in D . For convenience, we assume that c is positive. It is better to rename the diagram D_+ . Then we perform a crossing change and a smoothing at c . we denote by D_- and D_0 the altered diagrams, respectively. We connect D_+ and D_- (resp. D_+ and D_0) by drawing a line segment and assign v^2 (resp. vz) to the line segment. See Fig. 8. The appropriate assignment follows from the relation (2). We proceed this procedure for D_- and D_0 until the subsequent altered diagrams present trivial links, respectively. This completes a construction of a binary tree to calculate $P_L(v, z)$. We can now calculate $P_L(v, z)$ by summing up the HOMFLY polynomial of the each terminating trivial link (or knot) multiplied by the product of the coefficients (on the line segments) along the (uniquely determined) branch path that begins with the original diagram D and terminates with the diagram of this trivial link (or knot).

Fig. 8.

Definition 4.1. Let Γ_L be a resolution tree to calculate the HOMFLY polynomial for a link L . Then Γ_L is said to be a *positive resolution tree* if any diagram corresponding to each node is positive and any crossing which will be performed a crossing change or smoothing appears as a *positive clasp* (i.e., locally presented as a 2-string full-twist σ_i^2 for Artin's braid generator σ_i). See Fig. 9.

Fig. 9.

Lemma 4.2 ([16, Remark 2.5.]). *If a positive knot K has a positive resolution tree, then any coefficient polynomial $P_{2i}(v)$ is an alternating polynomial.* $\square$

A polynomial $f(t) = \sum a_j t^j \in \mathbb{Z}[t, t^{-1}]$ is said to be alternating if $a_j a_{j+1} < 0$ for any j . In this case we say $P_{2i}(v) = \sum a_j v^{2j}$ is alternating if $a_j a_{j+1} < 0$ for any j . For the positive knots derived from graphs $G_1, \dots, G_9$ in Fig. 7, we have the following lemma.

Lemma 4.3. *The positive knots derived from the graphs G_1, G_2, G_4, G_5, G_6 and G_7 in Fig. 7 have positive resolution trees.*

Proof. If the maximal number of valency is equal to 2, namely, graphs G_1 and G_4 , the positive knots derived from them are closed positive braids. By Franks and Williams [3] or Van Buskirk [24], they have positive resolution trees.

For the other graphs G_2, G_5, G_6 and G_7 , we reconsider each vertex as a disk. We call a graph consisting of disks and edges a *DE-graph*. If the knot diagram constructed by a DE-graph has a positive clasp (two successive edges in a DE-graph) then we can perform a crossing change and a smoothing, and proceed the next stage of a resolution tree. For edges in a DE-graph as in Fig. 10, which illustrates a part of a DE-graph, we can slide an edge. We call this deformation an *edge-slide*. This deformation corresponds to the Reidemeister move III for a knot diagram. We claim that we can obtain two successive edges by edge-slides or reduce the number of disks. $\square$

Fig. 10.

Claim 4.4. *Let $D_1, \dots, D_n$ be n -tuple disks in a DE-graph. If the partial positive knot diagram derived from the boundaries of the disk D_n and parts of disks $D_1, \dots, D_{n-1}$ form an n -braid presented as in Fig. 11, then we obtain one of the following; (1) the disk D_n is separated (i.e., D_n and D_{n-1} are not connected by an edge), (2) the disk D_n can be reduced by edge-slides, (3) there exist two successive edges.*

Proof of Claim 4.4. We denote by $e_{i,j}$ the j -th edge connecting the i -th disk D_i and the $(i+1)$ -th disk D_{i+1} ($1 \leq i \leq n-1, 0 \leq j \leq k_i$). Let W be the weight of this partial DE-graph defined as $W = \sum_{i=1}^{n-1} i k_i$, where k_i is the number of edges connecting the i -th disk D_i and the $(i+1)$ -th disk D_{i+1} . We assume that the weight W is minimal up to edge-slides.

