On Smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ -Actions on the Sphere of Dimension $2n - 1$

Fuichi UCHIDA*

*Department of Mathematical Sciences, Faculty of Science, Yamagata University,
Koshirakawa, Yamagata 990-8560, Japan*

Received October 14, 2002; final version accepted February 18, 2003

We shall study smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ actions on S^{2n-1} , each of which is an extension of the standard $\mathbf{U}(1) \times \mathbf{SO}(n)$ action. We shall show such an action is characterized by a smooth $\mathbf{R}^2$ -action on S^3 and a smooth mapping of S^3 onto $\mathbf{P}_1(\mathbf{C})$. Moreover, we shall show there are uncountably many topologically distinct such extensions.

KEYWORDS: non-compact Lie group action, sphere, codimension one

0. Introduction

Consider the standard $\mathbf{U}(1) \times \mathbf{SO}(n)$ action on the $(2n - 1)$ -sphere S^{2n-1} . Here we regard S^{2n-1} as the unit sphere in $\mathbf{C}^n$, the complex number space of n -dimension. This action has codimension one principal orbits and just two singular orbits. In this paper, we shall study smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ actions on S^{2n-1} , each of which is an extension of the standard $\mathbf{U}(1) \times \mathbf{SO}(n)$ action. We shall show such an action is characterized by a pair of a smooth $\mathbf{R}^2$ -action on S^3 and a smooth mapping of S^3 onto $\mathbf{P}_1(\mathbf{C})$. Practically, we shall construct a smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ action on S^{2n-1} from a pair of a smooth $\mathbf{R}^2$ -action on S^3 and a smooth mapping of S^3 onto $\mathbf{P}_1(\mathbf{C})$ satisfying certain condition. Moreover, we shall show there are uncountably many topologically distinct such extensions. More precisely, we may construct smooth actions $\Phi_{m,c}$ for any positive integer m and real number c , such that if $(m, c) \neq (m', c')$ then the corresponding actions are mutually distinct as continuous actions.

Let G be a non-compact semi-simple Lie group and K a maximal compact connected subgroup of G . For a smooth K action on the sphere S^N with codimension one orbits, we have studied smooth G actions each of which is an extension of the given K action. Asoh [1] has investigated smooth $\mathbf{SL}(2, \mathbf{C})$ actions on 3-manifolds. The maximal compact subgroup of $\mathbf{SL}(2, \mathbf{C})$ is $\mathbf{SU}(2)$ and it is known that the smooth action of $\mathbf{SU}(2)$ on 3-manifold is transitive or has a codimension one orbit. He has proposed certain new idea to investigate the latter case. His idea was improved and used by Uchida [5–7] and Mukōyama [2, 3]. In the case of these papers, G is a simple Lie group and the singular orbits of given K action are contained in open orbits of extended G action.

Uchida [8] has studied smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ actions on S^{m+n-1} each of which is an extension of the standard $\mathbf{SO}(m) \times \mathbf{SO}(n)$ action. In that case, $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ is not a simple Lie group and the singular orbits of the standard $\mathbf{SO}(m) \times \mathbf{SO}(n)$ action on S^{m+n-1} are invariant under any extended smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ action. On the contrary, one of the singular orbits of the standard $\mathbf{U}(1) \times \mathbf{SO}(n)$ action is contained in an open orbit of extended action and the other singular orbit is invariant under the extended action, for any smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ actions on S^{2n-1} , each of which is an extension of the standard $\mathbf{U}(1) \times \mathbf{SO}(n)$ action.

The author wishes to express gratitude to the referees and the editors for their advice.

1. Twisted Linear Actions

Denote by $\mathbf{C}^*$ the multiplicative group of non-zero complex numbers and $\mathbf{SO}(n, \mathbf{C})$ the Lie group consisting of complex matrices A of degree n such that ${}^tAA = I_n$ and $\det A = 1$. The twisted linear action ψ_c of $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} and the standard action ψ of $\mathbf{U}(1) \times \mathbf{SO}(n)$ on S^{2n-1} are defined by

$$\psi_c((z, A), \mathbf{v}) = e^{t(1+ic)}zA\mathbf{v}, \quad \psi((A, z), \mathbf{v}) = zA\mathbf{v}$$

where c is the given real number and $t = -\log \|zA\mathbf{v}\|$. Indeed, any twisted linear action ψ_c is an extended action of the standard action ψ . The general twisted linear actions are introduced by Uchida [4].

Denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the canonical basis of $\mathbf{C}^n$, and define $Q(\mathbf{v}) = v_1^2 + \dots + v_n^2$ for $\mathbf{v} = v_1\mathbf{e}_1 + \dots + v_n\mathbf{e}_n$.

Suppose $n \geq 3$. Let $\mathbf{v} \in S^{2n-1}$. Since $Q(z\mathbf{v}) = z^2Q(\mathbf{v})$, there exists $z \in \mathbf{U}(1)$ such that $Q(z\mathbf{v})$ is a non-negative real number. Put $z\mathbf{v} = X + iY$, where X, Y are real vectors such that $X \perp Y$. Then $\|X\| \geq \|Y\|$ and there exists $B \in \mathbf{SO}(n)$ such

that $zB\mathbf{v} = p\mathbf{e}_1 + iq\mathbf{e}_2$, where $p \geq q \geq 0$ and $p^2 + q^2 = 1$. Therefore, each orbit contains a point $p\mathbf{e}_1 + iq\mathbf{e}_2$, where $p \geq q \geq 0$ and $p^2 + q^2 = 1$.

By direct calculation, we obtain the condition that an element $(z, A) \in \mathbf{U}(1) \times \mathbf{SO}(n)$ belongs to the isotropy subgroup at $p\mathbf{e}_1 + iq\mathbf{e}_2$, where $p \geq q \geq 0$ and $p^2 + q^2 = 1$. In fact, if $p > q > 0$, then $z = \varepsilon$ and A is in the form

$$\left[\begin{array}{c|c} \varepsilon I_2 & 0 \\ \hline 0 & * \end{array} \right],$$

where $\varepsilon = \pm 1$ and the orbit is of codimension 1.

If $p = q = 1/\sqrt{2}$, then $z = e^{i\theta}$ and A is in the form

$$\left[\begin{array}{cc|c} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & \\ \hline 0 & & * \end{array} \right],$$

where θ is a real number and the orbit is of codimension 2.

If $p = 1, q = 0$, then $z = \varepsilon$ and A is in the form

$$\left[\begin{array}{c|c} \varepsilon & 0 \cdots 0 \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right],$$

where $\varepsilon = \pm 1$ and the orbit is of codimension $n - 1$.

Put

$$M(a, b) = \left[\begin{array}{cc|c} a & b & 0 \\ -b & a & \\ \hline 0 & & I_{n-2} \end{array} \right]$$

where $a, b \in \mathbf{C}$ and $a^2 + b^2 = 1$. Then $M(a, b) \in \mathbf{SO}(n, \mathbf{C})$. Denote by $N(n)$ the subgroup of $\mathbf{SO}(n, \mathbf{C})$ consisting of the matrices in the form $M(a, b)$.

Lemma 1.1. Suppose $n \geq 3$. If $Q(\mathbf{v}) \neq 0$, then there exists $(z, A) \in \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ such that $\psi_c((z, A), \mathbf{v}) = \mathbf{e}_1$. If $Q(\mathbf{v}) = 0$, then there exists $(z, A) \in \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ such that $\psi_c((z, A), \mathbf{v}) = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}$.

In particular, the twisted linear action ψ_c of $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} has just two orbits. One of them is an open orbit through $\mathbf{e}_1$ and the other is a compact orbit through $(\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}$.

Proof. Suppose $Q(\mathbf{v}) \neq 0$. Then

$$\exists (w, B) \in \mathbf{U}(1) \times \mathbf{SO}(n) : wB\mathbf{v} = p\mathbf{e}_1 + iq\mathbf{e}_2, \quad p, q \in \mathbf{R}, \quad p^2 + q^2 = 1, \quad p > q \geq 0.$$

Then

$$M(p/u, iq/u) \in \mathbf{SO}(n, \mathbf{C}), \quad M(p/u, iq/u)(p\mathbf{e}_1 + iq\mathbf{e}_2) = u\mathbf{e}_1$$

where $u = \sqrt{p^2 + q^2}$ is a positive real number. Then

$$u^{-1}wM(p/u, iq/u)B\mathbf{v} = \mathbf{e}_1.$$

Put $z = u^{-1}w$, $A = M(p/u, iq/u)B$. Then $zA\mathbf{v} = \mathbf{e}_1$. Hence, we obtain $\psi_c((z, A), \mathbf{v}) = \mathbf{e}_1$ for any real number c .

Suppose $Q(\mathbf{v}) = 0$. Then $\mathbf{v} = X + iY$, where X, Y are real vectors such that $\|X\| = \|Y\| \neq 0$ and $X \perp Y$, and hence

$$\exists B \in \mathbf{SO}(n) : B\mathbf{v} = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}.$$

Put $z = 1$, $A = B$. Then we obtain $\psi_c((z, A), \mathbf{v}) = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}$ for any real number c .

Notice that the condition $Q(\mathbf{v}) = 0$ is invariant under the twisted linear action of $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$. Hence the last statement is clear. $\square$

Theorem 1.2. Suppose $n \geq 3$. If $c \neq c'$, then the twisted linear action ψ_c is topologically distinct to $\psi_{c'}$. In fact, the isotropy subgroups at $\mathbf{e}_1$ are not mutually conjugate.

Proof. By direct calculation, we see that the element $(z, A) \in \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ belongs to the isotropy subgroups at $\mathbf{e}_1$ with respect to the twisted linear action ψ_c , if and only if $(1 - ic)\log(\varepsilon z)$ is a real number and A is in the form

$$\left[\begin{array}{c|c} \varepsilon & 0 \cdots 0 \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right].$$

Here, $\varepsilon = \pm 1$. □

2. Closed Subgroups

Denote by $\mathbf{I}(a, b)$ the isotropy subgroup at $a\mathbf{e}_1 + b\mathbf{e}_2$ with respect to the canonical representation of $\mathbf{SO}(n, \mathbf{C})$ on $\mathbf{C}^n$, where a, b are complex numbers with $(a, b) \neq (0, 0)$. Considering the corresponding Lie subalgebras, we see that $\mathbf{I}(a, b) = \mathbf{I}(c, d)$ if and only if $(c, d) = (\lambda a, \lambda b)$ for a complex number λ . Moreover, we see $\mathbf{I}(a, b) \cap \mathbf{I}(c, d) = \mathbf{SO}(n-2, \mathbf{C})$ for any $\mathbf{I}(a, b) \neq \mathbf{I}(c, d)$.

