Folding and Punching Paper

YASUHIKO ASAO¹,a) ERIK D. DEMAINÉ²,b) MARTIN L. DEMAINÉ²,c) HIDEAKI HOSAKA³,d)
AKITOSHI KAWAMURA⁴,e) TOMOHIRO TACHI⁴,f) KAZUNE TAKAHASHI¹,g)

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Abstract: We show how to fold a piece of paper and punch one hole so as to produce any desired pattern of holes. Given n points on a piece of paper (finite polygon or infinite plane), we give algorithms to fold the paper flat so that those n points and no other points of paper map to a common location, so that punching one hole and unfolding produces exactly the desired pattern of holes. Furthermore, we can forbid creases from passing through the points (allowing noncircular hole punches). Our solutions use relatively few creases (in some cases, polynomially many), and can be expressed as a linear sequence of folding steps of complexity O(1)—a generalization of simple folds which we introduce.

Keywords: origami, fold-and-punch problem, fold-and-cut problem, folding complexity, flat folding

1. Introduction

In the fold-and-cut problem introduced at JCDCG’98 [5], we are given a planar straight-line graph drawn on a piece of paper, and the goal is to fold the paper flat so that exactly the vertices and edges of the graph (and no other points of paper) map to a common line. Thus, one cut along that straight line (and unfolding the paper) produces exactly the given pattern of cuts. This problem always has a solution [3], [6], though so far the complexity of the crease pattern depends on both the number n of vertices and the ratio r of the largest and smallest distances between nonincident vertices and edges. (A rough estimate on the complexity is O(nr).)

In the fold-and-punch problem, we are given n points drawn on a piece of paper, and the goal is to fold the paper flat so that exactly those points (and no other points of paper) map to a common point. Thus, punching one hole at that point (and unfolding the paper) produces exactly the given pattern of holes. This problem is a natural analog of the fold-and-cut problem where we replace one-dimensional features and target (segments onto a common line) with zero-dimensional features and target (points onto a common point); thus, we also call the problem zero-dimensional fold and cut. This problem is also a special case of the multi-dimensional fold-and-cut problem posed in Ref. [6], after Open Problem 26.32.

Directly applying a fold-and-cut solution to the graph with n vertices and zero edges does not solve the corresponding fold-and-punch problem, because the n points would come to a common line but not a common point. This discrepancy can be fixed by then making n − 1 one-layer simple folds along perpendicular bisectors between consecutive points (all perpendicular to the common line).

Our goal in this paper is to find more efficient algorithms for the fold-and-punch problem. Indeed, unlike the fold-and-cut problem, we find solutions that use a number of creases depending only polynomially in n. We also consider four variations on the problem, based on two binary parameters:

Paper size: The paper can be either a bounded set or the infinite plane. Solutions to the fold-and-cut problem (with finitely many creases) are known only in the bounded case. The paper remains unbounded after any finite number of folds, making it difficult to prevent accidental alignment.

Creasing through points: Creases can be either permitted or forbidden to pass through the points to be aligned. Forbidding creases is useful, for example, if we want to punch a hole with a shape other than a circle. Even with a circular punch, creases through the hole give little tolerance for precise hole punching in practice. The solution above using fold-and-cut leads to creases at every point.

To quantify the efficiency of our solutions, we decompose our folding into a sequence of folding steps. In particular, our motivation is to generalize the notion of simple fold [1], [2], which allows folding along a single line or segment, and sequences of simple folds. More generally, we can decompose any folding of a piece of paper into a sequence of one or more folding steps. Each folding step starts from an already folded piece of paper (the result of the previous steps), and makes an arbitrary fold,
producing another folded state. Roughly, a folding step has complexity $k$ if the paper can be decomposed into $k+1$ clusters (possibly disconnected regions) that each get folded as a single unit not sandwiched within any other cluster; refer to Section 2 for formal definitions. We call a folding step flat if the unfolded flat state is flat before and after the step, and all-layers if it treats the folded state before the step as a piece of paper (not separating any previously collocated layers). In particular, every some-layers simple fold is a flat folding step of complexity 1, and every all-layers simple fold is an all-layers flat folding step of complexity 1. At the other extreme, any unfolded state with $k+1$ faces in its crease pattern can be viewed as a sequence of just one folding step, of complexity $k$.

We can now measure the number of folding steps, the maximum complexity of the folding steps in the sequence, the total complexity of all the folding steps, etc., in addition to the usual measure of the total number of creases in the final folded state. In particular, a natural class of foldings studied here decompose into a sequence of flat folds of $O(1)$ complexity — a natural generalization of a sequence of simple folds.

Table 1 summarizes the number of such folding steps that we use in our solution to each of the four variations of the problem, as well as the total number of creases in the final crease pattern (i.e., when viewing the folding as a single step). In some cases, we depend only on the number $n$ of points; in other cases, we depend (usually, only logarithmically) on the ratio $r$ of the largest and smallest distances between points.

For fun, we designed a typeface based on folding and one punch. Figure 1 illustrates one font in the series, showing both crease pattern and resulting hole punches. Presenting the crease pattern and just one circle results in an intriguing puzzle font; see our web implementation*1. See also Ref. [4] for related work on mathematical and puzzle fonts.

