Original

Ductility Criterion Applicable to The Prediction of
Sheet Formability and Central Bursting in Drawing
and Extrusion Under Plane Strain

by

Hitoshi Moritoki† and Eiki Okuyama†

ABSTRACT

General concept of criterion for ductility is presented, and its applications
to the prediction for cracking are shown: one is the formability in sheet metal
forming as an example of plane stress problem, and the other is the central
bursting in drawing and extrusion as an example of cracking in bulk speci-
mens. Here, the criterion is examined with respect to the collapse of the
unique solution. In general, there are two cases bringing about multiplicity:
one is statically admissible multiplicity and the other is kinematically admissi-
able multiplicity. After the appearance of statical instability, the strain path
cannot be controlled as freely as would be preferred. Ultimately, localized
necking occurs when the process satisfies the condition of kinematical instabil-
ity, the modes of which are consistent with the two modes required from the
condition permitting strain rate discontinuity under the continuity of velocity.
The general treatment about these criteria and the mode of planes permitting
strain rate discontinuity is discussed. Next, two examples for application of
the proposed criterion are shown: one is the prediction for the formability of
sheet metal, deforming according to the assumed process proceeding from
statical instability to kinematical one, and another shows the die angle reduc-
tion combination on which a chevron crack occurs in drawing and extrusion
under plane strain, where statical instability coincides with kinematical one.
These forming limits predicted are compared with several experimental results
published in the literature, and they are in very good agreement.

Key Words: Metal forming, Ductile Fracture, Sheet Formability, Chevron
Crack

1. Introduction

1.1 Prelude

It is remarkably common among ductile solids that when they are deformed sufficiently into the

Received June 30, 1992
† Department of Mechanical Engineering, Mining College, Akita University
1-1 Gakuen-cho, Tegata, Akita 010 Japan.
plastic range, deformation zone becomes gradually concentrated, and finally it shrinks to a plane on which cracking occurs. It is very important to predict how hard deformation metal can suffer without cracking. So, the prediction of forming limit has received much attention over the last two decades, and has been performed on numerous approaches, which are classified roughly into following two methods: one is based on plastic instability, and another is involved in the process proceeding with void nucleation, growth, and coalescence. The former is almost applied to sheet metal formability with some exception in upsetting, and the latter is used for the estimation of ductility in bulk materials, because the method of the estimation by means of plastic instability has not been sufficiently established for them, and on the other hand the phenomenon of void growth such as in a cylindrical tensile specimen has been observed.

Here, a criterion widely applicable to plastic instability is presented, which can be applied to predict both the ductility of sheet metals and that of bulk specimens.

1. 2 Sheet Metal Formability

Following general consideration with respect to the criterion for ductility, sheet formability will be dealt with. The formability of sheet metals is defined as their ability to be deformed into complex shapes without cracking. In general, formability depends on the formed shape, the forming conditions, the mechanical properties of the materials, and so on. When the material and the shape to be formed are given, it is very important for engineers to know what kind of forming conditions must be employed in order to secure good formability.

Swift\(^1\) and Hill\(^2\) are the first to have tried to predict formability of sheet metals in relation to plastic instability. Swift’s criterion is known as diffuse necking criterion and Hill’s is localized necking criterion. When a process is approaching its forming limit, most of the region other than necking part ceases plastic deformation, and suffers only elastic deformation. Then, the region which plastic deformation is still going on decreases in size and becomes local. Therefore, the localized necking is considered to correspond to the beginning of cracking. However, since the strain rate in minor principal direction must be negative for the application of Hill’s criterion, the formability of sheet metals cannot be evaluated under biaxial extension. Marciniak & Kuczynski\(^3\) postulated an initial inhomogeneity in the form of a little thin and narrow band region in a sheet, and when the deformation of sheet becomes concentrated in that thin band, it is assumed that the sheet is at the limit of forming. However, it is unreasonable that the forming limits should depend on the degree of the initial thinning. Subsequently, Stören & Rice\(^4\) supposed that localized necking occurs when multi-velocities become possible across a narrow band region at the common traction rate. However, they did not investigate whether or not a unique solution can be assured before the state of multi-velocities is realized, and whether or not the necking planes can take any mode. Moreover, reasonable forming limits cannot be obtained in the flow theory with the Mises’ yield criterion. Gotoh\(^5\) obtained forming limits using the Stören’s method with his own yield criterion, and tried to compare them with experimental data. Itoh\(^6\) explained the relationship between constitutive equation and plastic instability.

Thus, since the forming limit in press forming is concerned strongly with the instability phenomena, it is attempted to predicted the forming limit from the point of view of collapse for uniqueness of solution, by considering the full process including a normal process able to control strain path and an uncontrollable process from the occurrence of statical instability until kinematical
instability. It is shown that the necking planes of localized necking in sheet forming cannot have any other modes than either of two modes prescribed at kinematical criterion. The criteria due to statical and kinematical multiplicities are compared with the diffuse necking of Swift and the localized necking of Hill, and their differences are pointed out on the relation between assumptions and results obtained. Subsequently, forming limit curves are evaluated for several strain paths.

1.3 Central Bursting in Drawing and Extrusion

Finally, the occurrence of central bursting will be considered. Central bursts are internal defects, sometimes called chevron cracks, occasionally encountered in cold drawing and extrusion. Usually, it is very difficult to detect them by surface inspection of manufactured articles, and so the central bursting is very troublesome in forging processes. In automobile industries producing a large number of stepped shafts through cold forging, it is very important to understand how to prevent the occurrence of chevron cracks.

So far, many experimental studies have been performed on the influence of manufacturing conditions on central bursting, and some qualitative understanding has been obtained in detail (Jennison, Orbegozo). On the other hand, analytical studies have been performed by Avitzur and Avitzur et al., who postulate kinematically admissible velocity fields in a normal process and a process with central bursting respectively, and estimates plastic deformation energies based on these velocity field. They assume that central bursting occurs when the energy in a process with central burst is not larger than the energy in a normal process. These analyses result in good quantitative agreement with experimental data.

Even though manufacturing is done under a severe condition, a chevron crack does not always occur under the same conditions as it sometimes occurs, and the probability of cracking, in some cases, is less than one percent. This fact suggests that imperfections (voids or inclusions) dispersed in the matrices play the role as the starting point of cracking. Then, some researches have been performed from the viewpoint of such imperfections (Oh et al., Ayada et al.), and a correlation is observed between oxygen density in a copper specimen and occurrence of cracking (Tanaka et al.).

Thus, as mentioned above, numerous approaches which tried to predict the occurrence of central bursting from the viewpoint of imperfections have been published in the literature. However, we believe that the criterion for the collapse of unique solution has the ability to predict the ductility in the deformation processes of bulk materials. Then, the prediction for central bursting in drawing or extrusion will be discussed here using the proposed method. These formings are, in general, under axi-symmetrical deformation, but, in this paper, the formings under plane strain are considered, because the slip line method can be then applied, and it assures comparatively high accuracy for the evaluation of the stress state.

2. General Treatment of the Criterion for Instability and the Modes of Necking Plane

2.1 Collapses for Uniqueness

For rigid-plastic materials that plastic potential coincides with the yield function, Hill gives a sufficient condition for the uniqueness of solutions as follows:

\[ \int_V \left( \frac{2}{3} h \Delta \dot{\varepsilon}_{ij} \Delta \dot{\varepsilon}_{ij} - \sigma_{ij} \Delta \dot{u}_{ij} \right) dV > 0, \]  

(1)

where \( \sigma_{ij}, \dot{\varepsilon}_{ij}, \dot{u}_i \) are the components of true stress, Lagrangian strain rate, and velocity, and \( h \) is
the gradient of stress-strain curve. A dot represents the time derivative, and Δ the difference between any two multi-solutions. Eqn (1) starts from the following condition for multiplicity:

$$\int F_i \Delta \hat{\mathbf{u}}_i \, dS = \int \Delta \mathbf{S}_{ij} \Delta \hat{u}_{j,i} \, dV = 0,$$

where $F_i$ is the component of common surface traction, and $s_{ij}$ are of the nominal stress. Let the coordinate axes take the directions of principal axes. Then the vanishing of eqn (2) means

$$\Delta \hat{\mathbf{S}}_i \Delta \hat{\mathbf{u}}_{i,j} = \Delta \hat{\mathbf{S}}_i \Delta \hat{\mathbf{e}}_i = 0,$$

where the tensor components with only one subscript are referred to those with respect to principal directions. From the definition of nominal stress

$$\hat{\mathbf{s}}_i = \sigma_i - \sigma_i \hat{\mathbf{e}}_i, \quad \Delta \hat{\mathbf{s}}_i = \Delta \sigma_i - \sigma_i \Delta \hat{\mathbf{e}}_i.$$  

Here, we get started with eqn (3) as the condition for the collapses of uniqueness.

