On Contractible Edges in Convex Decompositions

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Abstract: Let Π be a convex decomposition of a set P of n ≥ 3 points in general position in the plane. If Π consists of more than one polygon, then either Π contains a deletable edge or Π contains a contractible edge.

Keywords: convex decomposition, convex deformation, contractible edge

1. Introduction

Let P be a set of n ≥ 3 points in general position in the plane. A convex decomposition of P is a set Π of convex polygons with vertices in P and pairwise disjoint interiors such that their union is the convex hull CH(P) of P and that no point in P lies in the interior of any polygon in Π. A geometric graph with vertex set P is a graph G, drawn in the plane in such a way that every edge is a straight line segment with ends in P.

Let Π be a convex decomposition of P. We denote by G(Π) the skeleton graph of Π, that is the plane geometric graph with vertex set P in which the edges are the sides of all polygons in Π. An edge e of Π is an interior edge if e is not an edge of the boundary of CH(P).

An interior edge e of Π is deletable if the geometric graph G(Π)e, obtained from G(Π) by deleting the edge e, is the skeleton graph of a convex decomposition of P. Neumann-Lara et al. [6] proved that if a convex decomposition Π of a set P of n points consists of more than (3n−2k)/2 polygons, where k is the number of vertices of CH(P), then Π has at least one deletable edge.

An interior edge e = uw of Π is contractible from u to v if the geometric graph G(Π)\{e\} = (G(Π) \{\{x_1u, x_2u, \ldots, x_mu, uw\}\} \cup \{x_1v, x_2v, \ldots, x_nv\}) is a skeleton graph of a convex decomposition of P\{u\}, where x_1, x_2, \ldots, x_m are the remaining vertices of G(Π) which are adjacent to u.

A simple convex deformation of Π is a convex decomposition Π' obtained from Π by moving a single point x along a straight line segment, together with all the edges incident with x, in such a way that at each stage we have a convex decomposition of the corresponding set of points. Deformations of plane graphs have been studied by several authors, both theoretically and algorithmically, see for instance Refs. [3], [4], [7] and [1],[2], [5], respectively.

Let P_1 and P_2 be sets of n ≥ 3 points in general position in the plane. A convex decomposition Π_1 of P_1 and a convex decomposition Π_2 of P_2 are isomorphic if there is an isomorphism of G(Π_1) onto G(Π_2), as abstract plane graphs, such that the boundaries of CH(P_1) and CH(P_2) correspond to each other with the same orientation.

Thomassen [7] proved that if Π_1 and Π_2 are isomorphic convex decompositions, then Π_2 can be obtained from Π_1 by a finite sequence of simple convex deformations. As a tool, Thomassen proved that if Π is a convex decomposition with at least two polygons, then there is an isomorphic convex decomposition Π' that can be obtained from Π by a finite number of simple convex deformations that preserve the boundary and such that Π' contains either a deletable edge or a contractible edge. In this note we prove that every convex decomposition Π with at least two polygons contains an edge which is deletable or contractible. Furthermore, if P contains at least one interior point, then Π contains a contractible edge.

2. Preliminary Results

Let Π be a convex decomposition of P containing no deletable edges. For every interior edge e of G(Π), the graph G(Π)e has an internal face Q_e which is not convex and at least one end of e is a reflex vertex of Q_e.

We define an abstract directed graph G(Π) with vertex set P in which \overrightarrow{uw} ∈ A(G(Π)) if and only if u is a reflex vertex of Q_w.

Notice that for each interior edge uw of G(Π), the directed graph G(Π) contains at least one of the arcs \overrightarrow{uw} and \overrightarrow{uw} (see Fig. 1).

Remark 1.

(1) The outdegree of every vertex u of G(Π) is at most 3.

(2) The outdegree of every vertex u in the boundary of CH(P) is 0.

(3) An interior vertex u of Π has outdegree 3 in G(Π) if and only if u has degree 3 in G(Π).

(4) If \overrightarrow{uw}, \overrightarrow{uw} ∈ A(G(Π)), then uw and uw lie in a common face of G(Π).

For two points α and β in the plane, we denote by r(αβ) the ray, with origin α, that contains the segment ab.