2. Folding Sequence Definition

In this section, we introduce the formal concept of a sequence of folds, and the complexity of the fold steps. This definition aims to generalize the idea of a sequence of simple folds in a way that will be useful beyond the work presented here.

First, we follow the definition of folded state by Ref. [6]:

Definition 2.1 (Folded State). A piece of paper $P$ is an orientable 2-manifold embedded in $\mathbb{R}^3$. A folded state or folding $(f, \lambda)$ of $P$ consists of an intrinsically isometric geometry $f : P \to \mathbb{R}^3$ and an ordering $\lambda : L \to \{-1, +1\}$, where $L = \{(p, q) \in P \times P | p \neq q$ are noncrease points of $f$, and $f(p) = f(q)$}, satisfying Antisymmetry, Transitivity, Consistency, and Noncrossing conditions [6]. Here $\lambda(p, q) = +1$ means that point $p$ is stacked above $q$ in the

Table 1 Results: The number of flat folding steps of complexity $O(1)$, and the number of resulting creases, in our solutions to each of the four problem variants.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Allow Creases Through Points</th>
<th>Forbid Creases Through Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded</td>
<td>$O(n)$</td>
<td>$O(n^2 \log r)$</td>
</tr>
<tr>
<td>Unbounded</td>
<td>$O(n)$</td>
<td>$O(n^2 r') = O(n^4 r)$</td>
</tr>
</tbody>
</table>

*1 http://erikdemaine.org/fonts/foldpunch/
direction of the normal $n_f(q)$ of the folded surface at $q$, and $\lambda(p, q) = -1$ means that $p$ is stacked below $q$ (in the direction $-n_f(q)$).

**Definition 2.2** (Folding Step). A folding step consists of a before folded state $(f_0, \lambda_0)$ and an after folded state $(f_1, \lambda_1)$: we call the folding step from $(f_0, \lambda_0)$ to $(f_1, \lambda_1)$. Let $L_i$ denote the domain of $\lambda_i$, i.e., all pairs of noncrease points mapped together by $f_i$.

A folding step is flat if both $f_0(P)$ and $f_1(P)$ lie in the xy plane.

**Definition 2.3** (Folding Step Complexity). A folding step of complexity $k$ from $(f_0, \lambda_0)$ to $(f_1, \lambda_1)$ consists of a clustering $C_0, C_1, \ldots, C_k$ satisfying the following properties:

(C1) **Clusters partition paper**: $C_0 \cup C_1 \cup \ldots \cup C_k$.

(C2) **Cluster geometry**: On each cluster $C_i$, $f_0$ and $f_1$ differ by only a rigid motion $g_i$, i.e., $(f_i)_{|C_i} = g_i \circ (f_0)_{|C_i}$. Furthermore, $g_i$ is the identity map, i.e., $C_0$ does not move geometrically. As a consequence, $L_i \cap (C_i \times C_i) = L_i \cap (C_\overline{i} \times C_\overline{i})$ where $\overline{X}$ denotes the closure of $X$.

(C3) **Cluster layering**: For each cluster $C_i$, $\lambda_0$ and $\lambda_1$ agree on all points in $C_i$ where they are defined, i.e., $(\lambda_0)_{|L_i \cap (C_i \times C_i)} = (\lambda_1)_{|L_i \cap (C_i \times C_i)}$.

(C4) **Clusters don’t sandwich**: For each cluster $C_i$, for both folded states $(f_0, \lambda_0)$ and for any point pair $(p, q) \in L_j \cap (C_i \times C_i)$, there is no point $b \in P \setminus C_i$ between $p$ and $q$ according to $(f_0, \lambda_0)$, i.e., no noncrease point $b \in P \setminus C_i$ with $f_0(b) = f_0(p) = f_0(q)$ satisfying $\lambda_0(p, b) \neq \lambda_0(q, b)$.

Note that the number of clusters is 1 larger than the complexity, so that complexity 0 corresponds to no folding whatsoever. This definition of folding step complexity is nicely general, but it still allows the layering to be complex even in a folding step of complexity 1. **Figure 2** shows how we may “weave” folded shapes without increasing the complexity.

For example, a (some-layers) simple fold is a flat folding step of complexity 1 satisfying three additional conditions:

(S1) Rigid motion $g_1$ is not the identity, and thus is a reflection about a line $\ell_1$.

(S2) For some sign $\sigma \in \{+1, -1\}$, there are no collisions during a continuous $\sigma \cdot 180^\circ$ rotation of $C_1$ around $\ell_1$, or equivalently, $C_1$ is on the same side of $C_0$ before and after

the folding step: $\lambda_1(p, q) n_f(q) = \sigma \mathbf{e}_1$ for all $(p, q) \in L_j \cap (C_1 \times C_0)$ and for both $j \in \{0, 1\}$. (Here $\mathbf{e}_1$ represents the vector $(0, 0, +1)$.)

(S3) Every point of $C_0 \cap C_1$ is a crease point of $f_1$ or on the boundary of $P$.

This definition exactly matches the definition of simple fold in Ref. [1]. Without Property S3, we call such a folding step a (some-layers) simple fold unfold.

At the other extreme in complexity, any target folded state $(f_1, \lambda_1)$ can be viewed as a folding step from the unfolded piece of paper (given by geometry $f_0(p) = p$) to $(f_1, \lambda_1)$. By choosing the clusters to be the $k + 1$ faces of the crease pattern of $f_1$, the complexity of this step is $k$ (assuming $f_1$ is a rigid motion on each face of the crease pattern). We refer to $k$ as the face complexity of $(f_1, \lambda_1)$.

