Consensus Formation based on Scenarios

Hiroaki Ishii*

*Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University, 2-1 Gakuen Sanda 669-1337 Hyogo, Japan

Abstract

The following ranking problem is considered.
(1) There exist $m$ candidates to be ranked with respect to $s$ criteria.
(2) Each candidate has $s$ characteristics and each one corresponds to each criterion. Based on the value of the characteristics, candidates are ranked with respect to the corresponding criterion. But this ranking may change due to the change of value of the characteristics. This change occurs according to the some probability. Therefore total change with respect to all criteria occurs with scenarios.
(3) Our theme is to make the total ranking among candidates considering not only the importance of criteria but the total change of ranking about candidates with respect to all criteria. That is, how to make a consensus formation is the main problem.

First for the fixed scenario case, that is, the fixed preference matrices, we define the preference matrix of candidates with respect to each criterion and distance measure of Cook [W.D.Cook, EJOR 172 (2006)369-385] based on the matrix. Next we extend the Cook distance measure toward the weighted distance measure and then formulate the equivalent assignment problem. The third we solve the assignment problem and based on the optimal assignment, that is, optimal ranking of candidate is found. The fourth we struggle to aggregate the result of each scenario case and make the final suitable ranking considering the risk. We also show our procedure can apply other ranking model, especially the case that preference matrix includes ambiguity in place of randomness.

Finally we conclude our results and discuss the further research problem including the efficient solution method for the assignment problem corresponding to the extended weighted distance measure.

Key words: Preference matrix, Ranking change, Scenario, Distance measure, Consensus formation

1. Introduction

For ranking of alternatives, one of the most familiar methods is to compare the weighted sum of their votes, after determining suitable weights of each alternative. Borda [1] initially proposed the “Method of Marks” more than two hundred years ago so as to obtain an agreement among different opinions. His method is surely a useful method evaluating consumers’ preferences of commodities in marketing, or in ranking social policies in political sciences. What is suitable weight of each alternative? In this context, Cook and Kress [2] formulated the measure to automatically decide on the total rank order weight in order to hold the advantage using Data Envelopment Analysis (DEA). DEA is originated by Charnes et al. [3] and extended by Banker et al. [4]. The basic DEA models are known as CCR and BCC named after the authors’ initials. Later, Green et al. [5] evolved the distance measure so as to make it possible to decide on the total rank order of all candidates. The distance measure is based on a different idea that individual preference for a set of candidates should be aggregated (see related works of W.D.Cook [6], Cook and Kress [2], Green et al. [5]. Tanabe and Ishii [7] extended the distance measure to construct a joint ballot model. This paper considers a consensus formation method aggregating ranking of candidate with respect to criteria based on scenarios using the measure of W. D. Cook [6].

Section 2 formulates the main problem. Section 3 reviews an ordinary consensus formation methods using results of our previous papers [7] based on the fixed
scenario. Based on Section 3, we discuss how to make a consensus formation under the risk described by

2. Problem Formulation

The following ranking problem is considered.
(1) There exist m candidates to be ranked with respect to s criteria.
(2) Each candidate has s characteristics and each one corresponds to each criterion. Based on the value of the characteristics, candidates are ranked with respect to the corresponding criterion. But this ranking may change due to the change of value of the characteristics. This change occurs according to the some probability. Therefore total change with respect to all criteria occurs with scenarios.
(3) Our theme is to make the total ranking among candidates considering not only the importance of criteria but the total change of ranking about candidates with respect to all criteria.

First for the fixed scenario case, we define the ranking matrices of candidates with respect to each criterion and distance measure of Cook [7] based on the matrix.

