On a Probabilistic Approach for Capricious Vague Perceptions of Random Phenomena*

Tokuo FUKUDA†

In this paper, the author investigates a class of fuzzy random sets as the vague perception of a crisp phenomenon or a crisp random phenomenon. First, considering that the vague perception of a crisp phenomenon fluctuates slightly but randomly due to the state of a capricious person’s mind, a new class of fuzzy random set is introduced. Secondly, the proposed fuzzy random set is generalized as the vague perception of a crisp random phenomenon. The expectations of the introduced fuzzy random sets are investigated from the viewpoint of the multivalued logic proposed by Kwakernaak[1].

1. Introduction

Motivated by the importance for treating the data exhibiting both vagueness and randomness, fuzzy random sets or fuzzy random variables have been intensively investigated for a long time by many researchers with various definitions. For instance, the concept of fuzzy random variables obtained as vague linguistic observations of crisp random data was firstly presented by Kwakernaak[1], and investigated by e.g., Kruse[2,3]. On the other hand, Puri and Ralescu[4] defined firstly fuzzy random variables as the generalized random sets and discussed their statistical properties by many researchers, e.g.,[5–7].

The purpose of this paper is to reconstruct fuzzy random sets as a model of the vague perceptions obtained from capricious persons, who may expresses the different feelings from one person to another with the same crisp observed values. For this purpose, a new class of fuzzy random sets is firstly introduced, considering that the vague perception of a crisp phenomenon fluctuates slightly but randomly due to the state of a capricious person’s mind. Secondly, the proposed fuzzy random set is generalized as the vague perception of a crisp random phenomenon. The expectations of the proposed fuzzy random sets are also investigated from the viewpoint of the multivalued logic proposed by Kwakernaak[1].

2. Set Representation Approach

In this paper, a fuzzy set \( \tilde{U} \) as the vague perception of \( u_0 \in \mathbb{R}^n \) is defined by the triple

\[
\tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\tilde{U}})
\]

with

\[
[\tilde{U}]_\alpha = \{ [\tilde{U}]_\alpha \in \mathcal{P}_0(\mathbb{R}^n) \mid \alpha \in I \},
\]

where \( I \) is the open interval between 0 and 1, i.e., \( I = (0,1) \); \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space called the basic space; \( s_{\tilde{U}} \) is the predicate, i.e., \( s_{\tilde{U}} : \mathbb{R}^n \rightarrow \mathcal{S} \) with \( \mathcal{S} \) the “universe of discourse” defined by a set of statements, which assigns a proposition

\[
s_{\tilde{U}}(u) = \{ u \text{ coincides with } u_0 \}
\]

to each element \( u \in \mathbb{R}^n \); and \([\tilde{U}]_\alpha \) is the family of subsets of \( \mathbb{R}^n \) satisfying

\[
L_\alpha \tilde{U} \subseteq [\tilde{U}]_\alpha \subseteq L_{\alpha} \tilde{U} \quad \text{for any } \alpha \in I,
\]

where \( L_\alpha \tilde{U} \) and \( L_{\alpha} \tilde{U} \) are the strong cut (strong level set) and the level set of \( \tilde{U} \) at the level \( \alpha \) defined respectively by

\[
L_\alpha \tilde{U} = \left\{ u \mid (u \in \mathbb{R}^n) \land ((\tilde{U})(u) > \alpha) \right\}
\]
\[
L_{\alpha} \tilde{U} = \left\{ u \mid (u \in \mathbb{R}^n) \land ((\tilde{U})(u) \geq \alpha) \right\}
\]

where \((\tilde{U})(u)\) is the membership function of \( \tilde{U} \) given by

\[
(\tilde{U})(u) = t(s_{\tilde{U}}(u))
\]

and \( t(\ast) \) in eq. (6) is the truth function of \( \ast \) in the sense of multivalued logic. The crisp point \( u_0 \in \mathbb{R}^n \), the vague perception of which gives the fuzzy set \( \tilde{U} \), is called the original point of \( \tilde{U} \).

In this paper, we restrict our attention to the following family of fuzzy sets denoted by \( \mathbb{F}_{cc}(\mathbb{R}^n) \), whose element \( \tilde{U} = (\mathbb{R}^n, [\tilde{U}], s_{\tilde{U}}) \) satisfies the following conditions:

(1) Any element \([\tilde{U}]_\alpha\) of the set representation \([\tilde{U}]\) of a fuzzy set \( \tilde{U} \) satisfies...
The quantity $\rho_p(\bar{U}, \bar{V})$ for any $\bar{U}, \bar{V} \in \mathbb{F}_{cc}^b(\mathbb{R}^n)$ is defined by

$$\rho_p(\bar{U}, \bar{V}) = \left( \int |\bar{s}(\bar{U}, \alpha, x) - \bar{s}(\bar{V}, \alpha, x)|^p \right)^{1/p}$$

and we can show that $\rho_p$ is a metric on $\mathbb{F}_{cc}^b(\mathbb{R}^n)$.\[8\]

### 3. Capricious Vague Perceptions of Crisp Phenomena

In this section, we consider that the vague perception of a crisp phenomenon fluctuates slightly but randomly by the state of a capricious person’s mind. Hence, the fuzzy set obtained as a vague perception of a crisp phenomenon may be some kind of ‘fuzzy random set’, i.e., it is a function of the generating point of some sample space.