Next we define a common type of folding step that treats the result of a previous folding step as its “piece of paper,” and thus does not separate any previously collocated layers of paper:

**Definition 2.4** (All-layers Folding Step). Given a folded state $(f_0 : P \to \mathbb{R}^3, \lambda_0)$, whose image $f_0(P)$ is an orientable manifold $P$ (with normals $n_f$ defined by an arbitrarily chosen “top” side), and given a second folded state $(f_{0-1} : P \to \mathbb{R}^3, \lambda_{0-1})$, define the all-layers folding step from $(f_0, \lambda_0)$ to $(f_1, \lambda_1)$ by

$$f_1 = f_{0-1} \circ f_0,$$

$$\lambda_1(p, q) = \begin{cases} \lambda_0(p, q) & (f_0(p) = f_0(q)), \\ \lambda_{0-1}(f_0(p), f_0(q)) \cdot \sigma_f(q) & (f_0(p) \neq f_0(q)), \end{cases}$$

where $\sigma_f(q) = n_f(q) \cdot n_f(q)$ indicates whether $q$’s normal vector on $P$ as mapped by $f_0$ matches or is flipped relative to the normal vector of $f_0(q)$ on the coalesced surface $P'$. If $(f_0, \lambda_0)$ has face complexity $k$, then there is an all-layers folding step from $(f_0, \lambda_0)$ to $(f_1, \lambda_1)$ of complexity $k$.

For example, an all-layers simple fold is a simple fold that is an all-layers folding step for some $f_{0-1}$. This definition exactly matches the definition of all-layers simple fold in Ref. [1].

**Definition 2.5** (Folding Sequence). A folding sequence is a sequence $(f_0, \lambda_0), (f_1, \lambda_1), \ldots, (f_m, \lambda_m)$ of folded states and a sequence $s_0, s_1, \ldots, s_m$ of folding steps, where each $s_i$ is a folding step from $(f_i, \lambda_i)$ to $(f_{i+1}, \lambda_{i+1})$. The result of the folding sequence is $(f_m, \lambda_m)$. The folding sequence is complete if $(f_0, \lambda_0)$ is the unfolded piece of paper, given by geometry $f_0(p) = p$.

**Remark 2.6.** A folded state $(f, \lambda)$ can be the result of multiple different folding sequences. **Figure 3** shows an example of folding which can be reached by either (1) a sequence of one simple fold (complexity 1) followed by one all-layers flat folding step of complexity 5; or (2) one flat folding step of complexity 11.
3. Problem and Results

In this section, we formally state the 0-dimensional fold-and-cut problem and describe our results.

**Problem 3.1 (0-dimensional fold and cut).** Given \( n \) points \( p_1, p_2, \ldots, p_n \) on a flat piece of paper \( P \), find a flat folding \((f, \lambda)\) satisfying the following conditions:

\[
\begin{align*}
(F1) & \quad f(p_1) = f(p_2) = \cdots = f(p_n), \\
(F2) & \quad f(q) \neq f(p_i) \text{ for any } q \in P \setminus \{p_1, p_2, \ldots, p_n\}.
\end{align*}
\]

We have four variations of this problem based on the following two criteria:

- The paper \( P \) is bounded or unbounded.
  - **(Bounded)** The paper \( P = \mathbb{R}^2 \) is bounded.
  - **(Unbounded)** The paper \( P = \mathbb{R}^2 \) is the infinite plane\(^2\).
- We can allow or forbid crease lines passing through given points:
  - **(Allowing)** Allow crease lines passing through \( p_i \).
  - **(Forbidding)** Forbid crease lines passing through \( p_i \).

Our main results are summarized by the following theorem:

**Theorem 3.2.** Problem 3.1 is always solvable in all four cases above. The solution can always be given by a sequence of flat folding steps, each of complexity \( O(1) \). Furthermore, every folding step is either a some-layers simple fold or an all-layers folding step; and in the bounded-paper cases, every folding step is a some-layers simple fold. Table 1 bounds the number of steps and the number of resulting creases in the crease pattern in each case.

**Remark 3.3.** For Problem 3.1 with bounded paper, any shape of paper \( P \) is accepted. If we solve the problem for larger finite paper \( P' \) containing original paper \( P \), then we have a solution for the original paper \( P \). Thus our solution is free to choose the shape of \( P' \), e.g., to be convex (say, the convex hull of \( P \)) or a rectangle (say, the bounding box of \( P \)).

3.1 Key Ideas

We present our solutions to the four cases in order of increasing solution complexity: bounded & allowing case (Section 4), unbounded & allowing case (Section 5), bounded & forbidding case (Section 6), and unbounded & forbidding case (Section 7). The latter two sections can be read independently from the others, so to see the most general result and techniques, the reader may skip to Section 7.

In the bounded & allowing case (Section 4), unbounded & allowing case (Section 5), and bounded & forbidding case (Section 6), we use the following fundamental technique. We perform some folds in order to place \( p_1, p_2, \ldots, p_n \) on the same horizontal line such that each \( p_i \) is not covered by any other layers of paper. Then, by just folding bisecting vertical lines, we can easily obtain a flat folding satisfying Properties F1 and F2; refer to Fig. 4.