If $a^2 + b^2 \neq 0$, then $\mathbf{I}(a, b)$ is conjugate with $\mathbf{I}(1, 0) = \mathbf{SO}(n-1, \mathbf{C})$, since

$$M(a', b')(a\mathbf{e}_1 + b\mathbf{e}_2) = \sqrt{a^2 + b^2}\mathbf{e}_1$$

for $(a', b') = (a/\sqrt{a^2 + b^2}, b/\sqrt{a^2 + b^2})$, and hence

$$M(a', b')\mathbf{I}(a, b)M(a', b')^{-1} = \mathbf{I}(1, 0).$$

If $a^2 + b^2 = 0$, then $\mathbf{I}(a, b) = \mathbf{I}(1, \varepsilon i)$, where $\varepsilon = \pm 1$. We see that the normalizer of $\mathbf{I}(1, \varepsilon i)$ is $N(n)\mathbf{I}(1, \varepsilon i)$.

Now we may state the following lemma. The proof is a simple routine work, so we omit the proof.

Lemma 2.1. *Suppose $n \geq 6$. Let S be a closed connected proper subgroup of $\mathbf{SO}(n, \mathbf{C})$ which contains $\mathbf{SO}(n-2)$ and $\dim \mathbf{SO}(n, \mathbf{C})/S \leq 2n-1$. Then, $S = \mathbf{I}(a, b)$ for some $(a, b) \neq (0, 0)$ or S contains $\mathbf{I}(1, \varepsilon i)$ as a normal subgroup for $\varepsilon = \pm 1$. In particular, if S contains $\mathbf{SO}(n-1)$, then $S = \mathbf{I}(1, 0)$.*

Remark. $\dim \mathbf{SO}(n, \mathbf{C})/\mathbf{I}(a, b) = 2n-2$.

Lemma 2.2. *Suppose $n \geq 6$. Let S be a closed connected subgroup of $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ which contains $\{1\} \times \mathbf{SO}(n-2)$.*

- (1) *Suppose $\dim\{\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})\}/S = 2n-1$. Then, (a) $S = S_1 \times \mathbf{I}(a, b)$ for some complex numbers a, b such that $a^2 + b^2 = 1$ and certain one-dimensional closed connected subgroup S_1 of $\mathbf{C}^*$, or (b) $S = S'_1\mathbf{I}(1, \varepsilon i)$ for $\varepsilon = \pm 1$ and certain one-dimensional closed connected subgroup S'_1 of $\mathbf{C}^* \times N(n)$.*
- (2) *Suppose $\dim\{\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})\}/S \leq 2n-2$. Then, (a) $S = \mathbf{C}^* \times \mathbf{I}(a, b)$ for some complex numbers a, b such that $a^2 + b^2 = 1$, or (b) $S = S'\mathbf{I}(1, \varepsilon i)$ for $\varepsilon = \pm 1$ and certain closed connected subgroup S' of $\mathbf{C}^* \times N(n)$ such that $2 \leq \dim S' \leq 4$.*

Proof. Let S_2 be the image of S by the natural projection $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C}) \rightarrow \mathbf{SO}(n, \mathbf{C})$. Let $S_{(2)}$ be the closed subgroup of $\mathbf{SO}(n, \mathbf{C})$ such that

$$\{1\} \times S_{(2)} = S \cap (\{1\} \times \mathbf{SO}(n, \mathbf{C})).$$

Then $\{1\} \times S_{(2)}$ is a normal subgroup of S . Moreover, we see $\dim \mathbf{SO}(n, \mathbf{C})/S_{(2)} \leq 2n-1$. Then, $S_{(2)}^0 = \mathbf{I}(a, b)$ or $S_{(2)}^0$ contains $\mathbf{I}(1, \varepsilon i)$ as a normal subgroup by Lemma 2.1, where $S_{(2)}^0$ is the identity component of $S_{(2)}$. Now we obtain the result by considering the normalizer of $\{1\} \times \mathbf{I}(a, b)$. □

Remark. $\mathbf{SO}(n, \mathbf{C}) = \mathbf{SO}(n)N(n)\mathbf{I}(a, b)$ for any $(a, b) \neq (0, 0)$.

Put

$$m(\theta) = \left[\begin{array}{cc|c} \cosh \theta & i \sinh \theta & 0 \\ -i \sinh \theta & \cosh \theta & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right]$$

for real number θ and denote by $M(n)$ the group of all matrices $m(\theta)$. Then, $M(n)$ is a closed subgroup of $N(n)$ and

$$N(n) = \mathbf{SO}(2)'M(n).$$

Here, $\mathbf{SO}(2)' = \mathbf{SO}(n) \cap N(n)$. Hence we obtain the following decomposition:

$$\mathbf{SO}(n, \mathbf{C}) = \mathbf{SO}(n)M(n)\mathbf{I}(a, b)$$

for any $(a, b) \neq (0, 0)$.

Lemma 2.3. *Suppose $km(\theta)u = k'm(\theta')u'$. Here $k, k' \in \mathbf{SO}(n)$ and $u, u' \in \mathbf{I}(a, b)$.*

If $(a, b) = (1, \varepsilon i)$ for $\varepsilon = \pm 1$, then $\theta = \theta'$ and $k^{-1}k' \in \mathbf{SO}(n-2)$. If $(a, b) = (1, 0)$, then one of the following conditions holds:

- (1) $\theta = \theta' = 0$, $k^{-1}k' \in \mathbf{SO}(n-1)$,
- (2) $\theta = \theta' \neq 0$, $k^{-1}k' \in \mathbf{SO}(n-2)$,
- (3) $\theta = -\theta' \neq 0$, $k^{-1}k' \in \mathbf{JSO}(n-2)$.

Here, J is the diagonal matrix with diagonal entries $(1, -1, -1, 1, \dots, 1)$.

Proof. Since $km(\theta)u(\mathbf{e}_1 + \varepsilon i\mathbf{e}_2) = e^{-\varepsilon\theta}k(\mathbf{e}_1 + \varepsilon i\mathbf{e}_2)$, we obtain the first part. Since $km(\theta)u\mathbf{e}_1 = \cosh \theta k\mathbf{e}_1 - i \sinh \theta k\mathbf{e}_2$, we obtain the second part. $\square$

Remark. $JM(a, b) = M(a, -b)J$, $Jm(\theta) = m(-\theta)J$.

3. Smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ Actions on S^{2n-1}

Put $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$, $K = \mathbf{U}(1) \times \mathbf{SO}(n)$ and $H = \{1\} \times \mathbf{SO}(n-2)$.

Let us consider a smooth G -action Φ on S^{2n-1} such that the restricted K -action of Φ coincides with the standard action ψ . Then the fixed point set $F(H)$ of restricted H -action is the 3-sphere as follows:

$$F(H) = \{x\mathbf{e}_1 + y\mathbf{e}_2 \mid x, y \in \mathbf{C}; |x|^2 + |y|^2 = 1\}.$$

We obtain a mapping $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ defined by the condition

$$f(\mathbf{v}) = (a : b) \quad \text{if and only if} \quad G_{\mathbf{v}} \supset \{1\} \times \mathbf{I}(a, b). \quad (*)$$

Since the isotropy subgroup $G_{\mathbf{v}}$ at $\mathbf{v} \in F(H)$ contains H , $G_{\mathbf{v}}$ contains a unique subgroup of the form $\{1\} \times \mathbf{I}(a, b)$ by Lemma 2.2. Here $\mathbf{P}_1(\mathbf{C})$ is the complex projective 1-space, on which $\mathbf{SO}(2, \mathbf{C})$ acts naturally. Since $N(n)$ is canonically isomorphic to $\mathbf{SO}(2, \mathbf{C})$, we can regard $\mathbf{P}_1(\mathbf{C})$ as an $N(n)$ -manifold.

The group $\mathbf{C}^* \times N(n)$ coincides with the identity component of the centralizer of $H = \{1\} \times \mathbf{SO}(n-2)$ in $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$. Hence the group $\mathbf{C}^* \times N(n)$ acts on $F(H)$ by the action Φ . On the other hand, we can regard the group $\mathbf{C}^* \times N(n)$ acting on $\mathbf{P}_1(\mathbf{C})$ by considering the trivial $\mathbf{C}^*$ -action.

Lemma 3.1. *The mapping $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ defined above is $\mathbf{C}^* \times N(n)$ -equivariant.*

Proof. Suppose $f(\mathbf{v}) = (a : b)$. That is, $G_{\mathbf{v}} \supset \{1\} \times \mathbf{I}(a, b)$. Let $g = (z, M(c, d))$ be an element of $\mathbf{C}^* \times N(n)$. Then

$$G_{g\mathbf{v}} = gG_{\mathbf{v}}g^{-1} \supset \{1\} \times M(c, d)\mathbf{I}(a, b)M(c, d)^{-1} = \{1\} \times \mathbf{I}(ca + db, -da + cb).$$

On the other hand, $g(a : b) = (ca + db : -da + cb)$. Hence we obtain $f(g\mathbf{v}) = gf(\mathbf{v})$. $\square$

Remark. The diagonal matrix J with diagonal entries $(1, -1, -1, 1, \dots, 1)$ is an element of the normalizer of H in G , hence J acts on $F(H)$. In fact, $J(a\mathbf{e}_1 + b\mathbf{e}_2) = a\mathbf{e}_1 - b\mathbf{e}_2$. Similarly, J acts on $\mathbf{P}_1(\mathbf{C})$ by $J(a : b) = (a : -b)$. Then, $Jf(\mathbf{v}) = f(J\mathbf{v})$ for $\mathbf{v} \in F(H)$.

Lemma 3.2. *The mapping $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ defined above is smooth.*

Proof. Denote by $\text{Lie } S$ the Lie algebra of S . Define 4 elements of $\text{Lie } \mathbf{SO}(n, \mathbf{C})$ as follows:

$$A = E_{n1} - E_{1n}, \quad A' = iA, \quad B = E_{n2} - E_{2n}, \quad B' = iB.$$

Then we see the following relation:

$$b_1A + b_2A' - a_1B - a_2B' \in \text{Lie } \mathbf{I}(a, b),$$

where each coefficients are real numbers defined by $a = a_1 + a_2i$, $b = b_1 + b_2i$.