Let the direction of subscript 1 take the direction of the maximum principal stress. When multiple solutions have just been realized, it would be able to put $\Delta \hat{\mathbf{e}}_i \neq 0$. The two conditions $\Delta \hat{\mathbf{e}}_1 = 0$ and $\Delta \hat{\mathbf{e}}_3 = 0$ cannot be set simultaneously owing to volume constancy. Then, eqn (3) gives the following three cases:

(a) $\Delta \hat{\mathbf{s}}_1 = 0, \quad \Delta \hat{\mathbf{s}}_2 = 0, \quad \Delta \hat{\mathbf{s}}_3 = 0$

(b) $\Delta \hat{\mathbf{s}}_2 = 0, \quad \Delta \hat{\mathbf{e}}_1 = 0, \quad \Delta \hat{\mathbf{s}}_3 = 0$

(c) $\Delta \hat{\mathbf{s}}_3 = 0, \quad \Delta \hat{\mathbf{s}}_2 = 0, \quad \Delta \hat{\mathbf{e}}_2 = 0$.

Since direction 1 is taken as that of maximum elongation, cases (b) and (c) have the same relations when directions 2 and 3 are exchanged. Therefore, cases (a) and (b) will be mainly discussed. Case (a) is a statically admissible condition for multiplicity, and the phenomenon based on it appears over a wide region; whilst cases (b) and (c) are kinematically admissible conditions, where the phenomenon is local.

In the present analysis, the Mises' yield condition and the Levy-Mises constitutive relations are used. Then, strain rates are related with stress rates as follows:

$$\dot{\varepsilon}_i = a_{ij} \sigma_j, \quad a_{ij} = \frac{9}{4n} \frac{\varepsilon_i \sigma_j}{\sigma^3_y},$$

where $\sigma_i$ is deviatoric stress, and the equivalent stress and the equivalent strain rate are

$$\sigma_y = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2},$$

$$\varepsilon_y = \frac{2}{\sqrt{3}} \sqrt{\dot{\varepsilon}_1 \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \dot{\varepsilon}_2 + \dot{\varepsilon}_3 \dot{\varepsilon}_3}.$$  

In the equation of $\varepsilon_y$, the relation of volume constancy is used in order to eliminate $\dot{\varepsilon}_3$. Equivalent strain $\varepsilon_y$ is defined as the integral of $d \varepsilon_y$. The material is assumed to strain-harden with the power law

$$\sigma_y = \sigma^{*} \dot{\varepsilon}_y^n,$$

where $\sigma^{*}$ and $n$ are material constants. The stress ratio, the strain rate ratio and the average normal stress divided by equivalent stress are defined as

$$\alpha = \frac{\sigma_2}{\sigma_1}, \quad \beta = \frac{\sigma_1}{\sigma_2}, \quad \gamma = \frac{\varepsilon_2}{\varepsilon_1}, \quad \tilde{\sigma} = \frac{1}{3 \sigma_y} (\sigma_1 + \sigma_2 + \sigma_3).$$

Eqn (6) gives the following relationship:
so that eqn (7) becomes

\begin{align}
\gamma &= \frac{2a - \beta - 1}{2 - a - \beta}, \\
\sigma_s &= \sigma, \frac{2 - a - \beta}{\sqrt{3}} \sqrt{1 + \gamma + \gamma^2} \\
\varepsilon_s &= \varepsilon, \frac{2}{\sqrt{3}} \sqrt{1 + \gamma + \gamma^2}.
\end{align}

Also, the average normal stress is

\begin{equation}
\tilde{\sigma} = \frac{1}{C} \frac{1 + a + \beta}{2 - a - \beta}, \quad C = \sqrt{3} \sqrt{1 + \gamma + \gamma^2}.
\end{equation}

Combining them with eqn (10)

\begin{align}
\alpha &= \gamma + C \tilde{\sigma} \\
\beta &= \frac{-1 - \gamma + C \tilde{\sigma}}{1 + C \tilde{\sigma}}
\end{align}

are obtained.

In order to distinguish each of them among multiple solutions, any two solutions being compared are specified by A or B. Subscript A or subscript B attached to physical quantities indicates that they belong to the corresponding solutions. Two cases occur: one is that both A and B are on plastic processes, and the other is that one of these processes is plastic and the other, denoted by B, is rigid. They will now be treated successively.

2. 1A Statically admissible condition for multiplicity

The multiplicity assigned in eqn (5a), that is, $\Delta s_1 = 0$, $\Delta s_2 = 0$, and $\Delta s_3 = 0$ is considered here.

2. 1A-1 Both A and B are on plastic processes

By substituting eqn (6) into eqn (4), the condition for the statical multiplicity becomes

\begin{equation}
(\delta_{ij} - \sigma_i, a_{ij}) \Delta \dot{\sigma}_j = 0.
\end{equation}

In order that non-zero $\dot{\varepsilon}_i$ exist, the determinant of the coefficient matrix in these equations must vanish. Hence, we get

\begin{equation}
\sigma_i a_{ij} = 1.
\end{equation}

Using eqn (6) and eqn (11a), the criterion

\begin{equation}
\varepsilon_{sd} = \frac{4}{3 \sqrt{3}} \frac{(2 - a - \beta)(\sqrt{1 + \gamma + \gamma^2})^3}{1 + a \gamma^2 + \beta (1 + \gamma)^3}
\end{equation}

is obtained, where subscript D is used to denote the values belonging to this criterion.

2. 1A-2 Solution B is on a rigid or a neutral process

On the assumption that solution B is not plastic, $\varepsilon_{ib} = 0$. hence,

\begin{equation}
s_{ib} = (\sigma_i a_{ij} a_{ij}, \sigma_i a_{ij} a_{ij}), \quad s_{ib} = (\sigma_i a_{ij} a_{ij}).
\end{equation}

Let $g_i$ take the base vectors of coordinate axes $\sigma_i$ in stress space. Using the statically admissible condition (eqn (5a)) and the above relations, the stress increment on B can be described as follows:

\begin{equation}
\dot{\sigma}_s = \sigma_{ie} g_i = s_{ib} g_i.
\end{equation}

The normal vector on yield surface is defined as,

\begin{equation}
P = \nabla \sigma_s = \frac{3}{2 \sigma_s} \sigma_s g_i.
\end{equation}
At the time when the multiplicity has just appeared, the restriction
\[ \dot{\sigma}_b \cdot P = \frac{3}{2 \sigma_e} \dot{\sigma}_i \dot{s}_{ik} \leq 0 \] (20)
must be imposed in order to fulfill the assumption that the process B is neutral or rigid. When B is neutral, the above equation vanishes, and the relation
\[ \dot{\sigma}_i \dot{s}_{ik} = \dot{\sigma}'_i (\dot{\sigma}_{ij} - \sigma_{ij} \dot{a}_{ij}) \dot{\sigma}'_{jk} = \dot{\sigma}'_i (1 - \sigma_{ij} \dot{a}_{ij}) \dot{\sigma}'_{jk} = 0 \] (21)
is obtained, using eqn (6a). Since normally \( \dot{\sigma}'_{jk} = 0 \), eqn (15) must hold in order to satisfy eqn (21). This leads to the same criterion, that is, eqn (16) with both solution being plastic.

2. 1B. Kinematically admissible criterion for (5b)

The multiplicity assigned in eqn (5b), that is, \( \Delta \dot{s}_{ij} = 0 \), \( \Delta \dot{\varepsilon}_{ij} = 0 \), and \( \Delta s_{ij} = 0 \) is considered here.