**Lemma 3.4 (Bisection Folding).** Let \( P \) be a piece of paper. If all \( p_i \)’s are on the same line \( \ell \), we can get such a folding satisfying Properties F1 and F2, without adding crease lines through \( p_i \). This construction results from a folding sequence of \( \Theta(n) \) some-layers simple folds, and works if \( P \) is bounded or unbounded.

\(^2\) \( \mathbb{R}^2 \) is the hardest case of any unbounded piece of paper: a solution to \( P = \mathbb{R}^2 \) can be applied to its subset.

![Fig. 4 Bisection folding to align collinear points.](image-url)
swapping \( p_i \) and \( p_j \), we have

\[
R_i = \sqrt{(x + d/2)^2 + y^2} = \sqrt{x^2 + y^2 + d^2/4 + xd},
\]

\[
R_j = \sqrt{(x - d/2)^2 + y^2} = \sqrt{x^2 + y^2 + d^2/4 - xd}.
\]

Dividing by \( D = \sqrt{x^2 + y^2 + d^2/4} = \Theta(L) \),

\[
\frac{R_i}{D} = 1 + \Theta\left(\frac{xd}{x^2 + y^2 + d^2/4}\right),
\]

\[
\frac{R_j}{D} = 1 - \Theta\left(\frac{xd}{x^2 + y^2 + d^2/4}\right).
\]

so

\[
\frac{R_i - R_j}{D} = \Theta\left(\frac{xd}{x^2 + y^2 + d^2/4}\right).
\]

Substituting various approximations, we obtain

\[
\frac{R_i - R_j}{\Theta(L)} = \Theta\left(\frac{\Omega(L/n^2)\Omega(L)}{\Theta(L^2) + O(L^2)}\right) = \Omega\left(\frac{\ell}{Ln^2}\right) = \Omega\left(\frac{1}{mn^2}\right).
\]

Taking the reciprocal, we have

\[
r' = \frac{\max_i R_i}{\min_i |R_i - R_j|} = \frac{\Theta(L)}{\min_i|R_i - R_j|} = O(rn^3)
\]
as desired. \( \square \)

Remark 3.9. Solutions to Problem 3.1 with unbounded paper can be applied to Problem 3.1 with bounded paper, and solutions to Problem 3.1 forbidding creases through points can be applied to Problem 3.1 allowing creases through points. Therefore, if we apply the solution of the unbounded & forbidding case to the bounded & forbidding case, we obtain the number of creases as \( O(rn^3) \), which is a better bound than \( O(rn^3r) \), but it does not consist of only simple folds.

4. Solution for Bounded Paper, Allowing Creases Through Points

Theorem 4.1. Let \( P \) be a bounded piece of paper and let \( p_1, p_2, \ldots, p_n \) be \( n \) distinct points on \( P \). Then there exists a flat folding satisfying Properties F1 and F2, allowing creases through \( \{p_i\} \). The number of creases is \( O(n^3) \). The flat folding is the result of a folding sequence of \( O(n) \) some-layers simple folds.

Proof. We can reduce to the case of Lemma 3.4 as follows. Rotate so that \( p_1, p_2, \ldots, p_n \) have distinct \( y \) coordinates \( y_1, y_2, \ldots, y_n \). Relabel the points so that \( y_1 > y_2 > \cdots > y_n \). If we fold the paper by horizontal mountain creases \( y = y_i \) for \( 1 \leq i \leq n \), and horizontal valley creases \( y = (y_i + y_{i+1})/2 \) for \( 1 \leq i \leq n - 1 \), then we will align the \( p_i \)'s onto a horizontal line, but other points
will fold to meet \( p_i \); see Fig. 6 (left). If we reverse-fold the mountain folds, turning horizontal crease \( y = (y_i + y_{i+1})/2 \) into a valley and adding two mountain crease lines at \( p_i \) with very small angle to the horizontal crease \( y = (y_i + y_{i+1})/2 \), then the creases will not intersect, so we obtain the desired result; see Fig. 6 (right). Now each \( p_i \) is not covered by any other points, so we can reduce to the case of Lemma 3.4.

We can implement essentially the same construction by a folding sequence of some-layers simple folds; refer to Fig. 7. For each \( i \), we fold the horizontal valley crease \( y = (y_{i-1} + y_i)/2 \) (one layer); then the horizontal mountain crease \( y = y_i \) (one layer); then each of two near-horizontal valley folds at \( p_i \) (two layers each). The valley folds need to be sufficiently close to horizontal to prevent additional overlap in future folds; in particular, it suffices to make the angle (between the valley folds and the horizontal crease \( y = (y_i + y_{i+1})/2 \)) at most half the angle from the reverse-fold method from Fig. 6 (which just needed to avoid intersecting creases). In Fig. 7, we fold the valley \( y = (y_{i-1} + y_i)/2 \) even for \( i = 1 \), letting \( y_0 \) be the top of the paper; and unfold this crease at the end of the sequence. Alternatively, we could skip the first valley fold so that each folding step is some-layers simple fold.