Now we define a Lie algebra homomorphism $\Phi^+ : \text{Lie } \mathbf{SO}(n, \mathbf{C}) \rightarrow \Gamma(S^{2n-1})$ by

$$\Phi^+(M)_{\mathbf{v}}(h) = \lim_{t \rightarrow 0} [h(\Phi(\exp(-tM), \mathbf{v})) - h(\mathbf{v})]/t,$$

where $\Gamma(S^{2n-1})$ denotes the Lie algebra consisting of smooth vector fields on S^{2n-1} , $M \in \text{Lie } \mathbf{SO}(n, \mathbf{C})$ and h is a smooth function defined on an open neighborhood of $\mathbf{v}$.

For $M \in \text{Lie } \mathbf{SO}(n, \mathbf{C})$, we see that $M \in \text{Lie } \mathbf{SO}(n, \mathbf{C})_{\mathbf{v}}$ if and only if $\Phi^+(M)_{\mathbf{v}} = 0$.

Now we consider the tangent vector fields $\Phi^+(A)$, $\Phi^+(A')$, $\Phi^+(B)$, $\Phi^+(B')$. By considering the isotropy algebras, we obtain the following result. Put $f(\mathbf{v}) = (a : b)$ for $\mathbf{v} \in F(H)$.

- $a \neq 0$ if and only if $\Phi^+(A)_{\mathbf{v}}, \Phi^+(A')_{\mathbf{v}}$ are linearly independent,
- $b \neq 0$ if and only if $\Phi^+(B)_{\mathbf{v}}, \Phi^+(B')_{\mathbf{v}}$ are linearly independent.

Put $U_A = \{\mathbf{v} \in F(H) \mid f(\mathbf{v}) = (1 : b)\}$, $U_B = \{\mathbf{v} \in F(H) \mid f(\mathbf{v}) = (a : 1)\}$. Then U_A , U_B are open sets of $F(H)$ and $F(H) = U_A \cup U_B$.

Let us denote by $(M, M')_{\mathbf{v}}$ the inner product of two tangent vector fields $\Phi^+(M)$, $\Phi^+(M')$ at $\mathbf{v}$ with respect to the

standard Riemannian metric on S^{2n-1} . Denote by $A[\mathbf{v}], B[\mathbf{v}]$ the Gram matrices as follows:

$$A[\mathbf{v}] = \begin{bmatrix} (A, A)_{\mathbf{v}} & (A, A')_{\mathbf{v}} \\ (A', A)_{\mathbf{v}} & (A', A')_{\mathbf{v}} \end{bmatrix}, \quad B[\mathbf{v}] = \begin{bmatrix} (B, B)_{\mathbf{v}} & (B, B')_{\mathbf{v}} \\ (B', B)_{\mathbf{v}} & (B', B')_{\mathbf{v}} \end{bmatrix}.$$

Then $A[\mathbf{v}]$ is non-singular at each point of U_A and $B[\mathbf{v}]$ is non-singular at each point of U_B .

Moreover, we see the following:

$$\begin{aligned} f(\mathbf{v}) = (1 : b_1 + b_2 i) \quad &\text{implies} \quad A[\mathbf{v}] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} (A, B)_{\mathbf{v}} \\ (A', B)_{\mathbf{v}} \end{bmatrix}, \\ f(\mathbf{v}) = (a_1 + a_2 i, 1) \quad &\text{implies} \quad B[\mathbf{v}] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (B, A)_{\mathbf{v}} \\ (B', A)_{\mathbf{v}} \end{bmatrix}. \end{aligned}$$

Hence we see that a_1, a_2 are smooth functions on U_B and b_1, b_2 are smooth functions on U_A .

Therefore the mapping $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ is smooth. $\square$

Lemma 3.3. $f(\mathbf{v}) = (1 : 0)$ if and only if $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$.

Proof. If $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$, then $G_{\mathbf{v}} \supset \mathbf{SO}(n-1)$. Then we obtain $G_{\mathbf{v}} \supset \mathbf{I}(1, 0)$ by Lemma 2.1, and hence $f(\mathbf{v}) = (1 : 0)$. If $f(\mathbf{v}) = (1 : 0)$, then $G_{\mathbf{v}} \supset \mathbf{I}(1, 0)$, and hence $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$. $\square$

4. Construction of $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ Actions on S^{2n-1}

Put $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$, $K = \mathbf{U}(1) \times \mathbf{SO}(n)$ and $H = \{1\} \times \mathbf{SO}(n-2)$ as in the previous section. Let us construct a G -action Φ on S^{2n-1} such that the restricted K -action of Φ coincides with the standard action ψ . Then the fixed point set $F(H)$ of restricted H -action is the 3-sphere as before.

Let ϕ be a smooth action of $\mathbf{C}^* \times N(n)$ on $F(H)$ and $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ be a smooth $\mathbf{C}^* \times N(n)$ -equivariant mapping. The matrix J acts on $F(H)$ and $\mathbf{P}_1(\mathbf{C})$ by $J(a\mathbf{e}_1 + b\mathbf{e}_2) = a\mathbf{e}_1 - b\mathbf{e}_2$ and $J(a : b) = (a : -b)$. Suppose the following conditions:

- (a) $\phi|_{\mathbf{U}(1) \times (N(n) \cap \mathbf{SO}(n))} = \psi|_{\mathbf{U}(1) \times (N(n) \cap \mathbf{SO}(n))}$,
- (b) $\phi(M(a, b), J\mathbf{v}) = J\phi(M(a, -b), \mathbf{v})$, $Jf(\mathbf{v}) = f(J\mathbf{v})$ for $\mathbf{v} \in F(H)$,
- (c) $f(\mathbf{v}) = (1 : 0)$ if and only if $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$.

Lemma 4.1. Suppose $f(\mathbf{v}) = (a : b)$ and $km(\theta)u = k'm(\theta')u'$, where $k, k' \in \mathbf{SO}(n)$ and $u, u' \in \mathbf{I}(a, b)$. Then the following equation holds:

$$\psi(k, \phi(m(\theta), \mathbf{v})) = \psi(k', \phi(m(\theta'), \mathbf{v})).$$

Proof. If $(a : b) = (1 : \varepsilon i)$, then we see $\theta = \theta'$ and $k^{-1}k' \in \mathbf{SO}(n-2)$ by Lemma 2.3, and hence we obtain the equation. In the case $(a : b) = (1 : 0)$, we see $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$ by Lemma 3.3. Now we consider the problem under the three cases (1), (2) and (3) of Lemma 2.3. Here we show the result under the case (3). Then, $\theta = -\theta'$ and $k^{-1}k' \in J\mathbf{SO}(n-2)$. In particular, $k' = kJh$ for some $h \in \mathbf{SO}(n-2)$. Therefore,

$$\begin{aligned} \psi(k', \phi(m(\theta'), \mathbf{v})) &= \psi(kJh, \phi(m(-\theta), \mathbf{v})) \\ &= \psi(k, J\phi(m(-\theta), \mathbf{v})) && h \in \mathbf{SO}(n-2) \\ &= \psi(k, \phi(m(\theta), J\mathbf{v})) && \text{by (b)} \\ &= \psi(k, \phi(m(\theta), \mathbf{v})) && \mathbf{v} \in \mathbf{Ce}_1 \cap F(H). \end{aligned}$$

Finally, we consider the general case $a^2 + b^2 \neq 0$. Then, we see $M(a', b')\mathbf{I}(a, b)M(a', b')^{-1} = \mathbf{I}(1, 0)$ for $(a', b') = (a/\sqrt{a^2 + b^2}, b/\sqrt{a^2 + b^2})$.

Put $\mathbf{v}' = \phi(M(a', b'), \mathbf{v})$. Then we obtain $f(\mathbf{v}') = (1 : 0)$, since f is $\mathbf{C}^* \times N(n)$ -equivariant. Decompose $M(a', b')^{-1} = k_0 m(\tau)$, where $k_0 \in \mathbf{SO}(2)'$ and certain real number τ . Put

$$v = k_0^{-1}m(-\tau)u m(\tau)k_0, \quad v' = k_0^{-1}m(-\tau)u' m(\tau)k_0.$$

Then, we see $v, v' \in \mathbf{I}(1, 0)$. Since $km(\theta)u = k'm(\theta')u'$, we obtain $(kk_0)m(\theta + \tau)v = (k'k_0)m(\theta' + \tau)v'$. Thus we obtain

$$\begin{aligned} \psi(k, \phi(m(\theta), \mathbf{v})) &= \psi(k, \phi(m(\theta), \phi(k_0 m(\tau), \mathbf{v}')) \\ &= \psi(kk_0, \phi(m(\theta + \tau), \mathbf{v}')) \\ &= \psi(k'k_0, \phi(m(\theta' + \tau), \mathbf{v}')) \\ &= \psi(k', \phi(m(\theta'), \phi(k_0 m(\tau), \mathbf{v}')) \\ &= \psi(k', \phi(m(\theta'), \mathbf{v})). \end{aligned}$$

Hence we obtain the desired result. $\square$

Lemma 4.2. Let $\mathbf{v}_1, \mathbf{v}_2 \in F(H)$. Suppose there exists $k \in \mathbf{SO}(n)$ such that $\psi(k, \mathbf{v}_1) = \mathbf{v}_2$. Then, one of the following conditions holds:

- (1) $\exists k_0 \in \mathbf{SO}(2)', h \in \mathbf{SO}(n-2) : k = k_0 h, \psi(k_0, \mathbf{v}_1) = \mathbf{v}_2,$
- (2) $\exists k_0 \in \mathbf{SO}(2)', h \in \mathbf{SO}(n-2) : k = k_0 Jh, \psi(k_0, J\mathbf{v}_1) = \mathbf{v}_2,$
- (3) $\exists s_1, s_2 \in \mathbf{SO}(2)', h \in \mathbf{SO}(n-1) : k = s_2^{-1} h s_1, \psi(s_t, \mathbf{v}_t) \in \mathbf{Ce}_1 \cap F(H).$

Proof. For any $\mathbf{v} \in F(H)$, we see that one of the following conditions holds:

$$(i) \mathbf{SO}(n)_{\mathbf{v}} = \mathbf{SO}(n-2) \quad (ii) \exists s \in \mathbf{SO}(2)' : \psi(s, \mathbf{v}) \in \mathbf{Ce}_1 \cap F(H).$$

Suppose $\mathbf{v}_1$ satisfies (i). Then $\mathbf{v}_2$ satisfies (i) also. By considering the isotropy subgroups, we see $k \in \mathbf{S}(\mathbf{O}(2) \times \mathbf{O}(n-2))$. Then $k = k_0 h$ or $k = k_0 Jh$ for $k_0 \in \mathbf{SO}(2)'$ and $h \in \mathbf{SO}(n-2)$. Then we obtain the cases (1) and (2).