2. 1B-1 Both A and B are on plastic processes

Using eqns (4) and (6), eqn (5b) becomes
\[ (\dot{\sigma}_{ij} - \sigma_{ij} \dot{a}_{ij}) \Delta \dot{s}_{ij} = \dot{\sigma}'_{ij} (1 - \sigma_{ij} \dot{a}_{ij}) \dot{\sigma}'_{jk} = 0 \] (22)
The existence of non-zero \( \Delta \dot{s}_{ij} \), requires that
\[ a_{ij} = 0. \] (23)
Form the definition of \( a_{ij} \) in eqn (6), this means
\[ \dot{\sigma}'_{ij} = 0, \quad \dot{\varepsilon}_{ij} = 0, \] (24)
that is, plane strain rate deformation with respect to direction 2. Moreover,
\[ a_{ij} = 0, \quad a_{ij} = 0 \] (25)
must hold from eqn (24a). Therefore, eqn (22) becomes
\[ (\dot{\sigma}_{ij} - \sigma_{ij} \dot{a}_{ij}) \Delta \dot{s}_{ij} = \dot{\sigma}'_{ij} (1 - \sigma_{ij} \dot{a}_{ij}) \dot{\sigma}'_{jk} = 0 \] (26)
From the requirement of non-zero \( \Delta \dot{s}_{ij} \),
\[ \sigma_{ij} \dot{a}_{ij} = 1 \] (27)
is derived. Eqn (10) gives
\[ \gamma = 0, \quad \alpha = \frac{1}{2} (1 + \beta) \] (28)
under plane strain rate deformation assigned in eqn (24), and eqn (27) becomes
\[ \frac{\varepsilon_{\text{pl}}}{n} = 2 \frac{1 - \beta}{\sqrt{3} (1 + \beta)}, \] (29)
which is the criterion for kinematical multiplicity, and subscript L refers to it.

2. 1B-2 Solution B is on a neutral or a rigid process

Since process B is rigid, \( \dot{\varepsilon}_{\text{in}} = 0 \). As \( \Delta \dot{\varepsilon}_{ij} = 0 \), \( \dot{\varepsilon}_{\text{in}} = 0 \) must also be in the plastic process. Therefore, eqns (24) and (25) hold true. The same treatment is done as in the case of statical collapses for uniqueness, and eqn (22) is obtained, where \( i = 2 \) must be omitted owing to eqn (24). This implies eqn (27), and then the same criterion as in eqn (29) is obtained.

2. 1C Kinematically admissible criterion for eqn (5c)

The criterion for eqn (5c) is derived as
\[ \frac{\varepsilon_{\text{pl}}}{n} = 2 \frac{1 - \alpha}{\sqrt{3} (1 + \alpha)}, \] (30)
where the requirements
are used of plane strain rate deformation in direction 3.

2.2 First Appearance of Instability

In the foregoing have been discussed both statical and kinematical conditions required for the collapse of the unique solution. Localized necking based on the kinematical collapse cannot occur unless equivalent strain is in excess of the value in eqn (29), or eqn (30), even if the process is under plane strain rate. Also, even if equivalent strain has larger value than that in eqn (29), or (30), localized necking cannot occur unless the process is under plane strain rate. Generally, it is very rare that the state satisfying the condition for localized necking occurs on strain path before the condition statical collapse is encountered. Hence, the value of eqn (29) or eqn (30) cannot be applied directly as the limit for localized necking in general forming processes.

Therefore, in any forming processes, attention must be paid to the appearance of the state satisfying statical criterion of eqn (16). After that, the strain path cannot be controlled as freely as would be preferred, and ultimately, localized necking occurs when the process satisfies either kinematical criterion of eqn (29) or eqn (30), the mode of which is fixed, as shown in eqn (28) or (31) respectively. These modes are consistent with the two modes required from the conditions permitting strain rate discontinuity which will be discussed in the next section.

2.3 Mode of Localized Necking

The conditions required kinematically for the mode of the necking plane are discussed here. Multiple strain rates must be able to exist on the necking plane. Let present time be the time when the conditions for multiplicity are satisfied, and assume that two kinds of strain rates occur on the necking plane N, passing at the point O. Then, the discontinuity of strain rate occurs across the plane N, the unit normal vector of which indicates the mode of necking. In order to distinguish the two sides divided by plane N, one side is called A and the other is called B.

The materials considered can be permitted to be unisotropic. But the principal directions for stress must be consistent with those for strain rate, and they are not allowed to rotate. Let coordinate axes x take the directions of principal axes, and let e be their base vectors. Hence, x are mutually perpendicular from this definition. σ is taken as the maximum stress of σ.

Let e take the unit vector normal to the plane N, and let e and e be unit vectors involved in it, being mutually perpendicular and defined as

\[
\begin{align*}
\bar{e}_1 &= \cos \phi \cos \psi e_1 + \cos \phi \sin \psi e_2 + \sin \phi e_3, \\
\bar{e}_2 &= -\sin \phi e_1 + \cos \phi e_3, \\
\bar{e}_3 &= -\sin \phi \cos \psi e_1 - \sin \psi \sin \phi e_2 + \cos \phi e_3,
\end{align*}
\]

where e is involved in the plane x, x. Fig. 1 shows the relationship of e with x.

Plastic deformation is assumed to take place here. The strain rate tensor ̂E is defined as

\[
\gamma = -1, \quad \beta = \frac{1}{2} (1 + \alpha)
\]
The difference of strain rate tensors between sides A and B is described as $\Delta \dot{E}$, where

$$\Delta \dot{E} = \dot{E}_B - \dot{E}_A = \Delta \dot{e}_{ij} e_i e_j = \Delta \dot{e}_{rs} e_r e_s$$

$$\Delta \dot{e}_{ij} = e_{ijB} - e_{ijA}, \Delta \dot{e}_{rs} = e_{rsB} - e_{rsA}$$

in which $\Delta$ denotes the difference of some physical quantities between the A and B, and $\dot{e}$ relates to the direction with base vector $e_i$.

Even if the discontinuity for strain rate field is permitted across plane N, their components parallel to N cannot take different values across N, because velocity must continue. Moreover, the normal component perpendicular to N cannot take a different value also, from the condition of incompressibility and the vanishing of the normal components of strain rate differences in the plane parallel to N. Therefore,

$$\Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0$$

must hold. Hence, amongst $\Delta \dot{e}_{nn}$, only $\Delta \dot{e}_{nn}$ and $\Delta \dot{e}_{nn}$ do not vanish. Substituting eqn (32) into eqn (34), and comparing the coefficients of dyadics, the strain rate components with respect to $x_i$ are obtained as

$$\Delta \dot{e}_{11} = -\Delta \dot{e}_{nn} \cos \phi \sin 2\phi - \Delta \dot{e}_{nn} \sin 2\phi \cos^2 \phi$$

$$\Delta \dot{e}_{22} = \Delta \dot{e}_{nn} \cos \phi \sin 2\phi - \Delta \dot{e}_{nn} \sin 2\phi \cos^2 \phi$$

$$\Delta \dot{e}_{33} = \Delta \dot{e}_{nn} \sin 2\phi$$

$$\Delta \dot{e}_{nn} = \Delta \dot{e}_{nn} \cos \phi \sin 2\phi - \frac{1}{2} \Delta \dot{e}_{nn} \sin 2\phi \cos \phi = 0$$

$$\Delta \dot{e}_{nn} = \Delta \dot{e}_{nn} \sin \phi \cos \phi + \Delta \dot{e}_{nn} \cos 2\phi \sin \phi = 0$$

$$\Delta \dot{e}_{nn} = -\Delta \dot{e}_{nn} \sin \phi \sin \phi + \Delta \dot{e}_{nn} \cos 2\phi \cos \phi = 0$$

Since $x_i$ are taken to the principal directions, all shear components in eqn (37) must vanish. In the second and third equations of eqns (37), if $\Delta \dot{e}_{nn} \sin \phi$ and $\Delta \dot{e}_{nn} \cos 2\phi$ are regarded as variables, the determinant of this coefficient matrix is unity, i.e., it is not zero. Hence, in order that these two equations in eqn (37) hold true, these variables must be zero. Combining this fact and first equation in eqn (37), the following two modes are obtained:

$$\Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0$$

$$\Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0, \quad \Delta \dot{e}_{nn} = 0$$

