Technological Note

Yosenabe is NP-complete

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Abstract: Yosenabe is one of Nikoli’s pencil puzzles, which is played on a rectangular grid of cells. Some of the cells are colored gray, and two gray cells are considered connected if they are adjacent vertically or horizontally. A set of connected gray cells is called a gray area. Some of the gray areas are labeled by numbers, and some of the non-gray cells contain circles with numbers. The object of the puzzle is to draw arrows, vertically or horizontally, from all circles to gray areas so that (i) the arrows do not bend, and do not cross other circles or lines of other arrows, (ii) the number in a gray area is equal to the total of the numbers of the circles which enter the gray area, and (iii) gray areas with no numbers may have any sum total, but at least one circle must enter each gray area. It is shown that deciding whether a Yosenabe puzzle has a solution is NP-complete.

Keywords: NP-complete, computational complexity, one-player game, Yosenabe

1. Introduction

Yosenabe is one of Nikoli’s pencil puzzles [20], which is played on a rectangular grid of cells (see Fig. 1 (a)). Some of the cells are colored gray, and two gray cells are considered connected if they are adjacent vertically or horizontally. A set of connected gray cells is called a gray area, which is regarded as a “deep pot” (“Nabe (鍋)” in Japanese). Some of the gray areas are labeled by numbers, and the remaining gray areas have no numbers. Some of the non-gray cells contain circles with numbers, where circles are regarded as “ingredients” (“Guzai (具材)” in Japanese). The Japanese word “Yosenabe (寄せ鍋)” means a “mixed stew.”

The object of the puzzle is to draw arrows, vertically or horizontally, from all circles to gray areas (see Fig. 1 (b)) so that (i) the arrows do not bend, and do not cross other circles or lines of other arrows, (ii) the number in a gray area is equal to the total of the numbers of the circles which enter the gray area, and (iii) gray areas with no numbers may have any sum total, but at least one circle (arrow tip) must enter each gray area. It should be noted that only one arrow tip can enter a gray cell.

In this paper, it is shown that deciding whether a Yosenabe puzzle has a solution is NP-complete. The puzzle is trivially in NP, since the puzzle can be solved by drawing an arrow from every circle one by one.

There has been a huge amount of literature on the computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [9]. After the publication of this book, the following Nikoli’s pencil puzzles were shown to be NP-complete: Hashiwokakero [1], Kurodoko [15], Shakashaka [6], Shikaku and Ripple Effect [19], Yajilin and Country Road [11].

Furthermore, it was also shown that Block Sum [8], Kaboozle [2], Magnet Puzzle [16], Pandemic [17], Shisen-Sho [14], String Puzzle [13], single-player UNO [5] and Zen Puzzle Garden [10] are NP-complete. As for higher complexity classes, Chat Noir [12], Rolling Block Maze [3], and two-player UNO [5] were shown to be PSPACE-complete.

Fig. 1 (a) A Yosenabe puzzle. (b) A solution of (a). All circles are moved, vertically or horizontally, so that they enter the gray areas. The number in a gray area is equal to the total of the numbers of the circles which enter the gray area. At least one circle enters every gray area with no number. (c) is not a solution, since pairs of arrows intersect at cells (B, h) and (F, e). (d) is not a solution, since there is a gray area (on cells (B, f) and (C, f)) which does not receive any arrow.

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2. NP-completeness of Yosenabe

We present a polynomial-time transformation from an arbitrary instance $C$ of PLANAR 3SAT to a Yosenabe puzzle $Y$ such that $C$ is satisfiable if and only if $Y$ has a solution.

2.1 PLANAR 3SAT Problem

The definition of PLANAR 3SAT is mostly from [LO1] of Ref. [7]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean variables. Boolean variables take on values 0 (false) and 1 (true). If $x$ is a variable in $U$, then $x$ and $\overline{x}$ are literals over $U$. The value of $x$ is 1 (true) if and only if $x = 0$ (false). A clause over $U$ is a set of literals over $U$, such as $\{x_1, x_3, x_4\}$. It represents the disjunction of those literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over $U$ such that (i) $|c_j| \leq 3$ for each $c_j \in C$ and (ii) the bipartite graph $G = (V, E)$, where $V = U \cup C$ and $E$ contains exactly those pairs $(x, c)$ such that either literal $x$ or $\overline{x}$ belongs to the clause $c$, is planar.

The PLANAR 3SAT problem asks whether there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $C$. This problem is known to be NP-complete. For example, $U = \{x_1, x_2, x_3, x_4\}, C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, \overline{x_3}\}, c_2 = \{\overline{x_1}, x_2, x_4\}, c_3 = \{\overline{x_1}, x_3, \overline{x_2}\}, c_4 = \{\overline{x_2}, x_3, \overline{x_4}\}$ provide an instance of PLANAR 3SAT. In this instance, the answer is “yes,” since there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses. It is known that PLANAR 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in $C$ [4].