Suppose $\mathbf{v}_1$ satisfies (ii). Then $\mathbf{v}_2$ satisfies (ii) also. Then we can assume

$$\exists s_1, s_2 \in \mathbf{SO}(2)' : \psi(s_t, \mathbf{v}_t) \in \mathbf{Ce}_1 \cap F(H), \quad (t = 1, 2).$$

By considering the isotropy subgroups, we see $s_2 k s_1^{-1} \in \mathbf{S}(\mathbf{O}(1) \times \mathbf{O}(n-1))$. Replacing s_2 by $-s_2$ if necessary, we can assume $s_2 k s_1^{-1} \in \mathbf{SO}(n-1)$. Put $h = s_2 k s_1^{-1}$. Then we obtain the case (3). $\square$

Let $(z, g) \in G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ and $p \in S^{2n-1}$. Decompose $p = \psi(k, \mathbf{v})$ for $k \in \mathbf{SO}(n)$, $\mathbf{v} \in F(H)$. Moreover, decompose $gk = k'm(\theta)u$, where $k' \in \mathbf{SO}(n)$, $u \in \mathbf{I}(a, b)$ for $f(\mathbf{v}) = (a : b)$.

Lemma 4.3. Under the above situation, the element

$$\psi(k', \phi(m(\theta), \phi(z, \mathbf{v})))$$

does not depend on the decompositions.

Proof. First we see that the element does not depend on the decomposition $gk = k'm(\theta)u$ by Lemma 4.1, because $f(\phi(z, \mathbf{v})) = f(\mathbf{v})$ for any $z \in \mathbf{C}^*$.

Next we assume

$$p = \psi(k_t, \mathbf{v}_t) : k_t \in \mathbf{SO}(n), \mathbf{v}_t \in F(H), gk_t = k'_t m(\theta_t) u_t, f(\mathbf{v}_t) = (a_t : b_t)$$

for $t = 1, 2$. Then $\psi(k_2^{-1} k_1, \mathbf{v}_1) = \mathbf{v}_2$. Here we can apply Lemma 4.2.

First we assume $k_2^{-1} k_1 = k_0 h$, $\psi(k_0, \mathbf{v}_1) = \mathbf{v}_2$, where $k_0 \in \mathbf{SO}(2)'$ and $h \in \mathbf{SO}(n-2)$. Since $k_1 = k_2 k_0 h$, we obtain

$$gk_1 = gk_2 k_0 h = k'_2 m(\theta_2) u_2 k_0 h = (k'_2 k_0) m(\theta_2) (k_0^{-1} u_2 k_0) h,$$

where $(k_0^{-1} u_2 k_0) h \in \mathbf{I}(a_1, b_1)$. Thus

$$\begin{aligned} \psi(k'_1, \phi(m(\theta_1), \phi(z, \mathbf{v}_1))) &= \psi(k'_2 k_0, \phi(m(\theta_2), \phi(z, \mathbf{v}_1))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(k_0, \phi(z, \mathbf{v}_1)))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(z, \mathbf{v}_2))). \end{aligned}$$

Next we assume $k_2^{-1} k_1 = k_0 Jh$, $\psi(k_0, J\mathbf{v}_1) = \mathbf{v}_2$, where $k_0 \in \mathbf{SO}(2)'$ and $h \in \mathbf{SO}(n-2)$. Since $k_1 = k_2 k_0 Jh$, we obtain

$$gk_1 = gk_2 k_0 Jh = k'_2 m(\theta_2) u_2 k_0 Jh = (k'_2 k_0 J) m(-\theta_2) (Jk_0^{-1} u_2 k_0 J) h,$$

where $(Jk_0^{-1} u_2 k_0 J) h \in \mathbf{I}(a_1, b_1)$. Thus

$$\begin{aligned} \psi(k'_1, \phi(m(\theta_1), \phi(z, \mathbf{v}_1))) &= \psi(k'_2 k_0 J, \phi(m(-\theta_2), \phi(z, \mathbf{v}_1))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(k_0, \phi(z, J\mathbf{v}_1)))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(z, \mathbf{v}_2))). \end{aligned}$$

Finally we assume $k_2^{-1} k_1 = s_2^{-1} h s_1$, $\psi(s_t, \mathbf{v}_t) \in \mathbf{Ce}_1 \cap F(H)$, $(t = 1, 2)$, where $s_1, s_2 \in \mathbf{SO}(2)', h \in \mathbf{SO}(n-1)$. Since $k_1 = k_2 s_2^{-1} h s_1$, we obtain

$$gk_1 = gk_2 s_2^{-1} h s_1 = k'_2 m(\theta_2) u_2 s_2^{-1} h s_1 = (k'_2 s_2^{-1} s_1) m(\theta_2) (s_1^{-1} s_2 u_2 s_2^{-1} s_1) (s_1^{-1} h s_1),$$

where $(s_1^{-1} s_2 u_2 s_2^{-1} s_1) (s_1^{-1} h s_1) \in \mathbf{I}(a_1, b_1)$. Thus

$$\begin{aligned} \psi(k'_1, \phi(m(\theta_1), \phi(z, \mathbf{v}_1))) &= \psi(k'_2 s_2^{-1} s_1, \phi(m(\theta_2), \phi(z, \mathbf{v}_1))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(s_2^{-1} s_1, \phi(z, \mathbf{v}_1)))) \\ &= \psi(k'_2, \phi(m(\theta_2), \phi(z, \mathbf{v}_2))). \end{aligned}$$

The proof is completed. $\square$

By Lemma 4.3, we can define a mapping $\Phi : G \times S^{2n-1} \rightarrow S^{2n-1}$ by

$$\begin{aligned}\Phi((z, g), p) &= \psi(k', \phi(m(\theta), \phi(z, \mathbf{v}))), \quad p = \psi(k, \mathbf{v}), \quad gk = k'm(\theta)u; \\ \text{for } k, k' \in \mathbf{SO}(n), \quad u \in \mathbf{I}(a, b), \quad \text{where } f(\mathbf{v}) &= (a : b), \mathbf{v} \in F(H).\end{aligned}$$

Theorem 4.4. *The mapping $\Phi : G \times S^{2n-1} \rightarrow S^{2n-1}$ is an abstract action and the restricted K -action of Φ coincides with the standard action ψ .*

Proof. Let $(z, k) \in K = \mathbf{U}(1) \times \mathbf{SO}(n)$ and $p \in S^{2n-1}$. Put $p = \psi(k_1, \mathbf{v})$ for $k_1 \in \mathbf{SO}(n)$, $\mathbf{v} \in F(H)$. Then

$$\Phi((z, k), p) = \psi(kk_1, \phi(z, \mathbf{v})) = \psi((z, k), \psi(k_1, \mathbf{v})) = \psi((z, k), p).$$

Thus the mapping Φ is an extension of ψ .

Let $z_1, z_2 \in \mathbf{C}^*$; $g_1, g_2 \in \mathbf{SO}(n, \mathbf{C})$ and $p \in S^{2n-1}$. Put $p = \psi(k, \mathbf{v})$ for $k \in \mathbf{SO}(n)$, $\mathbf{v} \in F(H)$. Suppose

$$g_2k = k'_2m(\theta_2)u_2; \quad k'_2 \in \mathbf{SO}(n), u_2 \in \mathbf{I}(a, b) \quad \text{for } f(\mathbf{v}) = (a : b).$$

Moreover

$$g_1k'_2 = k'_1m(\theta_1)u_1; \quad k'_1 \in \mathbf{SO}(n), u_1 \in \mathbf{I}(c, d) \quad \text{for } f(\phi(m(\theta_2), \mathbf{v})) = (c : d).$$

Then, $\Phi((z_2, g_2), p) = \psi(k'_2, \phi(m(\theta_2), \phi(z_2, \mathbf{v})))$. Hence we obtain

$$\begin{aligned}\Phi((z_1, g_1), \Phi((z_2, g_2), p)) &= \psi(k'_1, \phi(m(\theta_1), \phi(z_1, \phi(m(\theta_2), \phi(z_2, \mathbf{v})))))) \\ &= \psi(k'_1, \phi(m(\theta_1 + \theta_2), \phi(z_1z_2, \mathbf{v}))).\end{aligned}$$

On the other hand,

$$g_1g_2k = k'_1m(\theta_1)u_1m(\theta_2)u_2 = k'_1m(\theta_1 + \theta_2)(m(-\theta_2)u_1m(\theta_2))u_2,$$

where $(m(-\theta_2)u_1m(\theta_2))u_2 \in \mathbf{I}(a, b)$. Thus we obtain

$$\Phi((z_1z_2, g_1g_2), p) = \psi(k'_1, \phi(m(\theta_1 + \theta_2), \phi(z_1z_2, \mathbf{v}))).$$

Hence we obtain $\Phi((z_1, g_1), \Phi((z_2, g_2), p)) = \Phi((z_1z_2, g_1g_2), p)$.

Therefore, Φ is an abstract action of $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} . □

5. Smoothness of the Action Φ , Part I

Notice that the continuity and the smoothness of Φ defined in the previous section are unknown in this stage. Here we consider the smoothness of the action Φ .