From eqn (36), the defferences of normal strain rates for each modes are

$$\Delta \dot{e}_{11} = -\Delta \dot{e}_{nn}, \quad \Delta \dot{e}_{22} = 0, \quad \Delta \dot{e}_{33} = \Delta \dot{e}_{nn}$$

$$\Delta \dot{e}_{nn} = -\Delta \dot{e}_{nn}, \quad \Delta \dot{e}_{nn} = \Delta \dot{e}_{nn}, \quad \Delta \dot{e}_{nn} = 0$$

As mentioned above, in order to satisfy the conformity of deformation, the plane of localized necking which is the discontinuous plane of strain rate field cannot take any other modes except for two modes: one is $\phi = \pi / 4$, $\phi = 0$, and the other is $\phi = 0$, $\phi = \pi / 4$. When the sheet surface is the $x_1 x_2$ plane, for the former the intersections of necking plane with sheet surface is in the direction of $x_3$, and for the latter it is in the direction divided into equal angles between $x_1$ and $x_2$. Then, since these intersections are perpendicular and oblique with respect to $x_3$ axis, they are called perpendicular and oblique modes, respectively. Consequently, these two modes are summarized as:
(a) Perpendicular mode (mode P):
\[
\phi = \frac{\pi}{4}, \quad \phi = 0, \quad \Delta \dot{\varepsilon}_n = \text{non-zero}
\]
\[
\Delta \dot{\varepsilon}_1 = -\Delta \dot{\varepsilon}_n, \quad \Delta \dot{\varepsilon}_2 = 0, \quad \Delta \dot{\varepsilon}_3 = \Delta \dot{\varepsilon}_n, \quad (40)
\]
(b) Oblique mode (mode O):
\[
\phi = 0, \quad \phi = \frac{\pi}{4}, \quad \Delta \dot{\varepsilon}_{12} = \text{non-zero}
\]
\[
\Delta \dot{\varepsilon}_1 = -\Delta \dot{\varepsilon}_{12}, \quad \Delta \dot{\varepsilon}_2 = \Delta \dot{\varepsilon}_{12}, \quad \Delta \dot{\varepsilon}_3 = 0 \quad (41)
\]

3. Sheet Metal Formability

3.1 The Criterion under plane stress

In the discussion for the formability of sheet metals, let \( \varepsilon_3 \) coincide with the normal vector of sheet surface. Thus, in general, the plane stress condition \( \sigma_3 = 0 \) is fulfilled approximately. So \( \Delta \dot{\varepsilon}_3 = 0 \) cannot be usually realized. Hence, the localized necking in sheet forming takes only perpendicular mode with \( \Delta \dot{\varepsilon}_2 = 0 \). This mode is consistent with the mode, \( \gamma = 0 \) (eqn (28a)) for the kinematically admissible multiplicity of solutions. Then, since \( \beta = 0 \), eqn (10) becomes
\[
\alpha = \frac{1 + 2\gamma}{2 + \gamma}. \quad (42)
\]
From eqn (11), equivalent stress and strain rate are
\[
\sigma_\varepsilon = \sigma_1 - \sigma_3 \sigma_3 + \sigma_1^3 = \sigma_1 \sqrt{3} \frac{\sqrt{1 + \gamma + \gamma^3}}{2 + \gamma} \quad (43)
\]
\[
\dot{\varepsilon}_\varepsilon = \frac{2}{\sqrt{3}} \sqrt{1 + \gamma + \gamma^3} \quad (43)
\]
Also, eqns (16) and (29) become
\[
\frac{\varepsilon_{\varepsilon \theta}}{n} = \frac{4}{\sqrt{3}} \frac{(\sqrt{1 + \gamma + \gamma^3})^2}{(1 + \gamma)(2 - \gamma + 2\gamma^2)} \quad (44)
\]
\[
\frac{\varepsilon_{\varepsilon \phi}}{n} = \frac{2}{\sqrt{3}}. \quad (45)
\]
These are the criteria used for evaluating sheet formability.

3.2 Lack of Reasonability in Swift’s Diffuse Necking and Hill’s Localized Necking

3.2-1 Swift’s Diffuse Necking

The limit strain in eqn (44) due to statically admissible instability is just the same as the value of diffuse necking suggested by Swift\(^1\), but the basic concepts are quite different between this analysis and that of Swift. He started from the assumption that nominal stresses take stationary values simultaneously, but we assume that nominal stress rates do not differ for multiplicity. The difference between these analyses is very

![Fig. 2 Nominal stress rates at diffuse necking (linear path).](image-url)
important, as will be shown in the following.

The rates of nominal stresses are

\[
\begin{align*}
\frac{1}{s_0} \frac{ds_1}{d\varepsilon_1} &= \frac{1}{\sigma_1} \frac{d\sigma_1}{d\varepsilon_1} - 1, \\
\frac{1}{s_2} \frac{ds_2}{d\varepsilon_1} &= \frac{1}{\sigma_2} \frac{d\sigma_2}{d\varepsilon_1} - \gamma.
\end{align*}
\]

(46)

Since from eqns (42), (43) and (8)

\[
\frac{d\sigma_1}{d\varepsilon_1} = \frac{3}{4} \frac{1 + \gamma \frac{\dot{\varepsilon}}{\varepsilon}}{1 + \gamma + \gamma \frac{\dot{\varepsilon}}{\varepsilon}} \frac{d\sigma_1}{d\varepsilon_1} = \sigma_0 \frac{\dot{\varepsilon}}{\varepsilon},
\]

then

\[
\begin{align*}
\frac{1}{s_0} \frac{ds_1}{d\varepsilon_1} &= \frac{4}{\sqrt{3}} \frac{n}{\varepsilon_0} \frac{(\sqrt{1 + \gamma + \gamma \frac{\dot{\varepsilon}}{\varepsilon}})^2}{(2 + \gamma)(1 + \gamma \frac{\dot{\varepsilon}}{\varepsilon})} - 1, \\
\frac{1}{s_2} \frac{ds_2}{d\varepsilon_1} &= \frac{4}{\sqrt{3}} \frac{n}{\varepsilon_0} \frac{\frac{\dot{\varepsilon}}{\varepsilon}}{(2 + \gamma)(1 + \gamma \frac{\dot{\varepsilon}}{\varepsilon})} - \gamma,
\end{align*}
\]

(47)

where

\[
\varepsilon = \frac{\sigma_2}{\sigma_1}.
\]

(49)

As an example, let a linear strain path be taken. As \( \dot{\varepsilon} = \alpha \), the gradients of \( s_i \) in eqn (48) at the deformation state given by eqn (44) have the values shown in Fig. 2. This proves that the nominal stresses do not take stationary values simultaneously except for the case where \( \gamma = 1 \).

3. 2-2 Hill's Localized Necking

The reverse transformation in eqn (32) gives

\[
\begin{align*}
e_1 &= \cos \phi \cos \phi \hat{e}_1 - \sin \phi \hat{e}_2 - \sin \phi \cos \phi \hat{e}_3, \\
e_2 &= \cos \phi \sin \phi \hat{e}_1 + \cos \phi \hat{e}_2 - \sin \phi \sin \phi \hat{e}_3, \\
e_3 &= \sin \phi \hat{e}_1 + \cos \phi \hat{e}_3.
\end{align*}
\]

(50)

Substituting eqn (50) into eqn (34), and comparing the coefficients of dyadics with respect to \( \Delta \dot{\varepsilon}_1 \), \( \Delta \dot{\varepsilon}_2 \) and

\[
\Delta \dot{\varepsilon}_3 = (-\Delta \dot{\varepsilon}_1 - \Delta \dot{\varepsilon}_2 - \Delta \dot{\varepsilon}_3): \\
\Delta \dot{\varepsilon}_n = \Delta \dot{\varepsilon}_1 \sin^2 \phi + \Delta \dot{\varepsilon}_2 \cos^2 \phi \\
\Delta \dot{\varepsilon}_m = \Delta \dot{\varepsilon}_1 (\sin^2 \phi \cos^2 \phi - \cos^2 \phi) + \Delta \dot{\varepsilon}_2 (\sin^2 \phi \sin^2 \phi - \cos^2 \phi) \\
\Delta \dot{\varepsilon}_n = (\Delta \dot{\varepsilon}_1 - \Delta \dot{\varepsilon}_2) \sin \phi \sin \phi \cos \phi \\
\Delta \dot{\varepsilon}_n = \Delta \dot{\varepsilon}_1 \cos^2 \phi \sin^2 \phi + \Delta \dot{\varepsilon}_2 \cos^2 \phi \sin^2 \phi - \sin^2 \phi).
\]