If there exists no edge $e_{n-1,j}$ then D_n is separated. If there exists only one edge $e_{n-1,j}$ then D_n is reducible. For both cases Claim 4.4 holds. So we assume that there exist at least two edges $e_{n-1,j}$ ($j \geq 2$) connecting D_{n-1} and D_n .

Fig. 11.

If there exists no edge $e_{n-2,j}$ between $e_{n-1,1}$ and $e_{n-1,2}$ then $e_{n-1,1}$ and $e_{n-1,2}$ form two successive edges. If there exists only one edge $e_{n-2,j}$ between $e_{n-1,1}$ and $e_{n-1,2}$ then we can perform an edge-slide and decrease the weight W . We have a contradiction to minimality of the weight W . If there exist two edges e_{n-2,j_1} and e_{n-2,j_2} between $e_{n-1,1}$ and $e_{n-1,2}$ then we attend to e_{n-2,j_1} and e_{n-2,j_2} and proceed the same procedure for them. Finally we attend to edges e_{2,j_1} and e_{2,j_2} . For the cases such that the number of edges between e_{2,j_1} and e_{2,j_2} is zero or one, we have a conclusion or contradiction by a similar argument for $e_{n-2,1}$ and $e_{n-2,2}$. If there exist several edges $e_{1,j_1}, \dots, e_{1,j_m}$ ($m \geq 2$) between e_{2,j_1} and e_{2,j_2} then they form two successive edges. $\square$

There exist two types of disks of valency 3, such as type 1 or type 2 in Fig. 12.

Fig. 12.

For both types, the number of outer disks can be chosen suitably for each graph G_i ($i = 2, 5, 6, 7$). Figure 12 illustrates the case of G_5 .

For type 1. For graphs G_2 and G_5 , we see that the partial DE-graph in the dotted circle in Fig. 12(a) corresponds to that in Claim 4.4. Hence we can construct a positive resolution tree. For G_6 and G_7 , we have the same conclusion although there exist disks on the shaded disk in Fig. 12(a).

For type 2. For each graph except G_7 , we can suitably choose the disk corresponding to the disk D_n in Claim 4.4 from disks A or B in Fig. 12(b). Then we have the conclusion.

For the graph G_7 , there exists only one case such that we cannot obtain the partial DE-graph corresponding to that in Claim 4.4, that is, we cannot construct a positive resolution tree. The case is the following: There exists the type 1 disk of valency 3 on the disk B in Fig. 12(b). We obtain the DE-graph in Fig. 13 if we construct a DE-graph as few edges as possible so that it realizes the obstruction of the existence of a positive resolution tree. The other cases have positive resolution trees. We put one more edge to somewhere in the DE-graph in Fig. 13 to obtain the DE-graph of G_7 completely. Then we can see that any resulting DE-graph presents a link diagram. Hence it does not occur. This completes the proof of Lemma 4.3. $\square$

The HOMFLY polynomial of the positive knot S_1 is the following.

$$\begin{aligned} P_{S_1}(v, z) &= (3v^8 + 2v^{10} - 6v^{12} + 2v^{14}) \\ &\quad + z^2(15v^8 + 3v^{10} - 9v^{12} + v^{14}) \\ &\quad + z^4(19v^8 + 3v^{10} - 3v^{12}) \end{aligned}$$

Fig. 13.

$$+ z^6(8v^8 + v^{10})$$

$$+ v^8 z^8$$

Since each coefficient polynomial is not alternating, it is enough to check only the positive knot derived from graphs G_3 , G_8 , G_9 by Lemmas 4.2 and 4.3.

For a disk of valency 4 in a DE-graph, we have the four types as in Fig. 14, which illustrates the case of G_8 .

Fig. 14.

By a similar argument of the proof of Lemma 4.3, we have the following.

Lemma 4.5. *The positive knots derived from DE-graphs (a), (b), (c) in Fig. 14 have positive resolution trees.* $\square$

Hence for DE-graphs with the maximal valency 4 it is enough to check the positive knots derived from DE-graphs of type (d).

4.1 Graph G_3

For checking of the positive knots derived from the graph G_3 , we first construct DE-graphs from the graph G_3 according to the assignments of positive integers to edges.