This method consists of \( \Theta(n) \) simple folds, each folding through 1 or 2 layers. Thus the number of creases is \( \Theta(n) \), and all creases are nearly horizontal (as in Fig. 6 (right), but with a different mountain–valley assignment). When we apply the \( \Theta(n) \) vertical bisection folds of Lemma 3.4, each of the horizontal folds gets split into \( \Theta(n) \) pieces, so we end up with a final crease pattern with \( \Theta(n^2) \) creases. The folding sequence still consists of \( \Theta(n) \) some-layers simple folds. \( \Box \)

5. Solution for Unbounded Paper, Allowing Creases Through Points

**Theorem 5.1.** Let \( P = \mathbb{R}^2 \) and let \( p_1, p_2, \ldots, p_n \) be \( n \) distinct points on \( P \). Then there exists a flat folding satisfying Properties F1 and F2, allowing creases through \( (p_i) \). The number of creases is \( \Theta(n^2) \). The flat folding is the result of a folding sequence of \( \Theta(n) \) flat folding steps of complexity \( O(1) \).

**Proof.** The difference from Theorem 4.1 is that we cannot fold the near-horizontal folds in Figs. 6 or 7 because the paper \( P \) is the plane \( \mathbb{R}^2 \), so any near-horizontal lines would intersect all horizontal folds. Instead, we use the thorn gadget shown in Fig. 8. If the thorn gadget is applied to a point, the paper folds into a plane (with horizontal and vertical pleats) with a little triangle tab sticking out. By first rotating the point set, we can assume that all \( x \) and \( y \) coordinates are distinct, and thus sufficiently small horizontal and vertical pleats in the thorn gadget will avoid interaction with any other points.

Figure 9 illustrates the overall construction. We alternately apply the thorn gadget (as an all-layers flat folding step), and the bisection between horizontal lines (similar to Theorem 4.1), sequentially from the top to down. The result is a plane with \( n \) horizontally aligned tabs, with the \( n \) points at the tips of the tabs. One horizontal mountain fold beneath the triangular tabs leaves...
the tips of the tabs (where the points are) uncovered by other layers of paper, and on the horizontal line. Finally, we can apply Lemma 3.4 to align the points.

This method consists of Θ(n) flat folding steps, each of complexity O(1): 1 for the simple folds and 18 for the thorn gadget of Fig 8 (which is all-layers). For each point pi, the thorn gadget makes horizontal and vertical pleats that cross the pleats of all previous gadgets. Thus each gadget makes O(n) creases, for a total of O(n^2) creases. Similarly, Lemma 3.4 results in Θ(n^2) additional creases because Θ(n) vertical creases will cross Θ(n) horizontal pleats. We conclude that the number of resulting creases is Θ(n^2).

6. Solution for Bounded Paper, Forbidding Creases Through Points

Theorem 6.1. Let P be a bounded piece of paper and let p1, p2, ..., pn be n distinct points on P. Then there exists a flat folding satisfying Properties F1 and F2, forbidding creases through (pi). The number of creases is O(n(r^2)). The flat folding is the result of a folding sequence of O(n log r') some-layers simple folds.

Lemma 6.2 (Radial shrinking folding). Consider three noncollinear points p0, q1, q2 on a convex piece of paper P, defining a wedge (and angle) wq1q0q2 from ray pq0q1 clockwise to ray pq0q2. Assume that p0 is not interior to P, so that removing the fan wq1q0q2 separates P into two components. Then, for any ε > 0, there exists a folding f of ℝ^2 satisfying

(R1) the angle w(f(q1))f(p0)f(q2) ≤ ε;

(R2) the set of points q of P nonstrictly counterclockwise of ray pq0q1 (including q1) move as a single rotation around p0 and are singly covered in f;

(R3) the set of points q of P nonstrictly clockwise of ray pq0q2 (including q1) move as a single rotation around p0 and are singly covered in f; and

(R4) f(q) does not change the distance to p0 of any q ∈ P.

Also, f decomposes into a sequence of O(log(wq1p0q2/ε)) simple folds, all passing through p0, and the total number of creases is O(wq1p0q2/ε) = O(1/ε).

Proof. Let w = wq1p0q2 - ε, and assume w > 0. (Otherwise, Property R1 holds without any folding.) Refer to Fig 10.

To get started, keep fixed the set of points q nonstrictly counterclockwise of ray pq0q1, valley fold along the ray from p0 that is ε/2 clockwise of ray pq0q1, and mountain fold along the ray bisecting the wedge wq1p0q2. We obtain a folded state f where f1(q1) = q1, f1(p0) = p0, and angle wq1p0f1(q2) = wq1p0q2 - w = ε.

This folded state f1 has a two-layer flap of angle w/2. By applying a simple fold to the layers of this flap, bisecting its angle, we may shrink the flap angle by a factor of 2, without moving the rest of the paper. After k such simple folds, we shrink the angle of the tuck to w/2^k+1. Thus, if we apply log(wq1p0q2/ε) + 1 simple folds (resulting in Θ(wq1p0q2/ε) creases), we can shrink the flap to width < ε/2, clearing any overlap with points of P outside the wedge wq1p0q2. Thus, in the final folded state f, we obtain the single coverage required by Properties R2 and R3, without affecting Property R1 already true of f1. The rest of Properties R2 and R3, and Property R4, follow because our folds are all rays through p0 and strictly between rays pq0q1 and pq0q2.

Proof of Theorem 6.1. We assume that r' is bounded, i.e., R_i ≠ R_j for all i ≠ j. For example, this property can be achieved by the translation in Lemma 3.8, which guarantees that r' = O(rn^2).