From the viewpoint that localized necking occurs along the direction of zero extension rate on the plane of sheet in the tensile test of sheet specimens, Hill gives the following relations for the necking mode,

\[
\sin 2\phi = \frac{\sqrt{8}}{3}, \quad \cos 2\phi = \frac{1}{3}, \quad \Delta \dot{\varepsilon}_2 = -\frac{1}{2} \Delta \dot{\varepsilon}_1. \quad (52)
\]

Under this condition, each of the components in eqn (51) becomes

\[
\Delta \dot{\varepsilon}_n = 0, \quad \Delta \dot{\varepsilon}_m = -\Delta \dot{\varepsilon}_n = -\frac{1}{2} \Delta \dot{\varepsilon}_1 \cos 2\phi, \quad \Delta \dot{\varepsilon}_n = \frac{1}{\sqrt{2}} \Delta \dot{\varepsilon}_1 \sin \phi, \quad (53)
\]

which do not vanish simultaneously for any \( \phi \). Therefore, Hill's localized necking does not satisfy all of the requirements in eqn (35). Thus, it cannot be said that Hill's necking is reasonable for the
conformity of velocity discontinuity.

3.3 Strain Path to the Forming Limit

The limit due to statical criterion for multiplicity is called here diffuse necking after Swift\(^1\), and that due to kinematical criterion is called localized necking after Hill\(^2\).

Sheet metals can take any strain paths which are forced to trace beforehand, until the deformation process reaches instability. First of all, it is necessary to predict which kind of instabilities arise first, statical or kinematical. Since the kinematical instability cannot occur at any strain rates except for pure plane strain rate, the instability encountered first, in general, will be of the statical type. When the deformation in some part of sheet reaches the strain assigned by eqn (44), the statically admissible collapse for uniqueness begins there. At that time multiple deformations are possible to arise. Subsequently it is not possible to artificially control the strain of the sheet as it follows the strain path prescribed beforehand. Under such circumstances, it may be possible that in some region where statical criterion is satisfied plastic deformation stops and the stress state becomes neutral or rigid: this constrains the deformation of the region where plastic deformation has been developing, and gradually makes the region narrow. As the constraint intensifies, the strain rate in the plastic region would not be able to take the value prescribed in the strain path. The strain path followed depends on the setup of experimental apparatus, the mechanical properties, the configuration of the sheet metals, and so on. Hence, it is impossible to predict beforehand the strain path followed after diffuse necking. However, the narrower becomes the region where plastic deformation has been continuing, the more the stress state approaches that of plane strain rate so as to be consistent with the constraint of the rigid part where plastic deformation has just stopped.

This is the state required in kinematical instability.

As mentioned above, the development of strain rate after diffuse necking cannot be predicted beforehand, so that for the present, following assumption is made. Let the change of the nominal stress in first direction, that is, the value in eqn (48a) at the diffuse necking, be \(g_c\). Using \(\gamma_D\) at diffuse necking, \(g_c\) is defined from

\[
g_c = g_c \gamma_D.\tag{50}
\]

After diffuse necking, the first nominal stress is assumed to change with

\[
\frac{1}{s_1} \frac{d s_1}{d \varepsilon_1} = g_c \gamma_D.\tag{55}
\]

This and eqn (48a) give the stress rate ratio

\[
\xi = \frac{1}{\gamma} \left\{ \frac{4}{\sqrt{3}} \frac{n}{\varepsilon_s} \left( \frac{1}{2 + \gamma} \right) \right\} = \frac{(\sqrt{1 + \gamma + \gamma^2})}{(2 + \gamma)(1 + \gamma)}, \tag{56}
\]

Hence, using eqns (42) and (43a), the rate of strain rate ratio is obtained as

\[
\gamma = \frac{a - \xi_s}{c_1 \xi_s - c_2 a} \varepsilon_s, \tag{57}
\]

which is evaluated using eqn (50) and the specified value of \(g_c\). Usually, the deformation after diffuse necking has the tendency decreasing \(\gamma\), finally reaching the state of \(\gamma = 0\), where \(\dot{s}_1 = 0 (= \dot{s}_B)\) from eqn (55). Then, one condition \(\Delta \dot{s}_1 = 0\) for the kinematical instability holds, since \(\gamma = 0\). The other condition \(\Delta s_0 = 0\) requires that \(s_{0t} = 0\). When process B is plastic, from eqns (4a) and (6a)
with $i = 1$

$$\frac{d\sigma_{1A}}{d\sigma_{1A}} = 1 - q - q \gamma \xi_A = 0,$$

$$\frac{d\sigma_{2B}}{d\sigma_{1A}} = (1 - q - q \gamma \xi_B) \chi = 0$$

hold, where

$$q = \frac{\sqrt{3}}{4} \frac{\epsilon}{n} \left( \frac{2 + \gamma}{\sqrt{1 + \gamma + \gamma^2}} \right)^3, \quad \chi = \frac{\dot{\sigma}_{1B}}{\dot{\sigma}_{1A}}.$$  \hspace{1cm} (58)

In general, since $\chi \neq 0$, then $\xi_A = \xi_B$. Therefore, both processes A and B follow the same strain path. However, the magnitudes of these stress rates are different. When process B is rigid, $\dot{\sigma}_{1B} = 0$ since $\dot{\sigma}_{2B} = 0$. Though $\dot{\sigma}_{2B} (= \dot{\sigma}_{1B})$ is indefinite, it is assured that the process B is neutral because $\gamma = 0$. Therefore, when the process reaches the limit state of $\gamma = 0$, the condition of localized necking can be satisfied.

### 3. 4 Forming Limit under Various Strain Path

#### 3. 4-1 Linear Strain Paths

The forming limits under linear strain paths are shown in Fig. 3, where the limits are compared with the experimental results by Hecker\textsuperscript{16).} Until the diffuse necking occurs, the deformation proceeds with prescribed linear strain paths, but after the occurrence of diffuse necking, the strain paths begin to deviate from prescribed paths, finally reaching the state of plane strain rate, where the localized necking occurs and the limit of forming is determined. In this analysis, it is assumed that the process from diffuse necking to localized necking proceeds according to eqn (55), as an example. However, it is not obvious how localization develop. Hence, in general, the development of the localization must be assumed, based on the phenomena resulting from the experimental procedure.

#### 3. 4-2 Curved Strain Paths

Under biaxial tension, the strain rate ratio $\gamma$ is taken between $-1/2$ and 1. Two kinds of strain paths are used, chaging between these limits. The first path is as follows: $\gamma$ takes $-1/2$ at the beginning of deformation, and increases toward unity with increasing deformation; the second path being opposite to this. The two paths are called Path 1 and Path 2, respectively. Two fixed parameters, $a$ and $b$, and a variable parameter, $t$, are used to describe the increasing or the decreasing of $\gamma$ with deformation. Thus

**Path 1:**

$$\frac{\epsilon_1}{n} = \frac{1}{\sqrt{2}} \left\{ \left(\frac{t}{a}\right)^b + \frac{4}{3} t \right\}, \quad \frac{\epsilon_2}{n} = \frac{1}{\sqrt{2}} \left\{ \left(\frac{t}{a}\right)^b - \frac{2}{3} t \right\}.$$  \hspace{1cm} (60)

**Path 2:**

$$\frac{\epsilon_1}{n} = \frac{1}{\sqrt{5}} \left\{ \frac{5}{3} t + 2 \left(\frac{t}{a}\right)^b \right\}, \quad \frac{\epsilon_2}{n} = \frac{1}{\sqrt{5}} \left\{ \frac{5}{3} t - \left(\frac{t}{a}\right)^b \right\}.$$  \hspace{1cm} (61)

The forming limits for Path 1 are shown in Fig. 4, and those for Path 2 in Fig. 5. Consider first the phenomenon on Path 1. If stress state is close to $= -1/2$ over the whole of the process, the limits are not much different from those for linear paths, but if it is close to $\gamma = 1$ at the diffuse necking, considerable deformation is induced on the process leading to localization. Hence,
Hitoshi MORITOKI and Eiki OKUYAMA

these limits are much larger than those in linear paths. Next, examine Path 2. If the stress states are close to $\gamma = -1/2$ at relatively early stage, these limits are a little larger than those for linear paths, but if the strain paths are taken for which the values of $\gamma$ are positive at diffuse necking, the equivalent strain at diffuse necking is already in excess of the value in eqn (45). In these circumstances, if the process leading to localization is specified with the use of eqn (55), the strain rate ratio has the tendency not to approach $\gamma = 0$, but to depart from it, and hence the absolute value of $\dot{\varepsilon}_i$ increases. Therefore, since the localization process making $\dot{\varepsilon}_i$ stationary cannot be realized in such strain paths, their forming limits are assumed to be referred to those at diffuse necking. Generally, Path 1 gives much larger limit strains than does Path 2, which suggests that the strain path should be taken close to an uniaxial tension at the beginning of deformation and close to a biaxial tension at the end of process, in order to obtain good formability. This tendency shows good agreement with the available experimental results (e.g., by Müschenborn & Sonne").