3. Consensus formation based on the fixed Scenario

To each criterion, one candidate is assigned to one ranking and it is denoted by $m \times m$ ranking matrix $A_i = (a_{ij})$, $i = 1, 2, ..., s$ where

$$a_{ij} = \begin{cases} 
1 & \text{if as for the criterion } i \text{ candidate } j \text{ is ranked as to rank } j \\
0 & \text{otherwise} 
\end{cases}$$

Distance between ranking matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined as $\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij} - b_{ij}|$. We consider the aggregated decision among criteria is denoted by the ranking matrix $X = (x_{ij})$ and it should be determined by minimizing the total weighted distance defined as $\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{j=1}^{m} w_{ij} |a_{ij} - x_{ij}|$ where $w_{ij}$ is the importance of the criterion $i$. $w_{ij}$ is determined from the pair-wise comparison between each pair of positions by using AHP as follows:

First construct $s \times s$ pair-wise comparison matrix $D = (d_{ij})$ with respect to positions where scenarios in Section 4. Section 5 concludes this paper and discusses further research problems.

$$d_{ij} = \begin{cases} 
1 & (i = j) \\
\frac{1}{d_{ij}} & (i \neq j) 
\end{cases}$$

and $d_{ij}$ denotes how many times important criterion $i$ to criterion $j$, using the number among the numbers 1 to 9. Here we consider ideally $d_{ij} = \frac{w_i}{w_j}$ as a ratio between weights $w_i$ and $w_j$.

Therefore $j$-th element of an eigen vector corresponding to eigen value $s$ with respect to matrix $D$ gives weight $w_j$ since

$$\begin{pmatrix} w_1 \ w_2 \ \vdots \\
w_1 \ w_2 \ \vdots \\
\vdots \ \vdots \ \vdots \\
w_1 \ w_2 \ \vdots \\
w_1 \ w_2 \ \vdots \\
w_1 \ w_2 \ \vdots \\
w_1 \ w_2 \ \vdots \\
w_1 \ w_2 \ \vdots \\
\end{pmatrix} \begin{pmatrix} w_1 \\
w_2 \\
\vdots \\
w_j \\
\vdots \\
w_j \\
w_j \\
\end{pmatrix} = s \begin{pmatrix} w_1 \\
w_2 \\
\vdots \\
w_j \\
\vdots \\
w_j \\
w_j \\
\end{pmatrix}$$

Since the element of preference matrix is 0 or 1, minimizing $\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{i=1}^{m} \sum_{j=1}^{m} w_{ij} |a_{ij} - x_{ij}|$ is equivalent to maximizing $\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{j=1}^{m} w_{ij} a_{ij} x_{ij}$ as is easily shown.

Cook and Kress [2] has constructed relative positioning to express the degree of differences and this is basically equivalent to the difference $\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{j=1}^{m} |a_{ij} - x_{ij}|$ before weighting. Then according to [6], the position $j$ forward indicator vector $P^+(j)$ and the position $j$ backward indicator vector $P^-(j)$ are defined as