Now we introduce a hermitian matrix $P(\mathbf{v})$ of degree n for $\mathbf{v} \in F(H)$ by

$$P(\mathbf{v}) = \frac{1}{|a|^2 + |b|^2} \left[\begin{array}{cc|c} a\bar{a} & a\bar{b} & 0 \\ b\bar{a} & b\bar{b} & 0 \\ \hline 0 & 0 & 0 \end{array} \right]$$

where $f(\mathbf{v}) = (a : b)$. Then, $\text{trace } P(\mathbf{v}) = 1$ for any $\mathbf{v} \in F(H)$. We see $uP(\mathbf{v})u^* = P(\mathbf{v})$ for any $u \in \mathbf{I}(a, b)$. Moreover, we see

$$m(\theta)P(\mathbf{v})m(\theta)^* = \lambda(\theta, \mathbf{v})P(\phi(m(\theta), \mathbf{v})); \quad \lambda(\theta, \mathbf{v}) = \cosh 2\theta + \frac{i(\bar{a}b - a\bar{b})}{|a|^2 + |b|^2} \sinh 2\theta.$$

Define

$$\begin{aligned}S_+ &= \{\mathbf{v} \in F(H) \mid f(\mathbf{v}) = (1 : \beta), \text{Im } \beta > 0\}, \\ S_- &= \{\mathbf{v} \in F(H) \mid f(\mathbf{v}) = (1 : \beta), \text{Im } \beta < 0\}.\end{aligned}$$

These are open sets of $F(H)$ and $J(S_+) = S_-$, $J(S_-) = S_+$ by the condition (b). Then the function $\beta = \beta(\mathbf{v})$ is a smooth function on $S_+ \cup S_-$. Define

$$F(\beta) = \sqrt{(1 + |\beta|^2)^2 - 4(\text{Im } \beta)^2} / (1 + |\beta|^2).$$

Moreover, we define

$$W_+ = \{(g, \mathbf{v}) \mid g \in \mathbf{SO}(n, \mathbf{C}), \mathbf{v} \in S_+, \text{trace } gP(\mathbf{v})g^* \neq F(\beta(\mathbf{v}))\}.$$

Lemma 5.1. *For any $(g, \mathbf{v}) \in W_+$, there exist uniquely $k \in \mathbf{SO}(n - 2) \in \mathbf{SO}(n) / \mathbf{SO}(n - 2)$ and $\theta \in \mathbf{R}$ such that*

$$g = km(\theta)u; \quad u \in \mathbf{I}(1, \beta(\mathbf{v})), \phi(m(\theta), \mathbf{v}) \in S_+.$$

Proof. First, suppose $1 + \beta^2 = 0$, that is $\beta = \beta(\mathbf{v}) = i$. Then we see

$$\text{trace } gP(\mathbf{v})g^* = e^{-2\theta} > 0$$

for any decomposition $g = km(\theta)u$; $u \in \mathbf{I}(1, i)$ and $F(i) = 0$. So we see that $\text{trace } gP(\mathbf{v})g^* \neq F(\beta(\mathbf{v}))$ for $g \in \mathbf{SO}(n, \mathbf{C})$, and $\theta \in \mathbf{R}$ is uniquely determined by the equation $\text{trace } gP(\mathbf{v})g^* = e^{-2\theta}$. On the other hand, $f(\phi(m(\theta), \mathbf{v})) = (1 : i)$. Hence we obtain $\phi(m(\theta), \mathbf{v}) \in S_+$ for any $\theta \in \mathbf{R}$. Moreover, the uniqueness of $k\mathbf{SO}(n-2)$ is assured by Lemma 2.3.

Next, suppose $1 + \beta^2 \neq 0$ for $\beta = \beta(\mathbf{v})$. Then we see

$$\text{trace } gP(\mathbf{v})g^* = \lambda(\theta, \mathbf{v}) = \cosh 2\theta - \frac{2 \operatorname{Im} \beta}{1 + |\beta|^2} \sinh 2\theta \quad (**)$$

for any decomposition $g = km(\theta)u$; $u \in \mathbf{I}(1, \beta)$. Moreover, we see that the function $\lambda(\theta, \mathbf{v})$ with respect to the variable θ has the minimum value at $\theta = \theta_0$ such that $\tanh 2\theta_0 = \frac{2 \operatorname{Im} \beta}{1 + |\beta|^2}$, and $\lambda(\theta_0, \mathbf{v}) = F(\beta(\mathbf{v}))$. Therefore, $(g, \mathbf{v}) \in W_+$ if and only if

$$\theta \neq \theta_0 \text{ for } g = km(\theta)u; u \in \mathbf{I}(1, \beta).$$

On the other hand, we see

$$\phi(m(\theta), \mathbf{v}) \in S_+ (\text{resp. } S_-) \text{ if and only if } \theta < \theta_0 (\text{resp. } \theta > \theta_0).$$

Now, we assume $\phi(m(\theta), \mathbf{v}) \in S_-$ for some decomposition $g = km(\theta)u$; $u \in \mathbf{I}(1, \beta)$.

Since $1 + \beta^2 \neq 0$, we obtain $f(\phi(M(c, d), \mathbf{v})) = (1 : 0)$ for $c = 1/\sqrt{1 + \beta^2}$, $d = \beta/\sqrt{1 + \beta^2}$. Then, $\phi(M(c, d), \mathbf{v}) \in \mathbf{Ce}_1 \cap F(H)$ by the condition (c).

Therefore, we see $J\phi(M(c, d), \mathbf{v}) = \phi(M(c, d), \mathbf{v})$, and hence

$$M(c, d)^{-1}JM(c, d) \in \mathbf{I}(1, \beta).$$

Put

$$M(c, d) = k_1m(\tau), \quad k' = kJk_1^2, \quad u' = M(c, d)^{-1}JM(c, d)u$$

for $k_1 \in \mathbf{SO}(2)'$. Then, we see $g = km(\theta)u = k'm(2\tau - \theta)u'$, where $u' \in \mathbf{I}(1, \beta)$. Moreover, we see that

$$\phi(k_1^2m(2\tau - \theta), \mathbf{v}) = J\phi(m(\theta), \mathbf{v}) \in S_+.$$

Hence we obtain $\phi(m(2\tau - \theta), \mathbf{v}) \in S_+$, since S_+ is invariant under the action of $\mathbf{SO}(2)'$.

Therefore, we see that there exists a decomposition

$$g = km(\theta)u; k \in \mathbf{SO}(n), u \in \mathbf{I}(1, \beta), \phi(m(\theta), \mathbf{v}) \in S_+$$

in the case $1 + \beta^2 \neq 0$. In this case, $\theta \in \mathbf{R}$ is uniquely determined by the equation (**) and the uniqueness of $k\mathbf{SO}(n-2)$ is assured by Lemma 4.1 and the uniqueness of θ . $\square$

By this lemma, we can define a mapping $\Delta : W_+ \rightarrow \mathbf{SO}(n)/\mathbf{SO}(n-2) \times \mathbf{R}$ by the following condition

$$\Delta(g, \mathbf{v}) = (k\mathbf{SO}(n-2), \theta); g = km(\theta)u, k \in \mathbf{SO}(n), u \in \mathbf{I}(1, \beta(\mathbf{v})), \phi(m(\theta), \mathbf{v}) \in S_+.$$

Lemma 5.2. *The mapping Δ is smooth.*

Proof. Define $\gamma : \mathbf{R} \times \mathbf{SO}(n, \mathbf{C}) \times S_+ \rightarrow \mathbf{R}$ by $\gamma(\theta, g, \mathbf{v}) = \lambda(\theta, \mathbf{v}) - \text{trace } gP(\mathbf{v})g^*$. Then, γ is smooth and $\gamma(\theta, g, \mathbf{v}) = 0$ for a decomposition $g = km(\theta)u$, $u \in \mathbf{I}(1, \beta(\mathbf{v}))$. Moreover, we see

$$\frac{\partial \gamma}{\partial \theta}(\theta, g, \mathbf{v}) = 2 \cosh 2\theta \left(\tanh 2\theta - \frac{2 \operatorname{Im} \beta}{1 + |\beta|^2} \right) < 0 \quad \text{for} \quad \tanh 2\theta < \frac{2 \operatorname{Im} \beta}{1 + |\beta|^2}.$$

The last condition coincides with the condition $\phi(m(\theta), \mathbf{v}) \in S_+$. Therefore, $\theta = \theta(g, \mathbf{v})$ is smooth by the implicit function theorem.

Next, we show that the mapping $(g, \mathbf{v}) \rightarrow k\mathbf{SO}(n-2)$ is smooth. Put $\beta(\mathbf{v}) = a(\mathbf{v}) + ib(\mathbf{v})$ for $\mathbf{v} \in S_+$, where $a(\mathbf{v}), b(\mathbf{v}) \in \mathbf{R}$. Then $a(\mathbf{v}), b(\mathbf{v})$ are smooth functions on S_+ . Put

$$\begin{aligned} \mathbf{v}_1 &= (\cosh \theta - b(\mathbf{v}) \sinh \theta)k\mathbf{e}_1 + a(\mathbf{v}) \cosh \theta k\mathbf{e}_2, \\ \mathbf{v}_2 &= a(\mathbf{v}) \sinh \theta k\mathbf{e}_1 + (-\sinh \theta + b(\mathbf{v}) \cosh \theta)k\mathbf{e}_2 \end{aligned}$$

and

$$T(\mathbf{v}, \theta) = \begin{bmatrix} \cosh \theta - b(\mathbf{v}) \sinh \theta & a(\mathbf{v}) \sinh \theta \\ a(\mathbf{v}) \cosh \theta & -\sinh \theta + b(\mathbf{v}) \cosh \theta \end{bmatrix}.$$

Suppose $g = km(\theta)u, u \in \mathbf{I}(1, \beta)$. Then we see $\mathbf{v}_1 + i\mathbf{v}_2 = g(\mathbf{e}_1 + \beta\mathbf{e}_2)$ and $(\mathbf{v}_1, \mathbf{v}_2) = (k\mathbf{e}_1, k\mathbf{e}_2)T(\mathbf{v}, \theta)$. Moreover, $T(\mathbf{v}, \theta)$ is non-singular under the condition $\phi(m(\theta), \mathbf{v}) \in S_+$. Therefore, we obtain $(k\mathbf{e}_1, k\mathbf{e}_2) = (\mathbf{v}_1, \mathbf{v}_2)T(\mathbf{v}, \theta)^{-1}$ and hence the correspondence $(g, \mathbf{v}) \rightarrow (k\mathbf{e}_1, k\mathbf{e}_2)$ is smooth. $\square$

Put

$$W(\Phi) = \{(g, \psi(k, \mathbf{v})) \in \mathbf{SO}(n, \mathbf{C}) \times S^{2n-1} \mid k \in \mathbf{SO}(n), (gk, \mathbf{v}) \in W_+\}.$$

Lemma 5.3. *The restriction of Φ to $\mathbf{C}^* \times W(\Phi)$ is smooth.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{SO}(n, \mathbf{C}) \times \mathbf{SO}(n) \times S_+ & \xrightarrow{\mu \times 1} & \mathbf{SO}(n, \mathbf{C}) \times S_+ \\ \downarrow 1 \times \psi & & \downarrow \Phi \\ \mathbf{SO}(n, \mathbf{C}) \times S^{2n-1} & \xrightarrow{\Phi} & S^{2n-1}. \end{array}$$

Then, we see $W(\Phi) = (1 \times \psi)(\mu \times 1)^{-1}(W_+)$. By the property of the standard action ψ , we see the restricted mapping $\psi : \mathbf{SO}(n) \times S_+ \rightarrow S^{2n-1}$ is a smooth submersion onto an open set. Hence, we see the mapping $1 \times \psi : \mathbf{SO}(n, \mathbf{C}) \times \mathbf{SO}(n) \times S_+ \rightarrow \mathbf{SO}(n, \mathbf{C}) \times S^{2n-1}$ is a smooth submersion onto an open set. Since W_+ is an open set of $\mathbf{SO}(n, \mathbf{C}) \times S_+$, we see $W(\Phi)$ is an open set of $\mathbf{SO}(n, \mathbf{C}) \times S^{2n-1}$. By Lemma 5.2, $\Phi|_{W_+}$ is smooth, and hence $\Phi|_{W(\Phi)}$ is smooth.