4. Central bursting in Drawing and Extrusion

4.1 Assumption in the Analysis

Here, drawing and extrusion are assumed to be under plane strain deformation in width direction. Coordinate axes $x_i$ ($i = 1, 2, 3$) are taken in the directions of length, thickness, and width respectively. On the central plane with respect to the thickness direction, the coordinate axes coincide with the principal ones, and eqn (31) holds true. The substitution of eqn (31) into eqn (16) results in the same expression as eqn (30). This means that the statical condition and the kinematical one for the collapse of uniqueness agree with each other under plane strain. Using eqn (13a) where $\gamma = -1$ is substituted,

$$\frac{\tilde{\sigma}}{n} = \frac{2}{3}$$

is obtained from eqn (30). Moreover, using eqn (11), the relation for limit strain can be written as

$$\frac{\tilde{\sigma}}{n} = \frac{1}{\sqrt{3}}.$$
4.2 Slip Line Analysis

The slip line field is assumed here such that the rigid regions at the entrance and exit parts meet at the straight line along the width direction on the central plane. Since the field in which the plastic region on the central plane has finite area is closer to uniform deformation than the field mentioned above, central bursting may be expected to occur mostly in the assumed field.

The slip line field is shown in Fig. 6. The analysis is performed by a method similar to that proposed by HILL in drawing (HILL18). AB (= 1) is the die surface, and the deformation is uniform in the triangular region OAB. Let the x axis be the tangent to the \( \beta_s \) line at point O, and y to \( \alpha_s \). Shear yield stress is \( k \), and the quantities with an upper bar denote the quantities divided by \( 2k \). The friction stress on the die surface is \( \tau_f \). Then, the angle OBA (= \( \mu \)) is represented by

\[
\mu = \frac{1}{2} \tan^{-1} \frac{\sqrt{1 - 4 \tau_f^2}}{2 \tau_f}
\]

Points A and B are singular points. Let the angles OAA\(_a\) and OBB\(_B\) in the fan regions at points A and B be \( \phi_{\beta a} \) and \( \phi_{\alpha b} \) respectively. The radii of curvature, for the \( \alpha_s \) and \( \beta_s \) lines in the curved rectangle assigned by \( \phi_s \) and \( \phi_s \) in \( 0 \leq \phi_s \leq \phi_{\alpha b} \) and \( 0 \leq \phi_s \leq \phi_{\alpha b} \), are represented by

\[
R_{\alpha} = R_{\beta} I_{1}(2 \sqrt{\phi_s \phi_{\beta}}) + R_{\beta} \phi_{\alpha} I_{1}(2 \sqrt{\phi_s \phi_{\alpha}})
\]

\[
R_{\beta} = R_{\beta} I_{1}(2 \sqrt{\phi_s \phi_{\beta}}) + R_{\alpha} \phi_{\beta} I_{1}(2 \sqrt{\phi_s \phi_{\beta}})
\]

where \( I_{1} \) and \( I_{1} \) are modified Bessel functions. The angle between the x axis and the \( \beta_s \) line

\[
\phi = \phi_{\beta} - \phi_s, \quad \phi_{\beta} = \phi_{\alpha b} - \phi_{\beta b},
\]

and the relations

\[
\theta_{c} = \frac{\pi}{4} - \phi_{c} = \mu - \alpha_{D}, \quad \phi = \phi + \theta_{c}
\]

hold, where \( \alpha_{D} \) is the die-half angle, and \( \theta_{c} \) is the angle between the central plane and x axis. The subscript c refers to the values on the central plane. \( ds_{s} \) and \( ds_{s} \) are infinitesimal line elements of the \( \alpha_s \) and \( \beta_s \) lines respectively. Along \( \alpha_s \):

\[
x_{a} = R_{\alpha} \sin \phi_{\beta}, \quad y_{a} = R_{\beta} (1 - \cos \phi_{\beta})
\]

\[
ds_{s} = R_{\alpha} d\phi, \quad dx = -ds_{a} \sin \phi, \quad dy = ds_{a} \cos \phi.
\]

Along \( \beta_s \):

\[
x_{a} = -R_{\alpha} (1 - \cos \phi_{\alpha}), \quad y_{a} = R_{\alpha} \sin \phi_{\alpha}
\]

\[
ds_{s} = R_{\alpha} d\phi, \quad dx = ds_{s} \cos \phi, \quad dy = ds_{s} \sin \phi.
\]
Average pressure is
\[ \overline{p} = -\frac{1}{4k} (\sigma_1 + \sigma_2). \tag{70} \]

Along \( \alpha_s \) and \( \beta_s \), average pressure changes satisfying the well-known relation
\[ \alpha_s : d \overline{p} + d \phi = 0, \quad \beta_s : d \overline{p} - d \phi = 0. \tag{71} \]

**Drawing:** The following relation holds along the \( \alpha_s \) line
\[ \int_C^C \left( \frac{1}{2} \sin \phi - \overline{p} \cos \phi \right) ds = \frac{1}{2} \frac{t_s}{h_u}, \tag{72} \]
where \( t_s \) is backward tension.

**Extrusion:** Along the \( \beta_s \) line
\[ \int_C^C \left( \frac{1}{2} \cos \phi - \overline{p} \sin \phi \right) ds = \frac{1}{2} \frac{c_f}{h_u}, \tag{73} \]
where \( c_f \) is forward compression, and \( h_u \) and \( h_i \) denote the thicknesses before and after deformation. Rectangular coordinate axes \( x' \) and \( y' \) are set up with the \( x' \) axis being parallel to central plane.

The \( y' \) coordinate of point C is
\[ y' = x_s \sin \theta_s + y \cos \theta_s. \tag{74} \]

Then, the thicknesses before and after deformation are represented by
\[ h_u = 2 (y' + R_u \cos \theta_s), \]
\[ h_i = 2 (y' + R_u \sin \theta_s). \tag{75} \]

The area reduction and the tensile strain of the plate are
\[ r = 1 - \frac{h_u}{h_i}, \quad \varepsilon_1 = -\ln (1 - r). \tag{76} \]

On calculating the slip line field, \( \alpha_D \), and \( \overline{r} \) (or \( \mu \) in eqn (64)) are specified beforehand. Further, \( \phi_{\alpha_0} \) in drawing or \( \phi_{\beta_0} \) in extrusion are given arbitrarily. Then, \( \phi_{\alpha_0} \) in drawing or \( \phi_{\beta_0} \) in extrusion is found from the relation
\[ \phi_{\alpha_0} - \phi_{\beta_0} = \alpha_D - \mu + \frac{\pi}{4}. \tag{77} \]

When \( t_s \) or \( c_f \) is given, the pressure at the point C is determined using eqn (72) or (73).

When area reduction is small, the plastic region appears at the part of plate approaching the die (die entrance) or leaving the die (die exit), and the central part of the plate behaves as being rigid. They are called entrance bulge or exit bulge, and the slip line field at the transition from the normal process to the bulging process is shown in Fig. 7. The transition begins when the average pressure on the central plane reaches following values.