For the assignments $\{6, 2, 2, 2\}$ and $\{5, 3, 2, 2\}$ we have DE-graphs in Fig. 15 without 6 or 5 edges corresponding to the edge in G_3 assigned 6 or 5.

Fig. 15.

We put 6 or 5 edges between the disk A and the disk B (we also denote these disks by A and B for the other cases) as in Fig. 15 to construct DE-graphs completely. Then we must construct each DE-graph so that it does not present a link diagram or a composite knot diagram, its one of disks is not reduced by edge-slides as in Fig. 16.

Fig. 16.

Then we obtain 24 DE-graphs.

Next we consider the DE-graphs with the assignment $\{4, 4, 2, 2\}$. For this assignment, we have DE-graphs in Fig. 17.

Fig. 17.

We put 4 edges between the disk A and the disk B to construct DE-graphs completely. Then we can see that any resulting DE-graph presents a link diagram or a composite knot diagram, or can be deformed into one of the previous DE-graphs by edge-slides and ambient isotopy. Then we can see that this assignment does not contribute to yielding DE-graphs to be checked.

For the assignment $\{4, 3, 3, 2\}$, we have DE-graphs in Fig. 18.

Fig. 18.

Then we obtain 12 DE-graphs by putting 4 edges with the above conditions.

For the assignment $\{3, 3, 3, 3\}$ we have DE-graphs in Fig. 19 without 3 edges corresponding to one of the edges in G_3 assigned 3.

Then we obtain 11 DE-graphs with the above conditions.

We will check polynomial invariants for the positive knots derived from the above DE-graphs in Sect. 5.

4.2 Graph G_8

The assignment for the edges in the graph G_8 depends on which integers are assigned to the two edges joined by the vertex of valency 2, such as $(5, 2)$, $(2, 5)$, $(2, 2)$ in the assignment $\{5, 2, 2, 2, 2\}$, $(4, 3)$, $(3, 4)$, $(4, 2)$, $(2, 4)$, $(3, 2)$, $(2, 3)$, $(2, 2)$ in the assignment $\{4, 3, 2, 2, 2\}$ and $(3, 3)$, $(3, 2)$, $(2, 3)$, $(2, 2)$ in the assignment $\{3, 3, 3, 2, 2\}$.

Lemma 4.6. *If the assignment for the two edges joined by the vertex of valency 2 is $(3, 2)$, $(2, 3)$ and $(2, 2)$ then any DE-graph from the graph G_8 with such assignments is reducible.*

Proof. For any assignment in Lemma 4.6 for the two edges joined by the vertex of valency 2, we have a part of a DE-graph as in Fig. 16. Then we can reduce the number of disks by edge-slides as in Fig. 16. $\square$

Fig. 19.

We have the following for other assignments, similarly.

Lemma 4.7. *If the assignment for the two edges joined by the vertex of valency 2 is $(5, 2)$ and $(4, 3)$ (resp. $(4, 2)$ and $(3, 3)$) then any DE-graph from the graph G_8 with such assignments can be deformed into DE-graphs with the assignment $(2, 5)$ (resp. $(2, 4)$) for the two edges joined by the vertex of valency 2. $\square$*

Hence it is enough to check only the assignments for the two edges joined by the vertex of valency 2 such as $(2, 5)$ in the assignment $\{5, 2, 2, 2, 2\}$, $(2, 4)$ in the assignment $\{4, 3, 2, 2, 2\}$ and $(3, 3)$ in the assignment $\{3, 3, 3, 2, 2\}$.

We construct DE-graphs from the graph G_8 with considering conditions mentioned in the construction for G_3 . Then we can see that the graph G_8 does not contribute to yielding DE-graphs to be checked.

4.3 Graph G_9

For the graph G_9 , we construct DE-graphs by the following way: For each assignment, we first prepare 2 disks and connect them by edges. The number of those edges is the maximal integer in the assignment. Next we put a disk with 2 or 3 edges on the disk of valency 5 corresponding to the vertex of valency 5 in G_9 .

Fig. 20.