Lemma 2. An edge uw of Π is not contractible from u to v if and only if there are edges xy and xu, lying in a common face of G(Π)

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that contains vertex \( u \), such that the ray \( r(yx) \) meets the edge \( uv \) at point \( ut \), with \( u \neq u \neq v \), and that the triangular region defined by \( x, u, \) and \( u \) contains no point of \( P \) in its interior.

Proof. It is easy to see that the existence of such edges \( yx \) and \( xu \) implies that the edge \( uv \) cannot be contracted from \( u \) to \( v \). We proceed to prove the remaining part of the lemma. Let \( uv \) be an interior edge of \( \Pi \) with \( u \neq u \neq v \) not lying in the boundary of \( CH(\Pi) \) and let \( x_1, x_2, \ldots, x_m \) be the remaining vertices of \( G(\Pi) \) which are adjacent to \( u \). We contract the edge \( uv \) in a continuous way as follows: Slide the point \( u \) along the ray \( r(ut) \), together with the edges \( x_1u, x_2u, \ldots, x_mu \) (see Fig. 2).

If \( uv \) is not contractible from \( u \) to \( v \), then either the transformed graph \( T(G(\Pi)) \) becomes non planar or one of its faces becomes non convex. This implies that we must reach a point \( u_t = u + t(v - u) \), with \( 0 < t < 1 \), such that there are two edges \( yx_i \) and \( x_iu_t \) lying in a common face, which become collinear in \( T(G(\Pi)) \) (see Fig. 3).

Notice that two or more different pairs of edges \( yx_i, xu_t \) and \( y'x_j, x_ju_t \) may become collinear simultaneously; in such a case we may choose any of those pairs and proceed with the proof.

The triangular region defined by \( x_i, u_t \) and \( u \) is the region swept by the edge \( x_iu_t, 0 \leq s \leq t \) and therefore it contains no point of \( P \) in its interior. The lemma follows since the edges \( yx_i \) and \( xu_t \) lie in a common face of \( G(\Pi) \) and the ray \( r(yx_i) \) meets the edge \( uv \) at
the point $u_i$. \hfill \square

Let $N$ denote the set of arcs $\overparen{uv}$ of $G(\Pi)$ such that the edge $uv$ is not contractible from $u$ to $v$ in $\Pi$. For each $\overparen{uv} \in N$ let $y = y_{uv}$, $x = x_{uv}$ and $u_i$ be as in Lemma 2. Since the edges $y_{uw}, x_{uw}$ and $x_{uw}$ lie in a common face of $G(\Pi)$ and the triangular region, defined by $x_{uv}, u_i$ and $u$, contains no point of $P$ in its interior, the geometric graph $G(\Pi) - x_{uw}$ contains a face $Q_{x_{uv}}$ in which $x_{uw}$ is a reflex vertex and therefore $s_{x_{uw}} \in A(G(\Pi))$. This defines a function

$$f: N \rightarrow A(G(\Pi))$$

given by $f(\overparen{uv}) = s_{x_{uw}}$.

Notice that the arcs $f(\overparen{uv})$ and $\overparen{uw}$ form a directed path in $G(\Pi)$ with length 2 and middle vertex $u$. This implies that if $f(\overparen{uv_1}) = f(\overparen{uv_2})$, then $u_1 = u_2$. Moreover, if $\overparen{uv_1}$, $\overparen{uw}$ and $\overparen{uw}$ are distinct arcs such that $f(\overparen{uv_1}) = f(\overparen{uw}) = f(\overparen{uw}) = \overparen{ux}$, then $u$ is adjacent in $G(\Pi)$ to $v_1$, $v_2$, $v_3$ and to $x$, which is not possible by Remark 1, since $u$ has outdegree 3 in $G(\Pi)$. It follows that there are no three arcs in $N$ with the same image under the function $f$ and therefore $|\text{Im}(f)| = |N| - |U|$, where $U$ is the set of points $u$ of $P$ for which there is a pair of arcs $\overparen{uv}, \overparen{uw} \in N$ such that $f(\overparen{uv}) = f(\overparen{uw})$.

**Lemma 3.** Let $\Pi$ be a convex decomposition of $P$ with no deletable edges. If $U \neq \emptyset$, then there is a function $g: U \rightarrow A(G(\Pi))$

such that for each $u \in U$, $g(u)$ is not in the image of the function $f$.