By relabeling the points, we arrange to have R_i < R_j for all i < j. Also, we compute the permutation of the points p1, p2, ..., pn (according to this radius order) into their clockwise order q1, q2, ..., qn around p0. In addition, let q0 be the counterclockwise-most point of P relative to p0, and let qn+1 be the clockwise-most point of P relative to p0. Thus q0, q1, ..., qn, qn+1 appear in clockwise order around p0.

The fold sequence consists of two parts. In the first part, we bring q0, q1, ..., qn+1 to be almost collinear, as follows. Divide the paper P into wedges with apex at p0 which have p_i’s on their boundaries (and have no p_i’s strictly interior to a wedge). For each 0 ≤ i ≤ n, we apply Lemma 6.2 to shrink the angle of the wedge wq0p0q1 down to ≤ θ/n, while only rotating the outer wedges. After these folds, all of the paper lies within a wedge with apex at p0 of angle ≤ θ, for desired θ > 0. We use
Similarly, points on \( P \). Then there exists a flat folding satisfying Proper-

Thus this part consists of \( O(n \log(n/\theta)) = O(n \log(n')) = O(n \log r') \), simple folds (because \( r' = \Omega(n) \) and \( O(nn/\theta) = O(n^2r') \) creases.

In the second part of the fold sequence, we bring the \( b_k \)'s to a common point by bisection folding steps. Refer to Fig. 11. For \( 1 \leq k < n \), let \( b_k \) be the perpendicular bisector of the segment \( p_kp_{k+1} \). It suffices to fold the paper along these \( b_k \)'s, assigning \( b_k \) as mountain for \( k \) odd and valley for \( k \) even. This part consists of \( O(n) \) simple folds, each of which can cross the \( O(n^2r') \) creases from the first part. Thus the total number of creases is \( O(n^2r') \).

Let \( X \) and \( Y \) be any points on the arcs of circles centered at \( p_0 \) and with radii \( R_i \) and \( R_{i+1} \) (i.e., passing through \( p_i \) and \( p_{i+1} \)), respectively. By the definition of \( \theta \),

\[
p_iX < R_{i+1} - R_i < R_{i+1}X
\]

Similarly,

\[
p_i+1Y < p_iY.
\]

Therefore, all perpendicular bisectors lie between adjacent arcs, and do not cross each other. This completes the proof.

\[\Box\]

7. Solution for Unbounded Paper, Forbidding Creases Through Points

Theorem 7.1. Let \( P = \mathbb{R}^2 \) and let \( p_1, p_2, \ldots, p_n \) be \( n \) distinct points on \( P \). Then there exists a flat folding satisfying Properties F1 and F2, forbidding creases through \( \{p_i\} \). The number of creases is \( O(n^{2}r^{3}) \). The flat folding is the result of a folding sequence of \( O(n \log r^{3}) \) folding steps of complexity \( O(1) \).

Lemma 7.2 (Parallel shrink folding). Consider points \( q_1, q_2 \in \mathbb{R}^2 \) with distinct \( y \)-coordinates \( y_1 > y_2 \). Then, for any \( \epsilon > 0 \), there exists a folding \( f \) of \( \mathbb{R}^2 \) satisfying

\[
(P1) \quad \left| (y \text{ coordinate of } f(q_1)) - (y \text{ coordinate of } f(q_2)) \right| \leq \epsilon;

(P2) \quad \text{the set of points } q \text{ with } y \text{ coordinate } \geq y_1 \text{ (including } q_1) \text{ move as a single translation and are singly covered in } f;

(P3) \quad \text{the set of points } q \text{ with } y \text{ coordinate } \leq y_2 \text{ (including } q_2) \text{ move as a single translation and are singly covered in } f; \text{ and}

(P4) \quad f(q) \text{ does not change the x coordinate of any } q \in \mathbb{R}^2.

Also, \( f \) decomposes into a sequence of \( O(\log (|y_1 - y_2|/\epsilon)) \) simple folds, all parallel to the x axis.

The proof of Lemma 7.2 is essentially the same as that of Lemma 6.2 after a projective transformation to place \( p_0 \) at infinity, but we repeat it for completeness.

Proof. Assume by induction that \( f(q_1) = q_1 \) and the y-coordinate of \( f(q_2) \) is \( y_2 + w = y_2 + \epsilon \).

This folded state \( f_1 \) has a two-layer flap of width \( w/2 \). By applying a simple fold to the layers of this flap in the middle \( y \)-coordinate, we may shrink the flap width by a factor of 2, without moving the rest of the paper. After \( k \) such simple folds, we shrink the width of the tuck to \( w/2^{k+1} \). Thus, if we apply \( \log (|y_1 - y_2|/\epsilon) + 1 \) simple folds, we can shrink the flap to width \( < \epsilon/2 \), clearing any overlap with the points \( q \) having \( y \)-coordinate \( \geq y_1 \) or \( \leq y_1 \). Thus, in the final folded state \( f \), we obtain the single coverage required by Properties P2 and P3, without affecting Property P1 already true of \( f_1 \). The rest of Properties P2 and P3, and Property P4, follow because our folds are all horizontal with \( y \) coordinates strictly between \( y_1 \) and \( y_2 \).