Since the mapping f is $\mathbf{C}^* \times N(n)$ -equivariant and $\mathbf{C}^*$ -action on $\mathbf{P}_1(\mathbf{C})$ is trivial, we see that S_+ is $\mathbf{C}^* \times \mathbf{SO}(2)'$ -invariant. Then, we see $\Phi|(\mathbf{C}^* \times W(\Phi))$ is smooth. $\square$

Lemma 5.4. *The orbit $G(\mathbf{e}_1)$ through $\mathbf{e}_1$ is an open set. If the restricted mapping $\Phi|(G \times G(\mathbf{e}_1))$ is smooth, then the action Φ is smooth.*

Proof. By the definition of Φ ,

$$\begin{aligned} G(\mathbf{e}_1) &= \{\psi(k, \phi(m(\theta), \phi(z, \mathbf{e}_1))) \mid k \in \mathbf{SO}(n), z \in \mathbf{C}^*, \theta \in \mathbf{R}\} \\ &= \{\psi(k, \phi(M(a, b), \phi(z, \mathbf{e}_1))) \mid k \in \mathbf{SO}(n), z \in \mathbf{C}^*, a, b \in \mathbf{C}, a^2 + b^2 = 1\}. \end{aligned}$$

Hence we obtain

$$G(\mathbf{e}_1) = \{\psi(k, \mathbf{v}) \mid k \in \mathbf{SO}(n), \mathbf{v} \in F(H), f(\mathbf{v}) = (a : b), a^2 + b^2 \neq 0\}.$$

This is an open set of S^{2n-1} .

The set $\mathbf{C}^* \times W(\Phi)$ is an open set of $G \times S^{2n-1}$ and contains $G \times \{\psi(k, \mathbf{v}) \mid k \in \mathbf{SO}(n), \mathbf{v} \in F(H), f(\mathbf{v}) = (a : b), a^2 + b^2 = 0\}$. Therefore,

$$(G \times G(\mathbf{e}_1)) \cup (\mathbf{C}^* \times W(\Phi)) = G \times S^{2n-1}.$$

Then we obtain the result by Lemma 5.3. $\square$

6. Certain Equivariant Decompositions

In this section, we shall consider $\mathbf{U}(1) \times \mathbf{SO}(n)$ -equivariant decompositions of certain open sets of S^{2n-1} . Let r be a real number such that $0 < r \leq 1$. Define

$$\begin{aligned} V_n(r) &= \{(w, X, Y) \mid w \in \mathbf{U}(1); X, Y \in \mathbf{R}^n, X \perp Y, \|X\| = 1, \|Y\| < r\}, \\ S_n(r) &= \{\mathbf{v} \in S^{2n-1} \mid |Q(\mathbf{v})| > (1 - r^2)(1 + r^2)^{-1}\}. \end{aligned}$$

Since

$$Q(w(1 + \|Y\|^2)^{-1/2}(X + iY)) = w^2(1 - \|Y\|^2)(1 + \|Y\|^2)^{-1},$$

we can define a mapping $\pi : V_n(r) \rightarrow S_n(r)$ by

$$\pi(w, X, Y) = w(1 + \|Y\|^2)^{-1/2}(X + iY).$$

We see π is smooth double covering such that

$$\pi(w, X, Y) = \pi(-w, -X, -Y).$$

Then the quotient space $V_n(r)/\sim$ is diffeomorphic to $S_n(r)$, by the equivalence relation $(w, X, Y) \sim (-w, -X, -Y)$.

The $\mathbf{U}(1) \times \mathbf{SO}(n)$ -action on $V_n(r)$ is defined by $(z, k)(w, X, Y) = (zw, kX, kY)$, where $(z, k) \in \mathbf{U}(1) \times \mathbf{SO}(n)$. Then the action coincides with the standard action on $S_n(r)$.

Notice that $S_n(1)$ is G -orbit through $\mathbf{e}_1$ for any twisted linear action ψ_c , and the G -orbit through $\mathbf{e}_1$ for the action Φ

defined in the section 4 coincides with $S_n(r)$ for certain r .

Here we consider the twisted linear action ψ_c on $V_n(1)/\sim$. Let $e^s z \in \mathbf{C}^*$, where $s \in \mathbf{R}$, $z \in \mathbf{U}(1)$, and let $k \in \mathbf{SO}(n)$. Then

$$\begin{aligned}\psi_c((e^s z, km(\theta)), \mathbf{e}_1) &= e^{\tau(1+ic)} e^s z km(\theta) \mathbf{e}_1 = e^{\tau(1+ic)} e^s z (\cosh \theta k \mathbf{e}_1 - i \sinh \theta k \mathbf{e}_2) \\ &= \pi(z e^{ic\tau}, k \mathbf{e}_1, -\tanh \theta k \mathbf{e}_2),\end{aligned}$$

where $\tau = -s - \log \sqrt{\cosh 2\theta}$.

The set $F(H) \cap S_n(r)$ is canonically identified with $S_2(r)$. Denote by $X^\perp = -x_2 \mathbf{e}_1 + x_1 \mathbf{e}_2$, for the unit vector $X = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$. Then we can identify $V_2(r)$ with the following set

$$W(r) = \{(w, X, t) \mid w \in \mathbf{U}(1), X \in \mathbf{R}^2, t \in \mathbf{R}; \|X\| = 1, |t| < r\}$$

by the correspondence $(w, X, t) \rightarrow (w, X, tX^\perp)$. The equivalence relation $(w, X, Y) \sim (-w, -X, -Y)$ corresponds to the equivalence relation $(w, X, t) \sim (-w, -X, t)$. Therefore, we can identify $S_2(r)$ with $W(r)/\sim$.

Here we consider the $\mathbf{C}^* \times N(n)$ -action on $S_2(1)$ with respect to the twisted linear action ψ_c . We obtain

$$\psi_c((e^s z, km(\theta), [w, X, t]) = [zwe^{ic\tau}, kX, (t - \tanh \theta)(1 - t \tanh \theta)^{-1}],$$

where $k \in \mathbf{SO}(2)'$, $\tau = -s - (1/2) \log[(\cosh \theta - t \sinh \theta)^2 + (-\sinh \theta + t \cosh \theta)^2](1 + t^2)^{-1}]$.

7. Construction of Smooth $\mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ -Actions

First we prepare the following lemma [1, Lemma].

Lemma 7.1. *There exist smooth functions A, B defined on $\mathbf{R}$ satisfying the conditions*

- (1) $A(t)$: odd function, $B(t)$: even function,
- (2) $|A(t)| < 1$ ($|t| < 1$), $A(t) = 1$ ($t \geq 1$), $A(t) = -1$ ($t \leq -1$),
- (3) $B(t) = 0$ ($|t| \geq 1$),
- (4) $A'(t) > 0$ ($|t| < 1$),
- (5) $B(t)A'(t) = A(t)^2 - 1$ ($\forall t$).

For each positive integer m , define new smooth functions A_m, B_m by

$$\begin{aligned}A_m(t) &= A((2m+2)t), \\ B_m(t) &= (2m+2)^{-1} \sum_{j=-m}^m B((2m+2)t - j).\end{aligned}$$

Then, A_m is an odd function and B_m is an even function. Moreover, we obtain the following relations:

$$\begin{aligned}A_m(t) &= 1 \quad (t \geq (2m+2)^{-1}), & A_m(t) &= -1 \quad (t \leq -(2m+2)^{-1}), \\ |A_m(t)| &< 1 \quad (|t| < (2m+2)^{-1}), & B_m(t) &= 0 \quad (|t| \geq (2m+1)/(2m+2)), \\ A'_m(t) &> 0 \quad (|t| < (2m+2)^{-1}), & B_m(t)A'_m(t) &= A_m(t)^2 - 1 \quad (\forall t).\end{aligned}$$

Define a smooth $\mathbf{R}$ -action ϕ_m on $\mathbf{R}$ by the relation

$$B_m(t) = \lim_{\theta \rightarrow 0} (\phi_m(\theta, t) - t)/\theta.$$

Since B_m is an even function, we obtain

$$(7.2) \quad \phi_m(-\theta, -t) = -\phi_m(\theta, t) \quad \forall(\theta, t).$$

Moreover, we see

$$\frac{\partial}{\partial \theta} A_m(\phi_m(\theta, t)) = A_m(\phi_m(\theta, t))^2 - 1.$$

Hence we obtain

$$(7.3) \quad A_m(\phi_m(\theta, t)) = \frac{A_m(t) - \tanh \theta}{1 - A_m(t) \tanh \theta} \quad \forall(\theta, t).$$

Define $f_m : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ by

$$\begin{aligned}f_m([w, x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, t]) &= (x_1 - iA_m(t)x_2 : x_2 + iA_m(t)x_1) \text{ on } W(1)/\sim \\ f_m(a\mathbf{e}_1 + b\mathbf{e}_2) &= (a : b) \text{ for } a^2 + b^2 = 0.\end{aligned}$$

Then f_m is a smooth function on $F(H)$.