**Proof.** Let $u \in U$ and let $v, w$ and $x = x_{uw} = x_{uw}$ be points in $P$ such that $f(\overparen{uv}) = f(\overparen{uw}) = \overparen{ux}$. If $u$ has degree larger than 3 in $G(\Pi)$, let $z \notin \{v, w, x\}$ be such that $uz$ is an edge of $G(\Pi)$. By Remark 1, the outdegree of $u$ in $G(\Pi)$ is at most 2, therefore $\overparen{uz}$ is not an arc of $G(\Pi)$. It follows that $\overparen{ux}$ must be an arc of $G(\Pi)$. In this case $g(u) = \overparen{ux} \notin \text{Im}(f)$ since $z \neq x$ and $\overparen{ux}$ is the unique arc in $\text{Im}(f)$ that ends at $u$.

If $u$ has degree 3 in $G(\Pi)$, then $u$ has outdegree 3 in $G(\Pi)$, by Remark 1 and therefore $\overparen{ux}$ is an arc of $G(\Pi)$. We claim that in this case $g(u) = \overparen{ux} \notin \text{Im}(f)$. Let $l_{ux}$ denote the line containing the edge $ux$, and let $y_{ux}$ and $y_{uw}$ be points in $P$ such that the rays $r(y_{ux}, x)$ and $r(y_{uw}, x)$ intersect the edges $uw$ and $uw$, respectively.

Without loss of generality we assume that $l_{ux}$ is a vertical line such that $v$ and $y_{uw}$ lie to the left of $l_{ux}$ and $w$ and $y_{uw}$ lie to the right of $l_{ux}$ (see Fig. 4). Clearly the angles $\angle y_{uw}ux$ and $\angle y_{uw}ux$ are smaller than $\pi$, it is easy to see that $\angle y_{uw}ux$ is also smaller than $\pi$.

Therefore if $xz$ is an edge of $\Pi$ with $z \notin \{u, y_{uw}, y_{uw}\}$, then $\overparen{ux}$ is not an arc of $G(\Pi)$. This implies that if $\overparen{ux} \in \text{Im}(f)$, then $\overparen{ux} = f(\overparen{ux}) = f(\overparen{ux}) = \overparen{ux}$. Since both edges $uw$ and $x_{uw}$ lie in the right halfplane defined by $l_{ux}$, then $r(\overparen{ux})$ cannot intersect the edge $x_{uw}$ and therefore $y_{uw} \neq w$. Finally, since $r(y_{uw}, x)$ intersects the edge $uw$, $r(\overparen{ux})$ cannot intersect the edge $x_{uw}$. Therefore $\overparen{ux} \neq f(\overparen{ux})$; analogously $\overparen{ux} \neq f(\overparen{ux})$.

**3. Main Results**

In this section we prove our main results.

**Theorem 4.** Let $P$ be a set of points in general position in the plane. If $\Pi$ is a convex decomposition of $P$ consisting of more than one polygon, then either $\Pi$ contains a deletable edge or $\Pi$ contains a contractible edge.

**Proof.** Assume the result is false and $\Pi$ contains no deletable edges and no contractible edges. Define the directed graph $G(\Pi)$ as in the previous section, notice that $A(G(\Pi)) \neq \emptyset$ since $\Pi$ contains at least two polygons. Since $\Pi$ contains no contractible edges, $N = A(G(\Pi))$.

Let $B = B(G(\Pi))$ be the set of arcs of $G(\Pi)$ of the form $\overparen{uv}$, with $u$ in the boundary of $CH(P)$, and let $\overparen{uv} \in B$. By Remark 1, $u$ has outdegree 0 in $G(\Pi)$ which implies $\overparen{uv} \notin \text{Im}(f)$.

If $U = \emptyset$, then $\text{Im}(f) \subset A(G(\Pi)) \setminus B$, 

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Proof. Let \( \Pi \) be a convex decomposition of a set of points \( P \) in general position in the plane. If \( P \) contains at least one interior point, then \( \Pi \) contains at least one contractible edge.