- **Entrance bulging:**
\[ \overline{p}_e = -\frac{1}{2} + \frac{\pi}{4} - \alpha_D + \mu - \phi_{\alpha_0} - \phi_{\beta_0}. \tag{78} \]
Exit bulging:

\[
\frac{\bar{p}_e}{n} = \frac{1}{2} + \frac{3\pi}{4} + \alpha_D - \mu - \phi_{\sigma_0} - \phi_{\sigma_0}.
\]

On the Mises’ yield condition the relation

\[
\sigma_k = \sqrt{3} k
\]

holds. Then, eqns (62) and (63) become

\[
\frac{\bar{p}}{n} \frac{\varepsilon_k}{\varepsilon_1} = \frac{1}{\sqrt{3}}, \quad \frac{\bar{p}}{n} \frac{\varepsilon_1}{2}.
\]

4.3 Influence of Work Conditions on a Chevron Crack

The slip line field provides the area reduction \( r \) and the average pressure \( \bar{p} \) on the central plane depending on working conditions \( D, \tau_r, \) and \( \bar{t}_c \), where die length is assumed to be \( AB=1 \). In general, the information obtained by the slip line method is applicable only to non-hardening materials. Though the stress state in work-hardening materials will be slightly different from that in non-hardening, the difference of stress ratios \( \alpha \) and \( \beta \) between them is expected to decrease with increasing strain level.

Fig. 8 Comparison of working strain \( \varepsilon_{1^*} \) with cracking strain \( \varepsilon_{1^*} \).

(a) Influence of die-half angle \( \alpha_D \).

(b) Influence of friction stress \( \tau_r \).

(c) Influence of backward tension \( \bar{t}_b \).
to be smaller than that of the stresses themselves, and the normalized pressure \( \bar{p} \) will be so too. Here the assumption is made that \( \bar{p} \) in hardening materials has the same value as that obtained by the slip line method. If the product of tensile strain divided by \( n \) and the normalized pressure is not larger than \(-1/2\), eqn (81) suggests that central bursting will occur.

When the area reduction \( r \) is given, the geometrical configuration of manufacturing provides eqn (76b) determining tensile strain, which is denoted by \( \varepsilon \). The normalized pressure on the central plane and eqn (81b) give tensile strain too, which is denoted by \( \varepsilon \). If the former strain is larger than the latter, there is the possibility of central bursting. Drawing is discussed here mainly for the explanation of phenomena, since the qualitative relations are similar to those in extrusion. Relatively small \( n \) value, \( n=0.1 \) is taken here, because central bursting usually occurs in the

**Fig. 9** Chevron cracking in drawing.
materials that have been severely cold worked. \( \varepsilon^* \) and \( \varepsilon^{**} \) are represented in Fig. 8(a), (b), and (c), where different parameters are used respectively. In these figures there are two kinds of broken lines. One of them is the available limit of the assumed slip line field and the other is the bulge limit at the die entrance. In drawing no bulging appears at the die exit. The curves of \( \varepsilon^* \) and \( \varepsilon^{**} \) do not always intersect each other, but there are some cases where they intersect at two points. Under the working conditions where \( \varepsilon^{**} \) is larger than \( \varepsilon^* \), the tensile strain required from the geometrical configuration is larger than the strain which causes central bursting. So, under these working conditions there is the possibility that central bursting occurs. Therefore, the intersecting points of these curves set up the boundary between working conditions with and without central bursting. The boundary curves consisting of these intersecting points are shown in Fig. 9(a), (b), and (c). Here one side of the curve is the safety zone and the other side is the defect zone where central bursting occurs. Fig. 9(a) shows that the defect zone increases with decreasing \( n \). Also, the defect zone shown in Fig. 9(b) is spreading with the increase of friction stress, and that is the case with the increase of back tension, as shown in Fig. 9(c). The available limit of the assumed slip line field moves toward low reduction with the increase of die friction and back tension. Since the curves separating the safety and defect zones are almost straight near the available limit, the curves may be extended slightly over the limit.

Similar results for extrusion are shown in Fig. 10(a), (b), (c), where backward tension in drawing is replaced by forward compression. In many cases of extrusion processes, the cylindrical surface of billet is constrained by a container so that it cannot expand radially. Therefore, in these cases the bulging phenomenon occurs only at the exit region. But in free extrusion, for example in the production of stepped shafts, bulging
phenomenon at the entrance region will be present too. Then, as an example, the entrance bulge limit is shown in only Fig. 10(a). Under working conditions for extrusion usually encountered in manufacturing processes, the stress state can be represented by the assumed slip line field.

Upon comparison between Fig. 9(a) and 10(a), an important contrast is seen; that is, central bursting occurs at smaller $n$ in extrusion than in drawing. Therefore, central bursting is easier to occur in drawing than in extrusion; and in drawing, central bursting may be possible even if annealed materials are used. When the comparison is made between Fig. 9(b) and 10(b), it is seen that the defect zone spreads with the increase of friction stress in drawing, but in extrusion, the reverse is seen. These phenomena result from the fact that friction stress increases the working force which is responsible for tensile stress in drawing and compressive stress in extrusion. Moreover, the comparison between Fig. 9(c) and 10(c) shows that the influence of backward tension in drawing on central bursting is opposite to the influence of forward compression in extrusion. These facts can be easily understood upon considering that those additional forces have similar effects on working forces as friction stress.

The merit of our analysis is that the occurrence of central bursting is not predicted at relatively small reduction. This is different from the result of Avitzur's analysis\(^{11}\). From the fact that central bursting in usual manufacturing does not occur at such small reduction as bulging appears, our result would be regarded as reasonable.

By the analogy of the stress state at the central region of a cylindrical specimen necking in tension, it is generally accepted that the average pressure should be negative, that is, of hydrostatic tension, for the growth of voids. Considering the fact that fracture occurs after the growth of void and that it might closely relate with localized necking, it is interesting that eqn (81) requires hydrostatic tension for the occurrence of central bursting, though it is derived from the criterion of mechanical necking.

4.4 Comparison with Experimental Results

A comparison is made of our present analysis with experimental results from some researches that have been published in the literature. There are a lot of papers concerned with chevron cracks, but very few papers in which mechanical properties of a specimen before working is definitely described. So the two following papers are chosen.

4.4A The Drawing by Orbegozo

Orbegozo carried out experiments of various kinds in axi-symmetric drawing on 2011 aluminum alloy wires which had been severely cold worked previously, in order to assure the occurrence of chevron cracks in wire drawing. From the observations he concludes that fracture nucleates around the existing defects at the center of the
Ductility Criterion Applicable to the Prediction of Sheet Formability and Central Bursting in Drawing and Extrusion Under Plane Strain

Table 1 Experimental results by Zimerman & Avitzur

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_s$</td>
<td>$\gamma$</td>
<td>$\varepsilon_{\phi}$</td>
<td>$h$</td>
<td>$n$</td>
<td>$J$</td>
</tr>
<tr>
<td>2.75</td>
<td>19.6</td>
<td>1.5244</td>
<td>0.018</td>
<td>0.027</td>
<td>S</td>
</tr>
<tr>
<td>0.5</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>6.5</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>8.5</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>10.05</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>15.6</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>30.4</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>8.0</td>
<td>6.9</td>
<td>1.3450</td>
<td>0.042</td>
<td>0.056</td>
<td>D</td>
</tr>
<tr>
<td>9.0</td>
<td>7.1</td>
<td>1.4162</td>
<td>0.038</td>
<td>0.054</td>
<td>S</td>
</tr>
<tr>
<td>8.0</td>
<td>5.9</td>
<td>2.2052</td>
<td>0.005</td>
<td>0.011</td>
<td>S</td>
</tr>
<tr>
<td>&quot;</td>
<td>7.3</td>
<td>2.2660</td>
<td>&quot;</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>&quot;</td>
<td>2.5</td>
<td>1.3622</td>
<td>0.039</td>
<td>0.053</td>
<td>S</td>
</tr>
<tr>
<td>&quot;</td>
<td>15.2</td>
<td>1.3862</td>
<td>0.038</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>&quot;</td>
<td>4.3</td>
<td>1.5344</td>
<td>0.005</td>
<td>0.008</td>
<td>D</td>
</tr>
<tr>
<td>&quot;</td>
<td>6.1</td>
<td>1.5800</td>
<td>&quot;</td>
<td>0.009</td>
<td>S</td>
</tr>
<tr>
<td>&quot;</td>
<td>7.5</td>
<td>1.7174</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>&quot;</td>
<td>7.7</td>
<td>1.7948</td>
<td>&quot;</td>
<td>&quot;</td>
<td>D</td>
</tr>
<tr>
<td>&quot;</td>
<td>29.7</td>
<td>1.9714</td>
<td>&quot;</td>
<td>0.010</td>
<td>D</td>
</tr>
</tbody>
</table>