Proof of Theorem 7.1. Let \( x_i \) and \( y_i \) be the \( x \) and \( y \) coordinates of \( p_i \) respectively. By suitable rotation, assume that all \( x_i \)'s are distinct and that all \( y_i \)'s are distinct. By relabeling the points, assume \( y_1 > y_2 > \cdots > y_n \). Define \( g_{n+1} = -\infty \).

We apply an iterative method. Let \( p_i(k) = (x_i(k), y_i(k)) \) denote the position of \( p_i \) before the \( k \)th iteration. For each point \( p_i(k) \), we can uniquely define (although they may not exist) its left and right adjacent points in the projection of \( p_i(k), p_i(k), \ldots, p_i(k) \) to the \( x \)-axis; let \( i^-(k) \) and \( i^+(k) \) denote the index of the left and right adjacent points respectively.

Assume by induction that
Fig. 13 Downshifting gadget can be inserted if conditions (i)–(ii) are satisfied.

\[ y_1(k) = y_2(k) = \cdots = y_k(k) > y_{k+1}(k) = y_{k+2} \]
\[ > y_{k+2}(k) = y_{k+3} > \cdots > y_k = y_n, \]

and \( i'(k) = i'(0) \) and \( i''(k) = i''(0) \) for all \( i \). For an angle \( 0 < \theta < \pi/4 \), choose \( \varepsilon > 0 \) sufficiently small to satisfy the following conditions:

(i) \[
\frac{\varepsilon}{2 \tan \theta} < y_{k+1}(k) - y_{k+2}(k) = y_{k+1} - y_{k+2}.
\]

(ii) There is a sufficient horizontal gap between \( p_{k+1} \) and its left and right neighbors:

\[
\left( \frac{3}{2} + \frac{1}{2 \tan \theta} \right) \varepsilon < x_{k+1}(k) - x_{k+1'}(k)
\]
\[
\left( \frac{3}{2} + \frac{1}{2 \tan \theta} \right) \varepsilon < x_{k+1}(k) - x_{k+1}(k).
\]

By applying parallel shrink folding of Lemma 7.2, we can obtain that \( d(k) := y_1(k) - y_{k+1}(k) \leq \varepsilon \). Then we apply the downshifting gadget, a folding composed of two alternating twist folds of twist angle \( \theta \) and pleats width \( \frac{1}{3} d(k) \); see Fig. 13. The pleats of unfolded width \( \frac{1}{3} d(k) \) decompose the paper into six regions which remain single layered in the folding. The top three regions translate down by \( d(k) \); the left two regions translate right by \( d(k) \); and the right two regions translate left by \( d(k) \). Now, the downshifting gadget is constructed to weave between the given points, such that (1) the neighborhoods of \( p_1, p_2, \ldots, p_n \) are kept single layered, (2) already aligned points \( p_1, p_2, \ldots, p_k \) are in the top left and top right regions, (3) point \( p_{k+1} \) is in the bottom middle region, and (4) the rest of the points \( p_{k+2}, \ldots, p_n \) are in the bottom left and bottom right regions. Conditions (i)–(ii) guarantee the necessary clearance between points separated by the pleats. After folding the gadget, \( y_1, y_2, \ldots, y_k, y_{k+1} \) coincide, and each point of \( p_1, p_2, \ldots, p_n \) has no other points of paper on it.

Now we have
\[ y_1(k + 1) = y_2(k + 1) = \cdots = y_{k+1}(k + 1) > y_{k+2}(k + 1) = y_{k+2} \]
\[ > y_{k+3}(k + 1) = y_{k+3} > \cdots > y_k(k + 1) = y_k. \]

Because \( \left( \frac{1}{2} + \frac{1}{2 \tan \theta} \right) \varepsilon > \varepsilon \geq d(k) \), the horizontal ordering of points does not change, and thus \( i'(k + 1) = i' \) and \( i''(k + 1) = i''. \)

The horizontal gap shrinks by multiplying by a factor of at least \( \frac{1}{3 \tan \theta} < 1 \). By induction, the points \( p_1, p_2, \ldots, p_n \) become collinear after \( n - 1 \) iterations. Therefore we can apply the bisection fold of Lemma 3.4 to get the desired folding.

Before we count the number of resulting creases, we discuss how small \( y_k - y_{k+1} \) should be to apply the shrink fold \( f_i \) of Lemma 7.2. Let \( y'_i(k) \) denote the \( y \) coordinate of \( f_i(p_i(k)) \). For each step \( k \), we apply Lemma 7.2 so that \( y'_i(k) - y'_{i+1}(k) < \varepsilon \). To satisfy the conditions (i)–(ii) above for each \( k \), it suffices to satisfy the following:

\[
C \cdot \left( y'_i(k) - y'_{i+1}(k) \right) < \min \left( \min_{j \in k} |x_i - x_j|, \min_{k \neq n} |y_k - y_j| \right),
\]

where \( C \) is a constant satisfying the following:
\[
0 < C < \frac{1}{2 \tan \theta} < \frac{1}{2} + \frac{1}{2 \tan \theta}.
\]

Note that \( C \) depends only on \( \theta \). Here, we used that the horizontal displacement between horizontally adjacent points \( x_i \) and \( x_j \) can occur only twice in the whole sequence. This ensures that the right-hand sides of condition (ii) are bounded by the original gap scaled by a constant factor:
\[
x_i(m) - x_j(m) < \frac{1 + \tan \theta}{1 + 3 \tan \theta} (x_i - x_j),
\]
\[
x_i(m) - x_j(m) < \frac{1 + \tan \theta}{1 + 3 \tan \theta} (x_i - x_j).
\]

In Lemma 7.2, we make \( O(\log r') \) shrink folds. Then we have
\[
y'_i - y'_{i+1} \leq \frac{1}{C r'} (y_k - y_{k+1}).
\]

By the definition of \( r' \) in Eq. (2), Eq. (3) holds.