$$P^+(j) = \left[ \sum_{i=1}^{m} a_i \right], \quad P^-(j) = \left[ \sum_{i=1}^{m} a_i \right]$$
The relative distance function \( d_p(A, B) \) is given with 
\[
P^*_A(j), P^*_B(j) \quad \text{and} \quad P_A^r(j), P_B^r(j)
\]
as below.
\[
d_p(A, B) = m(m-1) - \sum_{j=1}^{m} \left[ \{P^*_A(j), P^*_B(j)\} + \{P_A^r(j), P_B^r(j)\} \right]
\]
where \( \{P^*_A(j), P^*_B(j)\} \) is inner products between \( P^*_A(j) \) and \( P^*_B(j) \) and \( \{P_A^r(j), P_B^r(j)\} \) that between \( P_A^r(j) \) and \( P_B^r(j) \). Minimizing the relative distance is equivalent to maximizing the inner product
\[
\sum_{j=1}^{m} \left[ \{P^*_A(j), P^*_B(j)\} + \{P_A^r(j), P_B^r(j)\} \right].
\]
So we seek the assignment matrix (ranking matrix) \( X = (x_{ij}) \) maximizing the total weighted sum
\[
\sum_{i=1}^{s} \sum_{j=1}^{m} w_i \left[ \{P^*_A(j), P^*_B(j)\} + \{P_A^r(j), P_B^r(j)\} \right].
\]
Rewriting it and divided by \((m-1)\) results
\[
\sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} w_i \left[ \{P^*_A(j), P^*_B(j)\} + \{P_A^r(j), P_B^r(j)\} \right]
\]
and further by changing the order of summation, finally we have
\[
\sum_{k=1}^{m} \sum_{i=1}^{s} \left( \sum_{j=1}^{m} \{P^*_A(j), P^*_B(j)\} \right) x_{ik} + \sum_{j=1}^{m} \{P_A^r(j), P_B^r(j)\} x_{ik} = \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} \left( \sum_{j=1}^{m} \{P^*_A(j), P^*_B(j)\} \right) x_{ik} + \sum_{j=1}^{m} \{P_A^r(j), P_B^r(j)\} x_{ik}
\]
Since \( x_{ij} = 0 \) or 1 and \( \alpha'_j(r) = 0 \) or 1. Extending the definition of the position \( j \) forward indicator vector \( P^*_p(j) \) and that of the position \( j \) backward indicator vector \( P^*_{p}^{-}(j) \) now we define as
\[
P^*_p(j) = \left[ \sum_{r=1}^{q} \sum_{j=1}^{m} \alpha'_j(r) \right],
\]
\[
P^*_{p}^{-}(j) = \left[ \sum_{r=1}^{q} \sum_{j=1}^{m} \alpha'_j(r) \right]
\]
So we seek the assignment matrix (ranking matrix) \( X = (x_{ij}) \) maximizing the total weighted sum
\[
\sum_{i=1}^{s} \sum_{j=1}^{m} w_i \left[ \{P^*_p(j), P^*_{p}^{-}(j)\} \right].
\]
Rewriting it and divided by \((m-1)\) results
\[
\sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} w_i \left[ \{P^*_p(j), P^*_{p}^{-}(j)\} \right] x_{ik}
\]
and further by changing the order of summation, finally we have
\[
\sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} \left( \sum_{j=1}^{m} \{P^*_A(j), P^*_B(j)\} \right) x_{ik} + \sum_{j=1}^{m} \{P_A^r(j), P_B^r(j)\} x_{ik} = \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} \left( \sum_{j=1}^{m} \{P^*_A(j), P^*_B(j)\} \right) x_{ik} + \sum_{j=1}^{m} \{P_A^r(j), P_B^r(j)\} x_{ik}
\]
Problem (3) can be solved efficiently using some algorithm for an assignment problem such as Munkres [8]. (Also refer to Penticio [9]). From an optimal solution of (3), we obtain a consensus formation among candidates.

4. Consensus formation based on the Scenarios

We assume that due to the change of candidates with respect to criteria, estimation of ranking candidates is changeable. This change of each ranking with respect to each criterion is denoted by possibility of the scenario. That is, with probability \( p(r) \), \( r \)-th scenario occurs and ranking matrix becomes \( A'(r) = (\alpha'_j(r)) \) for each criterion \( \ell, \ell = 1, \ldots, s \) and \( r=1,2,\ldots,q \). Note that \( p(r) > 0, \sum_{r=1}^{q} p(r) = 1 \). In this situation we propose two methods to make consensus formation. One is as follows:

First we calculate so called expected ranking matrix for each criteria \( \ell, E[A'] = \sum_{r=1}^{q} p(r) A'(r) = \left( \sum_{r=1}^{q} p(r) \alpha'_j(r) \right) \). We replace \( A' \) in Section 3 by \( E[A'] \). Again minimizing \( \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} w_i \sum_{j=1}^{m} p(r) \alpha'_j(r) x_{ij} \) is equivalent to maximizing \( \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{m} w_i \sum_{j=1}^{m} p(r) \alpha'_j(r) x_{ij} \) since \( x_{ij} = 0 \) or 1.
\[
\sum_{i=1}^{m} \sum_{k=1}^{m} \left\{ \sum_{j=1}^{q} w_j \sum_{i=1}^{k-1} \left( \sum_{j=1}^{n} \alpha_i^j(r) \right) + \sum_{j=k+1}^{m} \left( \sum_{i=1}^{n} \alpha_i^j(r) \right) \right\} x_{ik}.
\]