**Example 1** Consider the situation that a person feels the temperature, the humidity and the strength of wind simultaneously (all of those values are assumed to be fixed), and he expresses his feeling like “dry”, “comfortable”, “sultry”, “humid” and so on.

The linguistic data in Example 1 has two outstanding features. One is fuzziness due to the intrinsic vagueness of words, and another is the randomness caused by the capricious person’s feelings, i.e., with same combination of the temperature, the humidity and the strength of wind, some person will feel “humid” and another person will feel “comfortable”. Furthermore, according to the persons physical condition, even same person sometimes feels “comfortable” and sometimes feels “sultry”. This means that the vague perception would be changed randomly along with the state of capricious person’s feelings.

Secondly, we consider more complex example.

**Example 2** Consider the situation of evaluating the economic condition for forecasting the businesses activities, and it is expressed like “good”, “bad”, “neutral”, “no so bad” and so on.

It is needless to say that the economic condition is evaluated based on various economic indices such as stock prices, exchange rates of money, the unemployment rate, etc. However, each economic index shows only one aspect of the highly complex economic system and it is often observed that at first glance, some of the indices seem to show the contradictory values each other. Therefore, deliberating the whole aspect of the economic system indicated by various indices, we have to evaluate the economic state and have to express it like “good”, “bad”, “neutral”, “no so bad” and so on. However, even if the professionals evaluate the economic conditions, its decision will fluctuate slightly but randomly one by one due to the complexity of economic conditions.

In order to define a fuzzy random set within the framework mentioned above, the elementary discrete
fuzzy random set is defined as follows.

**Definition 1.** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be an elementary probability space, where \(\Omega = \{\omega_1, \omega_2, \ldots, \omega_M\}\) and \(\mathcal{A}\) be a \(\sigma\)-algebra given by the subsets of \(\Omega\). Then, an elementary fuzzy random set \(\tilde{U}(u_0, \omega) \in \mathbb{F}^n_c(\mathbb{R}^n)\) as a capricious vague perception of the crisp vector \(u_0 \in \mathbb{R}^n\) is defined as follows:

\[
\tilde{U}(u_0, \omega) = \sum_{i=1}^{M} \omega_i(\omega) u_i,
\]

where \(\omega_i(\omega)\) is the characteristic function of \(\omega_i\), i.e., \(\omega_i(\omega) = 1\) if \(\omega = \omega_i\) and otherwise 0; and \(\tilde{U}_i\) is the fuzzy set given by the triple \(\tilde{U}_i = (\mathbb{R}^n, [\tilde{U}_i], s_{\tilde{U}_i}) \in \mathbb{F}^n_c(\mathbb{R}^n)\)

with

\[
\tilde{U}_i = \left\{ [\tilde{U}_i]_{\alpha} | \alpha \in \Omega \right\} \text{ for } i = 1, 2, \ldots, M.
\]

The predicate \(s_{\tilde{U}_i}\) in eq. (16) is given through the proposition

\[
s_{\tilde{U}_i}(u) = \left\{ u \text{ is an original point } u_0 \right\},
\]

where \(u\) is a realization of the random vector \(u(\omega)\) given by

\[
u(\omega) = \sum_{i=1}^{M} \omega_i(\omega) u_i \quad (u_i \in \mathbb{R}^n)
\]
defined on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\).

As mentioned in Section 2, the crisp vector \(u_0 \in \mathbb{R}^n\) perceived vaguely by a fuzzy random set \(\tilde{U}(u_0, \omega)\) is called the original point of \(\tilde{U}(u_0, \omega)\). The random vector \(u(\omega)\) defined by eq. (19) is called as the (random) possible original point of \(u_0\), whose admissible class is given by

\[
\mathbb{U} = \left\{ u | u(\omega) = \sum_{i=1}^{M} \omega_i(\omega) u_i; u_i \in \mathbb{R}^n \right\}.
\]

The measurability of \(\tilde{U}(u_0, \omega)\) is given through its \(\mathcal{A} - \mathcal{B}\) measurability, i.e.,

\[
\tilde{U}^{-1}(u_0, \omega)(\tilde{B}) \in \mathcal{A} \quad \text{for any } \tilde{B} \in \mathbb{B},
\]

where \(\mathbb{B}\) is a \(\sigma\)-algebra generated by the subsets of \(\mathbb{F} = \{\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_M\}\).