Let c be a real number. Define a smooth mapping $\phi_{m,c} : \mathbf{C}^* \times N(n) \times W(1)/\sim \rightarrow W(1)/\sim$ by

$$\phi_{m,c}((e^s z, km(\theta)), [w, X, t]) = [zwe^{-ic(s+\tau_m(\theta,t))}, kX, \phi_m(\theta, t)]$$

where $z \in \mathbf{U}(1)$, $k \in \mathbf{SO}(2)'$; $s, \theta \in \mathbf{R}$ and

$$\tau_m(\theta, t) = (1/2) \log[(\cosh \theta - A_m(t) \sinh \theta)^2 + (-\sinh \theta + A_m(t) \cosh \theta)^2] / (1 + A_m(t)^2).$$

Then we see $\phi_{m,c}$ is an action of $\mathbf{C}^* \times N(n)$ on $W(1)/\sim$ by (7.3). Moreover, we see

$$\begin{aligned} \phi_{m,c}(m(\theta), [w, X, t]) &= [we^{ic\theta}, X, t] \text{ for } t \geq (2m+1)/(2m+2) \\ \phi_{m,c}(m(\theta), [w, X, t]) &= [we^{-ic\theta}, X, t] \text{ for } t \leq -(2m+1)/(2m+2). \end{aligned}$$

Now we modify $\phi_{m,c}$ by

$$\varphi_{m,c}((e^s z, km(\theta)), [w, X, t]) = [zwe^{-ic(s+\rho(t)\tau_m(\theta,t))}, kX, \phi_m(\theta, t)].$$

Here, $\rho(t)$ is a smooth even function such that

$$\begin{aligned} \rho(t) &= 1 \text{ } (|t| \leq (6m+4)/(6m+6)), \quad \rho(t) = 0 \text{ } (|t| \geq (6m+5)/(6m+6)), \\ 0 &< \rho(t) < 1 \text{ } ((6m+4)/(6m+6) < |t| < (6m+5)/(6m+6)). \end{aligned}$$

Then we see $\varphi_{m,c}$ is also an action of $\mathbf{C}^* \times N(n)$ on $W(1)/\sim$ and

$$\varphi_{m,c}(m(\theta), [w, X, t]) = [w, X, t] \text{ for } |t| \geq (6m+5)/(6m+6).$$

Therefore, we can extend smoothly the action $\varphi_{m,c}$ on $F(H)$ by

$$\varphi_{m,c}((e^s z, km(\theta)), \mathbf{v}) = e^{-ics} z k \mathbf{v}$$

for $\mathbf{v} \in F(H)$ such that $|Q(\mathbf{v})| < (12m+11)/[(6m+6)^2 + (6m+5)^2]$.

Notice that the action of the matrix J can be described by

$$J[w, X, t] = [w, JX, -t].$$

Now, we shall show the pair $(\varphi_{m,c}, f_m)$ satisfies the conditions on (ϕ, f) stated in the Sect. 4.

Lemma 7.4. *Let $(\varphi_{m,c}, f_m)$ be the pair defined above. Then the pair satisfies the following conditions:*

- (1) $f_m : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ is $\mathbf{C}^* \times N(n)$ -equivariant mapping from $F(H)$ with the action $\varphi_{m,c}$ to $\mathbf{P}_1(\mathbf{C})$ with the standard action.
- (2) $\varphi_{m,c} = \psi$ with respect to the restricted $\mathbf{U}(1) \times \mathbf{SO}(2)'$ -action.
- (3) $\varphi_{m,c}(M(a, b), J(\mathbf{v})) = J\varphi_{m,c}(M(a, -b), \mathbf{v})$ for any complex numbers a, b such that $a^2 + b^2 = 1$.
- (4) $f_m J = J f_m$.
- (5) $f_m(\mathbf{v}) = (1 : 0)$ if and only if $\mathbf{v} \in \mathbf{Ce}_1 \cap F(H)$.

Proof. (1) Clearly f_m is $\mathbf{U}(1) \times \mathbf{SO}(2)'$ -equivariant, and it is $\mathbf{C}^* \times N(n)$ -equivariant by (7.3). (2) $\varphi_{m,c}((z, k), [w, X, t]) = [zw, kX, t]$ for $z \in \mathbf{U}(1)$, $k \in \mathbf{SO}(2)'$. (3) is shown by (7.2) and the facts: $\tau_m(-\theta, -t) = \tau_m(\theta, t)$, $\rho(-t) = \rho(t)$. (4) is shown by the fact that $A_m(t)$ is an odd function. (5) is shown by the fact that $A_m(t) = 0$ if and only if $t = 0$. $\square$

By this lemma and Theorem 4.4, we obtain an abstract action $\Phi_{m,c}$ of $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} for each positive integer m and each real number c .

Theorem 7.5. *The action $\Phi_{m,c}$ is smooth, has just $m+1$ open orbits, and the isotropy subgroup at $\mathbf{e}_1$ coincides with the isotropy subgroup at $\mathbf{e}_1$ with respect to the twisted linear action ψ_c .*

Proof. We see that the function $A_m(t)$ is an odd function and the restriction $A_m : (-1/(2m+2), 1/(2m+2)) \rightarrow (-1, 1)$ is a diffeomorphism. Then there exists an even smooth function $a_m(t)$ such that $A_m(t) = ta_m(t)$ for any t . Hence we obtain a smooth $\mathbf{U}(1) \times \mathbf{SO}(n)$ -equivariant mapping

$$h : V(1/(2m+2))/\sim \rightarrow V(1)/\sim$$

by $h[w, X, Y] = [w, X, a_m(\|Y\|)Y]$. We see that h is a diffeomorphism, $V(1/(2m+2))/\sim$ is the G -orbit through $\mathbf{e}_1$ with respect to the action $\Phi_{m,c}$ and $V(1)/\sim$ is the G -orbit through $\mathbf{e}_1$ with respect to the twisted linear action ψ_c . By (7.3), we see that h is a G -equivariant diffeomorphism from $V(1/(2m+2))/\sim$ with the G -action $\Phi_{m,c}$ to $V(1)/\sim$ with the twisted linear G -action ψ_c . In particular, this fact shows the G -action $\Phi_{m,c}$ on $G(\mathbf{e}_1)$ is smooth, and hence the G -action $\Phi_{m,c}$ is smooth by Lemma 5.4.

Final statement is clear by the equivariant diffeomorphism h , and the number of open orbits is known by the construction of ϕ_m . $\square$

Corollary 7.6 *The two actions $\Phi_{m,c}, \Phi_{m',c'}$ are topologically distinct for any $(m, c) \neq (m', c')$.*

Remark. By considering the restricted $\mathbf{SO}(n, \mathbf{C})$ -action of $\Phi_{m,c}$, we obtain infinitely many topologically distinct smooth

actions of $\mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} (cf. [9, 4.2.3]).

8. Singular Orbits

Return to the situation in the Sect 3.

Let $\Phi : G \times S^{2n-1} \rightarrow S^{2n-1}$ be a smooth action which is an extension of the standard action $\psi : K \times S^{2n-1} \rightarrow S^{2n-1}$. Here, $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ and $K = \mathbf{U}(1) \times \mathbf{SO}(n)$. The standard K -action ψ has just two singular orbits, the one is $K(\mathbf{e}_1)$ and the other is $K(\mathbf{v}_0)$, where $\mathbf{v}_0 = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}$.

We show $K(\mathbf{e}_1)$ is contained in an open orbit $G(\mathbf{e}_1)$ for any extended G -action Φ in Lemma 5.4. Now we shall show the following.

Theorem 8.1. *The singular orbit $K(\mathbf{v}_0)$ is invariant under any extended G -action Φ .*

Proof. Denote by ϕ the restricted $\mathbf{C}^* \times N(n)$ -action of Φ on $F(H)$. Put

$$R(\tau) = \left[\begin{array}{cc|c} \cos \tau & \sin \tau & 0 \\ -\sin \tau & \cos \tau & 0 \\ \hline 0 & 0 & I_{n-2} \end{array} \right] \in \mathbf{SO}(2)'.$$

Then we see $\phi(R(\tau), \mathbf{v}_0) = e^{i\tau} \mathbf{v}_0$. Put

$$\phi((e^s, m(\theta)), \mathbf{v}_0) = x\mathbf{e}_1 + iy\mathbf{e}_2; |x|^2 + |y|^2 = 1, \quad x = x(s, \theta), \quad y = y(s, \theta) \in \mathbf{C}.$$

Since $\mathbf{C}^* \times N(n)$ is an abelian group, we obtain

$$\phi(R(\tau), \phi((e^s, m(\theta)), \mathbf{v}_0)) = \phi((e^s, m(\theta)), \phi(R(\tau), \mathbf{v}_0)).$$

Now we analyze the equation by the fact that the restricted $\mathbf{U}(1) \times \mathbf{SO}(2)'$ -action is standard action.

$$\begin{aligned} \phi(R(\tau), \phi((e^s, m(\theta)), \mathbf{v}_0)) &= \phi(R(\tau), x\mathbf{e}_1 + iy\mathbf{e}_2) \\ &= (x \cos \tau + iy \sin \tau)\mathbf{e}_1 + (-x \sin \tau + iy \cos \tau)\mathbf{e}_2, \\ \phi((e^s, m(\theta)), \phi(R(\tau), \mathbf{v}_0)) &= \phi((e^s, m(\theta)), e^{i\tau} \mathbf{v}_0) \\ &= \phi(e^{i\tau}, \phi((e^s, m(\theta)), \mathbf{v}_0)) \\ &= e^{i\tau}(x\mathbf{e}_1 + iy\mathbf{e}_2) \\ &= (x \cos \tau + ix \sin \tau)\mathbf{e}_1 + (-y \sin \tau + iy \cos \tau)\mathbf{e}_2. \end{aligned}$$

We obtain $(x - y) \sin \tau = 0$ for each s, θ, τ , and hence $x = y$. Put $w(s, \theta) = \sqrt{2}x(s, \theta)$. Then we obtain

$$(+) \quad \phi((e^s, m(\theta)), \mathbf{v}_0) = w(s, \theta)\mathbf{v}_0; w(s, \theta) \in \mathbf{U}(1)$$

On the other hand, we see that the isotropy subgroup $G_{\mathbf{v}_0}$ contains $\mathbf{I}(1, i\delta)$ for $\delta = \pm 1$, by Lemma 2.2 and the form of $K_{\mathbf{v}_0}$. By the decomposition $\mathbf{SO}(n, \mathbf{C}) = \mathbf{SO}(n)M(n)\mathbf{I}(1, i\delta)$, we see

$$G(\mathbf{v}_0) = \{\psi(k, \phi((ze^s, m(\theta)), \mathbf{v}_0)) \mid k \in \mathbf{SO}(n), z \in \mathbf{U}(1)\}.$$

Then we obtain by (+),

$$G(\mathbf{v}_0) = \{\psi(k, \phi(z', \mathbf{v}_0)) \mid k \in \mathbf{SO}(n), z' \in \mathbf{U}(1)\} = K(\mathbf{v}_0). \quad \square$$

9. Smoothness of the Action Φ , Part II

Return to the situation in the Sect. 4. Let ϕ be a smooth $\mathbf{C}^* \times N(n)$ -action on $F(H)$, and $f : F(H) \rightarrow \mathbf{P}_1(\mathbf{C})$ be a smooth $\mathbf{C}^* \times N(n)$ -equivariant mapping. Suppose the pair (ϕ, f) satisfies the conditions (a), (b) and (c) stated in the Sect. 4.

We shall show the following result in this section.