Experimental results by Zimerman & Avitzur

Column J
- $S$: no defect (safety zone)
- $D$: chevron cracking (defect zone)

(c) Electrolytic iron (CB-3)

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_s$</td>
<td>$\gamma$</td>
<td>$\varepsilon_{\phi}$</td>
<td>$h$</td>
<td>$n$</td>
<td>$J$</td>
</tr>
<tr>
<td>30.0</td>
<td>49.5</td>
<td>2.0882</td>
<td>&gt;0.16</td>
<td>&gt;0.33</td>
<td>S</td>
</tr>
<tr>
<td>45.0</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>60.0</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
<tr>
<td>8.0</td>
<td>21.9</td>
<td>1.7128</td>
<td>0.14</td>
<td>0.24</td>
<td>S</td>
</tr>
<tr>
<td>22.5</td>
<td>25.1</td>
<td>1.9602</td>
<td>&gt;0.15</td>
<td>&gt;0.29</td>
<td>S</td>
</tr>
<tr>
<td>30.0</td>
<td>26.1</td>
<td>&quot;</td>
<td>&quot;</td>
<td>&quot;</td>
<td>S</td>
</tr>
</tbody>
</table>

Column J
- $S$: no defect (safety zone)
- $D$: chevron cracking (defect zone)

wire, but that the main factor contributing to crack nucleation is the tri-axiality arising at the central axis of a wire when it is drawn through a die. The latter conclusion supports our criterion in eqn [8] that limit strain is mainly influenced by hydro-static tension. A typical stress-strain curve for the swaged wire used as the specimen of his experiments is shown in Fig. 10 (OP) in his paper (OP means to be in the original paper referred), from which the $n$ value can be evaluated as $n = 0.04$. He detects central bursting by using either of the following methods; one is by surface inspection, and the other is with the aid of X-rays, which is done for only a part of his experiments,
and the results of which are given in Table 2(OP). Only X-ray inspection data are taken up for the comparison, because there are some cases where chevron cracks may occur inside of the wire even if any indication of a crack cannot be observed on the surface. The comparison is shown of his results with ours in Fig. 11, where boundary curves separating safety and defect zones gives good agreement with his experimental date, although our analysis is obtained under plane strain condition which is different from the axisymmetric condition of his experiments.

4. 4B The Extrusion by Zimerman & Avitzur

They used three different iron alloys for their experimental work: AISI 1024 steel (CB—1, CB—4), AISI 12L14 steel (CB—2) and Electrolytic iron (CB—3). The extrusion was carried out in a hydrostatic extrusion process without receiver pressure (in exit chamber). The specimens have been cold worked beforehand through drawing and extrusion. The stress-strain curve of specimens is assumed to be linear strain hardening in their paper, but we postulate it to be power law as in eqn (8). Then, the correspondence between them must be

---

(a) AISI 1024 steel (CB—1)

(b) AISI 12L14 (Ledloy A) (CB—2)

(c) AISI 1024 steel (CB—4)

Fig. 12 Comparison with experimental results in extrusion (by Zimermen & Avitzur)

---
made for comparing each data. Their assumption is

$$\sigma_e = \sigma_{eb} (1 + n \epsilon_e - \epsilon_{eb})$$

(82)

where \( h \) is the gradient of the linear stress-strain curve, and the subscript \( b \) means to belong to the cold worked state before the extrusion process. When \( \epsilon_e - \epsilon_{eb} \ll \epsilon_{eb} \), eqn (8) can be approximated as

$$\sigma_e = \sigma_{eb} (1 + n \frac{\epsilon_e - \epsilon_{eb}}{\epsilon_{eb}})$$

(83)

Therefore, the following relation yields.

$$h = \frac{n}{\epsilon_{eb}}$$

(84)

The drawn bars and extruded billets were examined for central bursting by X-ray inspection in their experiment, whose data are shown in Table 1.

Our analytical predictions concerning with their results are shown in Fig. 12 (a), (b), (c), where the indicated numerical values denote \( n \) converted by eqn 84, and the symbols \( G \) and \( B \) denote whether the agreement of our prediction with these data are good or bad respectively. Though the data CB-3 is not shown there, it is clear from Table 1 that good agreement is obtained between them. Although the agreement is not always good in all these figures, we may say that our analysis can give approximate estimation for the occurrence of chevron cracks.

5. Conclusions

5.1 Proposed Criterion

The Limit of ductility has been discussed in connection with the collapse for uniqueness of solution. There are two cases making the multiple solutions possible, one of which is statically admissible, the other being kinematically admissible.

In the process of localization, multiple strain rate must be consistent with the compatibility of velocity on a boundary plane of localization. The plane can take no other mode than just two modes conformable with strain rate discontinuity.

The kinematically admissible multiplicity is local, and it occurs under pure plane strain rate, which is consistent with the requirement for the compatibility of strain rate.

5.2 Sheet formability

The forming limit obtained from the conditions of statically admissible multiplicity is just the same as that of diffuse necking suggested by Swift, but their basic concepts are quite different.

Under plane stress it is possible for only the perpendicular mode to be realized. This mode is different from the mode of localized necking suggested by Hill, which satisfies not all of the requirements of velocity compatibility.

Before diffuse necking, forming sheets can take any strain paths prescribed beforehand, but after it strain paths cannot be controlled artificially. Hence, it would be very rare that the deformation happens to satisfy the kinematically admissible condition before the diffuse necking. In general, localized necking would finally occur at the stage that deforming region shrinks to a plane ultimately in a localization process.

Forming limit diagrams are obtained for three kinds of strain paths including a linear path. In this analysis, the processes from diffuse necking to localized necking are assumed, in which the
nominal stress rate in first principal direction is proportional to strain rate ratio $\gamma$. The results show reasonable dependencies of the formability on the strain paths.

5. 3 Central Bursting

The central bursting in drawing and extrusion under plane strain deformation is investigated with respect to localized necking. The following conclusions are obtained.

1. The criterion for the central bursting is given as $p \varepsilon / n = -1/2$.
2. In the processes with smaller reduction than that responsible for the entrance bulge in drawing and the exit bulge in extrusion, central bursting is not predicted.
3. The possibility of central bursting is higher in drawing than in extrusion, when identical reduction of area and die angle are used in these manufacturing processes.
4. The smaller the die angle is, the lower is the risk of occurrence of central bursting.
5. In drawing, the possibility of central bursting increases with the increase of die friction and backward tension, while in extrusion it does so with the decrease of die friction and forward compression.

Acknowledgement

The author wishes to thank Professor Betzalel Avitzur for his kindness in offering us his report at Lehigh University in which the details of his experiment and valuable data are described.

References


**Appendix**

When a sheet or a strip is loaded in tension, it deforms uniformly until the load reaches a maximum. After that, the deformation begins to be non-uniform and a neck or a waist is formed where plastic deformation continues, but the other region is unloaded where plastic deformation ceases. Such a phenomenon is called necking. Plastic instability identifies what happens when a state of nearly uniform strain gives way to one in which straining is more or less localized. The profile curvature of a neck increases from zero as straining continues. This is diffuse necking, so labeled because its initiation is truly diffuse. As it grows, it takes in a length no less than about specimen width. Under such circumstance, there is possibility that another type of necking occurs which is called localized necking. In localized necking the specimen is thinned along a narrow band inclined at a certain angle to the pulling axis. Ultimately thinning region takes a plane in shape. The plane is called necking plane, whose normal direction specifies its mode, that is the mode of necking plane.

In one dimensional state of loading, the occurrence of plastic instability is assigned by the condition of maximum load. However, in three dimensional state of loading, the general condition has not been obtained in which plastic instability occurs. That condition is designated as the criterion for instability. For the criterion of instability under place stress, Swift proposed the condition that both loads in two dimensional loading have the stationary state simultaneously. It is called Swift's diffuse necking. The localized necking in the tensile test of a strip is called Hill’s localized necking.