By Corollary 5, if \( U \neq \emptyset \), by Lemma 3 no arc in \( \text{Im}(g) \) lies in \( \text{Im}(f) \), therefore
\[
|\text{Im}(f)| \leq |A(G(\Pi))| - |\text{Im}(g)| - |B|,
\]
which is not possible since \( \Pi \) contains no deletable edges and \(|B| \geq 3\).

And if \( U \neq \emptyset \), by Lemma 3 no arc in \( \text{Im}(g) \) lies in \( \text{Im}(f) \), therefore
\[
|\text{Im}(f)| \leq |A(G(\Pi))| - |\text{Im}(g)| - |B|,
\]
which is not possible since \( \Pi \) contains no deletable edges and \(|B| \geq 3\).

Corollary 5. Let \( \Pi \) be a convex decomposition of a set of points \( P \) in general position in the plane. If \( P \) contains at least one interior point, then \( \Pi \) contains at least one contractible edge.

Proof. Let \( \Pi' \) be a convex decomposition of \( P \) obtained from \( \Pi \) by removing deletable edges, one at a time, until no such edges remain, and let \( G(\Pi') \) be the corresponding directed abstract graph. Since \( P \) contains an interior point, \( \Pi' \) contains at least one interior edge.

By Theorem 4, there is an arc \( \overrightarrow{uv} \in A(G(\Pi')) \) such that \( uv \) is contractible from \( u \) to \( v \) in \( \Pi' \). If \( uv \) is not contractible in \( \Pi \), then by Lemma 1 there are edges \( xy \) and \( xu \) lying in a common face of \( G(\Pi) \) such that the ray \( r(yx) \) meets the edge \( uv \) at an interior point \( u_t \) and that the triangular region \( yuv \) contains no point of \( P \) in its interior. This implies that the geometric graph \( G(\Pi) - xu \) contains a face \( Q_x \) in which \( x \) is a reflex vertex and therefore \( xu \) is not deletable in \( \Pi \) and \( \overrightarrow{ux} \) is an arc of \( G(\Pi) \).

Let \( R \) be the face of \( G(\Pi) \) which contains both edges \( yx \) and \( xu \). Since \( \Pi' \) is obtained from \( \Pi \) by deleting edges but no points, then there is a face \( R' \) of \( G(\Pi') \) which contains the edge \( xu \) and the region bounded by \( R \), let \( y' \in P \) be such that \( y'x \) is an edge of \( R' \). Notice that \( y' \neq y \) otherwise \( uv \) could not be a contractible edge of \( \Pi' \) because the ray \( r(yx) \) meets the edge \( uv \) at the point \( u_t \) (Fig. 5, left). Nevertheless, since the face \( R' \) contains the edge \( xu \) and the region bounded by \( R \), the ray \( r(y'x) \) also meets the edge \( uv \) at an interior point \( u_t \) (Fig. 5, right) which again is a contradiction. \( \square \)

Corollary 6. Let \( \Pi \) be a convex decomposition of a set of points \( P \) in general position in the plane and \( Q \) be the set of points in the boundary of \( CH(P) \). There is a sequence \( P = P_0, P_1, \ldots, P_m = Q \) of subsets of \( P \), and a sequence \( \Pi_0, \Pi_1, \ldots, \Pi_m \) of convex decompositions of \( P_0, P_1, \ldots, P_m \), respectively, such that \( \Pi_0 = \Pi, \Pi_{i+1} \) consists of the boundary of \( CH(P_i) \) and \( P_{i+1} \) is obtained from \( P_i \) by contracting an edge and for \( i = 0, 1, \ldots, k \), \( \Pi_{i+1} \) consists of the boundary of \( CH(P_i) \) and for \( i = k, k+1, \ldots, m-1 \), \( \Pi_{i+1} \) is obtained from \( \Pi_i \) by deleting an edge.

Proof. By Corollary 5, if \( P_1 \) contains interior points, then \( \Pi_1 \) has a contractible edge. If \( P_1 \) contains no interior points, then each interior edge of \( \Pi_1 \) is a deletable edge. \( \square \)

References
Ferran Hurtado was born in Valencia, Spain in 1951 and lived most of his life in Barcelona where he studied and worked first as a high school teacher and a later as a university professor at Universitat Politècnica de Catalunya where he reached the highest academic position of Catedrático de Universidad. His research interest was mainly Discrete and Computational Geometry.

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