Then setting
\[
c_{ik} = \sum_{j=1}^{m} \left\{ \sum_{i=1}^{q} p(r) \left( \sum_{j=1}^{n} \alpha_i^j(r) \right) + \sum_{j=k+1}^{m} \left( \sum_{i=1}^{n} \alpha_i^j(r) \right) \right\},
\]
\[i = 1, 2, ..., m, k = 1, ..., m\]
again we have the assignment problem to be solved.

Maximize
\[
\sum_{i=1}^{m} \sum_{k=1}^{m} c_{ik} x_{ik}
\]
subject to
\[\sum_{i=1}^{m} x_{ik} = 1, k = 1, 2, ..., m,\]
\[\sum_{k=1}^{m} x_{ik} = 1, i = 1, 2, ..., m,\]
\[x_{ik} = 0 \text{ or } 1,\]
\[i = 1, 2, ..., m, k = 1, 2, ..., m\]  \hspace{1cm} (6)

The other is as follows:
For assignment matrix \(X = (x_{ij})\) and each scenario \(r\), we calculate so called consistency value
\[C^*(X) = \sum_{i=1}^{m} \sum_{k=1}^{m} w_i \left[ \sum_{r=1}^{n} \alpha_{ik}^r (\sum_{r=1}^{n} x_{ri}) \right] + \sum_{r=1}^{n} \alpha_{ik}^r (\sum_{r=1}^{n} x_{ri})\]
and consider the following bottleneck assignment problem:

Maximize \[\min \{ p(r)C^*(X) \mid r = 1, 2, ..., q\}\]
subject to \[\sum_{j=1}^{m} x_{ij} = 1, k = 1, 2, ..., m, \sum_{i=1}^{m} x_{ij} = 1, i = 1, 2, ..., m,\]
\[x_{ij} = 0 \text{ or } 1, i = 1, 2, ..., m, k = 1, 2, ..., m\] \hspace{1cm} (7)

that is,

Maximize \[\min \{ \sum_{k=1}^{m} c_{ik}(r) x_{ik} \mid r = 1, 2, ..., q\}\]
subject to \[\sum_{i=1}^{m} x_{ik} = 1, k = 1, 2, ..., m,\]
\[\sum_{i=1}^{m} x_{ik} = 1, i = 1, 2, ..., m,\]
\[x_{ij} = 0 \text{ or } 1,\]
\[i = 1, 2, ..., m, k = 1, 2, ..., m\] \hspace{1cm} (8)

where
\[c_{ik}(r) = p(r) \sum_{j=1}^{m} \left\{ \sum_{i=1}^{q} \alpha_{ik}^j(r) \right\} + \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} \alpha_{ik}^j(r) \right\},\]
\[i = 1, 2, ..., m, k = 1, ..., m\]
It can be solved efficiently (see the book [10]).

5. Conclusion

We tried to make a consensus formation under the risk that some ranking matrix may change, that is, rank of candidates changes. This trial to make a consensus formation under the change of ranking may be new but in a real situation it may be usual in many situation of business since material price may change often recently and pay extra money to buy etc. In order to make a good assignment of candidates to ranks, we should investigate better methods. One is to make a higher weight to the higher rank compared to the lower rank since usually consistency of higher rank is more important. Another is to aggregate the captured number for each rank among criteria, further weight by probabilities of scenarios and then we use DEA like ranking method such as [11-14]. In an actual situation, some candidates have same rank and in that case, how to solve the ties is an important issue. Also there exist many other factors such as linguistic factors. In order to assure the fairness, evenness and effectiveness in the real situation, we should endeavour to construct the suitable mathematical evaluation methods though I suspect there exists no perfect evaluation method, that is, no almighty method.

References