2.2 Transformation from an Instance of PLANAR 3SAT to a Yosenabe Puzzle

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed into the variable gadget (as illustrated in Fig. 2(a)), which consists of two gray areas having number 2 (labeled with $x_i$ and $\overline{x_i}$), one circle with number 2 (labeled with $r$, and two pairs of circles with number 1 (labeled with $s$, $t$ and $u$, $v$).

Suppose that there are gray areas $c_j$ and $c_{j_1}, c_{j_2}$ to the left and right sides of the variable gadget (see Fig. 2(b)). If gray area $x_i$ is filled by circles $s$, $t$, then gray area $\overline{x_i}$ must be filled by circles $u$, $v$. In this case, gray area $c_j$ can be filled by circle $r$, but $c_{j_1}, c_{j_2}$ cannot be filled by circles $u$, $v$. This configuration corresponds to $x_i = 1$.

On the other hand, if gray area $\overline{x_i}$ is filled by circles $s$, $t$ (see Fig. 2(c)), then gray area $x_i$ must be filled by circle $r$. In this case, $c_{j_1}, c_{j_2}$ can be filled by circles $u$, $v$, but $c_j$ cannot be filled by $r$. This corresponds to $x_i = 1$.

Figure 3 is the right-and-left turn gadget, which consists of two gray areas having number $k$ (labeled with $A$ and $B$) and three circles with number $k$ (labeled with $p$, $q$, and $r$), where $k \in \{1, 2\}$. If gray area $c_j$ is filled by circle $p$, then gray areas $A$ and $B$ must be filled by circles $q$ and $r$, respectively. Conversely, if $x_i$ is filled by circle $r$, then gray areas $A$ and $B$ must be filled by circles $q$ and $p$, respectively.

Figure 4 is a Yosenabe puzzle $Y$ transformed from $C = \{c_1, c_2, c_3, c_4\}$ and $U = \{x_1, x_2, x_3, x_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}, c_2 = \{\overline{x_1}, x_2, x_4\}, c_3 = \{\overline{x_1}, x_3, \overline{x_2}\}$, and $c_4 = \{\overline{x_2}, x_3, \overline{x_4}\}$. If either literal $x_i$ or $\overline{x_i}$ belongs to the clause $c_j \in C$, then the variable gadget for $x_i \in U$ is connected to gray area $c_j$ via right-and-left turn gadgets.

Let $G = (V, E)$ be a graph, where $V = U \cup C$ and $E$ contains exactly those pairs $(x, c)$ such that either literal $x$ or $\overline{x}$ belongs to the clause $c_j$. Now one can see that $E$ corresponds to white regions of Fig. 4, and $V = U \cup C$ corresponds to red regions (labeled with $x_i$ and $\overline{x_i}$) and gray rectangle areas (labeled with $c_j$), respectively. Regions labeled with color $l \in \{3, 4, 5\}$ as in Fig. 4 correspond to $G$’s faces.

Regions labeled with color $l \in \{3, 4, 5\}$ are used as “walls” (see also Fig. 5), which are composed of pairs of gray areas (squares) with number $l$ and circles having number $l$. For example, the center region labeled with color 3 in Fig. 4 is composed of seven gray areas (squares) with number 3 and seven circles having number 3 (see Fig. 5).

Each gray square of Fig. 5 must be filled by a single circle (and not by two or more circles), since it is composed of a single cell. The center region of color 3 (see Fig. 5) can be filled up with seven arrows connecting circles to gray squares in the region.
Fig. 4  Yosnabe puzzle $Y$ transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_2, \overline{x_4}\}$, and $c_4 = \{\overline{x_1}, \overline{x_2}, x_3\}$. From the solution indicated by solid arrows, one can see that the assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfies all clauses of $C$.

Fig. 5  The center region labeled with color 3 of Fig. 4 is composed of seven gray areas (squares) with number 3 and seven circles having number 3. This region will be filled up with seven arrows connecting circles to gray squares in the region.

By using the quadratic-time four-coloring algorithm [18], we can assign four colors $\{3, 4, 5, 6\}$ to every face of $G$ so that no two adjacent faces have the same color. (In Fig. 4, color 6 is not used.) Since the maximum degree of $G$ is three, such a coloring satisfies that no two faces sharing a single vertex have the same color. Therefore, any pair of regions having the same color are sepa-
rated by a region having a different color. For example, in Fig. 4, there are two regions of color 3. The center region of color 3 is completely surrounded by colors 4 and 5, and is separated from the outer region of color 3.

Hence, every circle in a region of color $l \in \{3, 4, 5, 6\}$ has an arrow to a gray area (square) in the same region (see Fig. 5) (and not to a gray area (square) in a different region).

From this construction, the instance $C$ of PLANAR 3SAT is satisfiable if and only if Yosenabe puzzle $Y$ has a solution. From the solution indicated by solid arrows in Fig. 4, one can see that the assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfies all clauses of $C$.

References