Then there exist two ways to put a disk with 2 or 3 edges on the disk of valency 5, namely, the composition or plumbing. See Fig. 21. (By the primeness of the positive knot S_1 , the composition must be done before the plumbing.)

So we can construct DE-graphs from the graph G_9 with considering conditions mentioned in the construction for G_3 . We obtain 2 DE-graphs for the assignment $\{5, 2, 2, 2, 2\}$, 21 DE-graphs for the assignment $\{4, 3, 2, 2, 2\}$ and 71 DE-graphs for the assignment $\{3, 3, 3, 2, 2\}$.

Fig. 21.

5. Distinction

In Sect. 4, we obtained 141 DE-graphs (see Appendix) to check their polynomial invariants in order to distinguish the positive knot S_1 from the positive knots derived from them. We denote by 5_i (resp. 6_i) the i -th DE-graph with 5 disks (resp. 6 disks) in the tables in Appendix. There exist, however, positive knots which are mutant to each other among the positive knots derived from them. Two knots are mutant if they can be deformed to each other by flipping a 2-strand tangle (see Fig. 22). It is known that two mutant knots share the same polynomial invariants.

Fig. 22.

Hence we calculate polynomial invariants of the positive knots derived from DE-graphs up to mutant. Then we have 19 positive knots to calculate. For DE-graphs with 5 disks, we have $5_1, 5_2, 5_3, 5_4, 5_5, 5_6, 5_8, 5_{10}, 5_{13}, 5_{15}, 5_{23}, 5_{36}$ and 5_{40} , for DE-graphs with 6 disks, we have $6_2, 6_3, 6_4, 6_{18}, 6_{26}$ and 6_{34} . The positive knots derived from the other DE-graphs are mutant to them.

So we calculate the Conway polynomial of them. They are in Table 1.

Table 1. Conway polynomial

5_1	$1 + 12z^2 + 18z^4 + 8z^6 + z^8$	5_2	$1 + 11z^2 + 17z^4 + 8z^6 + z^8$
5_3	$1 + 10z^2 + 18z^4 + 8z^6 + z^8$	5_4	$1 + 9z^2 + 16z^4 + 8z^6 + z^8$
5_5	$1 + 9z^2 + 17z^4 + 8z^6 + z^8$	5_6	$1 + 10z^2 + 17z^4 + 8z^6 + z^8$
5_8	$1 + 8z^2 + 16z^4 + 8z^6 + z^8$	5_{10}	$1 + 7z^2 + 14z^4 + 8z^6 + z^8$
5_{13}	$1 + 9z^2 + 16z^4 + 8z^6 + z^8$	5_{15}	$1 + 8z^2 + 17z^4 + 8z^6 + z^8$
5_{23}	$1 + 8z^2 + 15z^4 + 8z^6 + z^8$	5_{36}	$1 + 8z^2 + 14z^4 + 8z^6 + z^8$
5_{40}	$1 + 7z^2 + 13z^4 + 7z^6 + z^8$	6_2	$1 + 11z^2 + 20z^4 + 9z^6 + z^8$
6_3	$1 + 9z^2 + 18z^4 + 9z^6 + z^8$	6_4	$1 + 9z^2 + 17z^4 + 9z^6 + z^8$
6_{18}	$1 + 9z^2 + 19z^4 + 9z^6 + z^8$	6_{26}	$1 + 7z^2 + 14z^4 + 9z^6 + z^8$
6_{34}	$1 + 11z^2 + 18z^4 + 9z^6 + z^8$		

The positive knot S_1 has the Conway polynomial

$$\nabla(S_1) = 1 + 10z^2 + 19z^4 + 9z^6 + z^8,$$

hence S_1 is a distinct knot from the positive knots derived from DE-graphs constructed in Sect. 4. Thus we can see S_1 has no positive diagram with 13 crossing or less by the above observations. This completes the proof of Theorem 3.1.

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Appendix

The following is tables of DE-graphs constructed in Sect. 4. (Continued on next page.)

Table 2. DE-graphs with 5 disks

Table 3. DE-graphs with 6 disks

Table 4. DE-graphs with 6 disks