Theorem 9.1. *The abstract action Φ of $G = \mathbf{C}^* \times \mathbf{SO}(n, \mathbf{C})$ on S^{2n-1} defined by (ϕ, f) is always smooth.*

Proof. We see

$$f(\phi((z, m(\theta)R(\tau)), \mathbf{e}_1)) = (\cos \tau - i \sin \tau \tanh \theta : -\sin \tau - i \cos \tau \tanh \theta).$$

Put $G' = \mathbf{U}(1) \times N(n)$. If $\phi((z, m(\theta)R(\tau)), \mathbf{e}_1) = \mathbf{e}_1$ for $(z, m(\theta)R(\tau)) \in G'$, then $\sin \tau + i \cos \tau \tanh \theta = 0$, and hence $\sin \tau = \tanh \theta = 0$. Therefore, we obtain

$$G'_{\mathbf{e}_1} = \{(\varepsilon, \varepsilon I_2) \mid \varepsilon = \pm 1\} = Z_2.$$

Since $\dim G' = \dim F(H) = 3$, we obtain a G' -equivariant diffeomorphism from G'/Z_2 onto an open set of $F(H)$, by the correspondence from $[z, m(\theta)R(\tau)]$ to $\phi((z, m(\theta)R(\tau)), \mathbf{e}_1)$. On the other hand, there is a $\mathbf{U}(1) \times \mathbf{SO}(2)'$ -equivariant decomposition $S_2(1) = W(1)/\sim$. So we can denote

$$\phi((1, m(\theta)), \mathbf{e}_1) = [w(\theta), X(\theta), t(\theta)] \in W(1)/\sim.$$

Under the $\mathbf{U}(1) \times \mathbf{SO}(2)'$ -actions, the orbit space of G'/Z_2 is diffeomorphic to an open set of the orbit space of $W(1)/\sim$. In particular, we see that the correspondence $\theta \rightarrow t(\theta)$ is a diffeomorphism from $\mathbf{R}$ onto an open set of $(-1, 1)$.

Considering the action of the matrix J , we obtain

$$J\phi((1, m(\theta)), \mathbf{e}_1) = \phi((1, m(-\theta)), J\mathbf{e}_1) = \phi((1, m(-\theta)), \mathbf{e}_1).$$

Put $X(\theta) = a(\theta)\mathbf{e}_1 + b(\theta)\mathbf{e}_2$. Then we see

$$[w(\theta), a(\theta)\mathbf{e}_1 - b(\theta)\mathbf{e}_2, -t(\theta)] = [w(-\theta), a(-\theta)\mathbf{e}_1 + b(-\theta)\mathbf{e}_2, t(-\theta)].$$

Therefore, $w(\theta), a(\theta)$ are smooth even functions, and $b(\theta), t(\theta)$ are smooth odd functions.

Since the diffeomorphism $\theta \rightarrow t(\theta)$ is an odd function, we see that there is a real number r such that the correspondence $\theta \rightarrow t(\theta)$ is a diffeomorphism from $\mathbf{R}$ to the interval $(-r, r)$ where $0 < r \leq 1$.

Moreover, there exist smooth even functions $t_1(\theta), b_1(\theta)$ such that $t(\theta) = \theta t_1(\theta), b(\theta) = \theta b_1(\theta)$, since $t(\theta), b(\theta)$ are smooth odd functions. We see that the function $t_1(\theta)$ is everywhere non-zero, because the function $t(\theta)$ is strictly monotone function. Put $c(\theta) = b_1(\theta)/t_1(\theta)$. Then $c(\theta)$ is a smooth even function. Define a matrix $A(\theta)$ by

$$A(\theta) = \begin{bmatrix} a(\theta) & -t(\theta)^2 c(\theta) \\ c(\theta) & a(\theta) \end{bmatrix}.$$

Then the correspondence $\theta \rightarrow A(\theta)$ is a smooth mapping from $\mathbf{R}$ to $\mathbf{SL}(2, \mathbf{R})$ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & t(\theta) \end{bmatrix} A(\theta) = \begin{bmatrix} a(\theta) & -t(\theta)b(\theta) \\ b(\theta) & t(\theta)a(\theta) \end{bmatrix}.$$

Notice that each component of $A(\theta)$ is an even function.

Denote by $\Theta : (-r, r) \rightarrow \mathbf{R}$ the inverse function of the correspondence $\theta \rightarrow t(\theta)$. Define

$$\bar{A}(t) = A(\Theta(t)) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

Then the correspondence $t \rightarrow \bar{A}(t)$ is a smooth mapping from $(-r, r)$ to $\mathbf{SL}(2, \mathbf{R})$ and each component of $\bar{A}(t)$ is an even function. By the construction, we see

$$a_{11}(t)^2 + a_{21}(t)^2 t^2 = 1, \quad a_{12}(t)^2 + a_{22}(t)^2 t^2 = t^2, \quad a_{11}(t)a_{12}(t) + a_{21}(t)a_{22}(t)t^2 = 0.$$

Now we consider the triple

$$(w, a_{11}(\|Y\|)X - a_{21}(\|Y\|)Y, a_{12}(\|Y\|)X - a_{22}(\|Y\|)Y)$$

for $(w, X, Y) \in V_n(r)$. We see the triple is an element of $V_n(r)$. Hence we obtain a $\mathbf{U}(1) \times \mathbf{SO}(n)$ -equivariant diffeomorphism $h_1 : V_n(r)/\sim \rightarrow V_n(r)/\sim$ by the correspondence

$$[w, X, Y] \rightarrow [w, a_{11}(\|Y\|)X - a_{21}(\|Y\|)Y, a_{12}(\|Y\|)X - a_{22}(\|Y\|)Y].$$

In particular, we see

$$h_1([w(\theta), X(\theta), t(\theta)]) = [w(\theta), \mathbf{e}_1, t(\theta)].$$

Next we define a mapping $\sigma : (-r, r) \rightarrow (-1, 1)$ by $\sigma(t) = -\tanh \Theta(t)$. Then σ is a diffeomorphism and an odd function. Then there exists a smooth even function $\sigma_1(t)$ such that $\sigma(t) = t\sigma_1(t)$.

Now we consider the function $z : \mathbf{R} \rightarrow \mathbf{U}(1)$ defined by

$$z(\theta) = w(\theta)^{-1} \exp(-ic(1/2) \log \cosh 2\theta).$$

Here c is a real number. Define $\bar{z}(t) = z(\Theta(t))$. We see $z(\theta), \bar{z}(t)$ are smooth even functions.

Then we obtain a $\mathbf{U}(1) \times \mathbf{SO}(n)$ -equivariant diffeomorphism $h_2 : V_n(r)/\sim \rightarrow V_n(1)/\sim$ by the correspondence

$$[w, X, Y] \rightarrow [w\bar{z}(\|Y\|), X, \sigma_1(\|Y\|)Y].$$

In particular, we see

$$h_2([w(\theta), \mathbf{e}_1, t(\theta)]) = [\exp(-ic(1/2) \log \cosh 2\theta), \mathbf{e}_1, -\tanh \theta] = \psi_c((1, m(\theta)), \mathbf{e}_1).$$

Put $h = h_2 \circ h_1$. Then h is a $\mathbf{U}(1) \times \mathbf{SO}(n)$ -equivariant diffeomorphism from $V_n(r)/\sim$ to $V_n(1)/\sim$ satisfying the condition:

$$h(\phi((1, m(\theta)), \mathbf{e}_1)) = \psi_c((1, m(\theta)), \mathbf{e}_1).$$

Now we shall decide the real number c . Since $f(\phi((e^s, 1), \mathbf{e}_1)) = f(\mathbf{e}_1) = (1 : 0)$, we obtain

$$\phi((e^s, 1), \mathbf{e}_1) = v(s)\mathbf{e}_1, \text{ for } v(s) \in \mathbf{U}(1).$$

Moreover, we see there exists a real number c such that $v(s) = e^{-ics}$. Then we see

$$h(\phi((e^s, 1), \mathbf{e}_1)) = h(e^{-ics}\mathbf{e}_1) = e^{-ics}\mathbf{e}_1 = \psi_c((e^s, 1), \mathbf{e}_1).$$

By making use of the decomposition $\mathbf{SO}(n, \mathbf{C}) = \mathbf{SO}(n)M(n)\mathbf{I}(1, 0)$ and the definition of Φ , we obtain

$$h(\Phi((z, g), \mathbf{e}_1)) = \psi_c((z, g), \mathbf{e}_1) \text{ for } z \in \mathbf{C}^*, g \in \mathbf{SO}(n, \mathbf{C}).$$

Therefore, the diffeomorphism $h : V_n(r)/\sim \rightarrow V_n(1)/\sim$ is compatible with G -actions.

Then we see the G -action Φ on $V_n(r)/\sim$ is smooth. Combining this result and Lemma 5.4, we see that the G -action Φ on S^{2n-1} is smooth. $\square$

REFERENCES

- [1] Asoh, T., “On smooth $\mathbf{SL}(2, \mathbf{C})$ actions on 3-manifolds,” *Osaka J. Math.*, **24**: 271–298 (1987).
- [2] Mukōyama, K., “Smooth $\mathbf{Sp}(2, \mathbf{R})$ -actions on the 4-sphere,” *Tohoku Math. J.*, **48**: 543–560 (1996).
- [3] Mukōyama, K., “Smooth $\mathbf{SU}(p, q)$ -actions on the $(2p + 2q - 1)$ -sphere and on the complex projective $(p + q - 1)$ -space,” *Kyushu J. Math.*, **55**: 213–236 (2001).
- [4] Uchida, F., “On a method to construct analytic actions of non-compact Lie groups on a sphere,” *Tohoku Math. J.*, **39**: 61–69 (1987).
- [5] Uchida, F., “On smooth $\mathbf{SO}_0(p, q)$ -actions on S^{p+q-1} ,” *Osaka J. Math.*, **26**: 775–787 (1989).
- [6] Uchida, F., “On smooth $\mathbf{SO}_0(p, q)$ -actions on S^{p+q-1} , II,” *Tohoku Math. J.*, **49**: 185–202 (1997).
- [7] Uchida, F., “On smooth $\mathbf{Sp}(p, q)$ -actions on $S^{4p+4q-1}$,” *Osaka J. Math.*, **39**: 293–314 (2002).
- [8] Uchida, F., “On smooth $\mathbf{SL}(m, \mathbf{R}) \times \mathbf{SL}(n, \mathbf{R})$ -actions on S^{m+n-1} ,” *Interdisciplinary Inf. Sci.*, **8**: 123–128 (2002).
- [9] Uchida, F., and Mukōyama, K., “Smooth actions of non-compact semi-simple Lie groups,” *Current Trends in Transformation Groups*, Bak, A., *et al.*, Ed., K-Monographs in Mathematics, Vol. 7, 201–215, Kluwer Academic Publishers (2002).