We divide the whole folding process into two parts: folding steps before applying Lemma 3.4 and folding steps in Lemma 3.4. The latter part consists of \( n - 1 \) vertical simple folds. The former part is divided into \( n - 1 \) inductive steps above. For each inductive step, we first apply Lemma 7.2 to make \( O(\log r') \) simple folds, resulting in \( O(r') \) total creases. Next we use the downshifting gadget, which consists of \( O(1) \) horizontal, vertical, and diagonal creases. The total number of folding steps is thus \( O(n \log r') \) for the former part plus \( O(n) \) for the latter part, for a total of \( O(n \log r') \). To bound the total number of resulting creases, we consider diagonal creases as both vertical and horizontal creases. Considering creases between horizontal and vertical creases, we conclude that the total number of resulting creases is \( O(n(n + m r')) = O(n^2 r'). \)

\[ \square \]

8. Open Problems

The natural open problems are to improve the bounds on the number of folding steps of complexity \( O(1) \), and the resulting number of creases, from Table 1. When allowing creases through
points, are $\Theta(n^2)$ creases necessary, or do $O(n)$ creases suffice, e.g., by trying to fold along the Voronoi diagram? When forbidding creases through points, is a dependence on $r$ necessary or is a polynomial dependence on $n$ possible? Can unbounded paper be solved by simple folds, or are folding steps of complexity $> 1$ necessary?

Another peculiar open problem is the worst-case ratio between $r$ and either $r'$ or $r''$. We proved that the ratio is $O(n^2)$, and simple examples (e.g., a regular $n$-gon) show that the ratio can be $\Omega(n)$. Is the tight bound one of these extremes, or something in between?

Finally, Fig. 2 illustrates that our definition of folding step complexity does not intuitively capture the complexity in the layering between clusters. Can we define a notion of layer complexity to measure this?

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References


Erik D. Demaine received a B.Sc. degree from Dalhousie University in 1995, and M.Math. and Ph.D. degrees from University of Waterloo in 1996 and 2001, respectively. Since 2001, he has been a professor in Computer Science at the Massachusetts Institute of Technology. His research interests range throughout algorithms, from data structures for improving web searches to the geometry of understanding how proteins fold to the computational difficulty of playing games. In 2003, he received a MacArthur Fellowship as a “computational geometer tackling and solving difficult problems related to folding and bending—moving readily between the theoretical and the playful, with a keen eye to revealing the former in the latter”. He cowrote a book about the theory of folding, together with Joseph O’Rourke (Geometric Folding Algorithms, 2007), and a book about the computational complexity of games, together with Robert Hearn (Games, Puzzles, and Computation, 2009). With his father Martin, his interests span the connections between mathematics and art.

Martin L. Demaine is an artist and computer scientist. He started the first private hot glass studio in Canada and has been called the father of Canadian glass. Since 2005, he has been the Angelika and Barton Weller Artist-in-Residence at the Massachusetts Institute of Technology. Martin works together with his son Erik in paper, glass, and other material. Their artistic work includes curved origami sculptures in the permanent collections of the Museum of Modern Art in New York, and the Renwick Gallery in the Smithsonian. Their scientific work includes over 60 published joint papers, including several about combining mathematics and art.

Hideaki Hosaka is a highschool teacher in mathematics at the Azabu Junior-High School. He received a Bachelor degree of Science and Master degree of Mathematical Sciences from the University of Tokyo in 2009 and 2011, respectively. His interest is algebraic representation theory and combinatorics.

Yasuhiko Asao is a Ph.D. student at Graduate School of Mathematical Sciences, the University of Tokyo. He is interested broadly in geometry and topology.
Akitoshi Kawamura received his Ph.D. in Computer Science from the University of Toronto in 2011. He then worked as research associate at the Department of Computer Science, the University of Tokyo, and since 2015 he is an assistant professor at the Department of General Systems Studies, the University of Tokyo. His research interests are in theory of computation, algorithm design, and discrete geometry.

Tomohiro Tachi is an assistant professor in Graphic and Computer Sciences at the University of Tokyo. He studied architecture and received his Ph.D. degree in Engineering from the University of Tokyo. He has been designing origami from 2002 and keeps exploring three-dimensional and kinematic origami through computation. He developed software tools including “rigid origami simulator”, “origamizer”, and “freeform origami”, which are available from his website. His research interests include origami, structural morphology, computational design, and fabrication.

Kazune Takahashi is a Ph.D. course student at Graduate School of Mathematical Sciences, the University of Tokyo. He studied pure mathematics and received his M.Sc. degree in Mathematical Sciences from the University of Tokyo in 2015. His research interests include elliptic partial differential equations, critical exponents and variational methods. He is strong at programming contests. He has won in a domestic AI competition “Code Runner 2015” and got a prize in an international AI competition “SamurAI Coding 2014-15”.

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