According to the Kwakernaak’s approach [1, 2], the fuzzy random set variable may be given by \(\tilde{U} = (\mathbb{R}, (\tilde{U})(u), s_{\tilde{U}})\), and the measurability of \(\tilde{U}\) is given through those of \(\inf L_{\mathcal{F}} \tilde{U}\) and \(\sup L_{\mathcal{F}} \tilde{U}\) for all \(\alpha \in \Omega\). However, it should be noted that in Kwakernaak approach, there is no consideration about capricious property of persons’ minds, and only one-dimensional fuzzy random variables are possible to investigate.

### 4. Expectation of \(\tilde{U}(u_0, \omega)\)

Since \(\tilde{U}(u_0, \omega)\) is a some kind of random quantities, it should be possible to consider its statistical moments like its expectation, its variance and so on. In this paper, only the expectation of \(\tilde{U}(u_0, \omega)\) is discussed. Consider the proposition given by

\[
s_{\mathcal{E}[\tilde{U}]}(x) = \left\{ (x = E\{u\}) \text{ and } (u(\omega) = u_0) \right\}
\]

for some \(u \in \Omega\) with

\[
E\{u\} = \sum_{i=1}^{M} u_i P(u_0, \omega_i)
\]

where \(P(u_0, \omega_i)\) is the probability of \(u = u_i\). Then, the above proposition may be given by a composite one such as

\[
s_{\mathcal{E}[\tilde{U}]}(x) = \bigvee_{u \in \Omega} \left\{ (x = E\{u\}) \text{ and } (u(\omega) = u_0) \right\}
\]

\[
= \bigvee_{u \in \Omega} \left\{ (x = \sum_{i=1}^{M} u_i P(u_0, \omega_i)) \wedge \left( \bigwedge_{i=1}^{M} (u_i = u_0) \right) \right\}.
\]

Therefore, applying the concept of the extension principle for fuzzy sets proposed by Zadeh [9–11], the truth value of \(s_{\mathcal{E}[\tilde{U}]}(x)\) is given by

\[
\max \left\{ \left. s_{\mathcal{E}[\tilde{U}]}(x) \right| x = E\{u\} \right\}
\]

\[
= \sup_{u \in \Omega} \left\{ \left. \left( \left( x = \sum_{i=1}^{M} u_i P(u_0, \omega_i) \right) \wedge \left( \bigwedge_{i=1}^{M} (u_i = u_0) \right) \right) \right| \right\}
\]

\[
= \sup_{u \in \Omega} \left\{ \min_{i=1, 2, \ldots, M} \left( s_{\tilde{U}_i}(u_i) \right) \right\} x = E\{u\},
\]

where \(s_{\tilde{U}_i}(u_i)\) is given by eq. (18), i.e., the proposition of the fuzzy set \(\tilde{U}_i\) with a possible original point \(u_i\) of \(\tilde{U}_i\), and hence, its truth value is given by the membership function \(\tilde{U}_i(u_i)\). Then, we can confirm the following property holds:

**Proposition 1.** Let \(\tilde{U}(u_0, \omega)\) be an elementary fuzzy random set. Then,

\[
\mathbb{E}\{\tilde{U}(u_0, \omega)\} = \sum_{i=1}^{M} \tilde{U}_i(\alpha) P(u_0, \omega_i).
\]

The summation and the scalar multiplication in eq. (26) are given by those of Minkowski.

(Proof) It is clear from eq. (4) that
for each \( \alpha \in \mathbb{I} \). Then, we have that
\[
\sum_{i=1}^{M} L_{\alpha} \bar{U}_i \cdot P(u_{0}, \omega_i) \subseteq \sum_{i=1}^{M} [\bar{U}_i]_{\alpha} \cdot P(u_{0}, \omega_i)
\]
which means
\[
E[L_{\alpha} \bar{U}(u_{0}, \omega)] \subseteq E[[\bar{U}(u_{0}, \omega)]_{\alpha}] \subseteq E[L_{\bar{U}}(u_{0}, \omega)].
\]

(i) If there exists a vector \( \zeta \) such that
\[
\zeta \in \left\{ x \Big| \min_{i=1,2,\ldots,M} \left\{ (\bar{U}_i)(u_i) \right\} \geq \alpha \right\}
\]
there is some element \( \xi \) of \( \mathcal{U} \) satisfying
\[
(\bar{U}_i)(\xi_i) = \zeta \quad \text{for any } i = 1,2,\ldots,M.
\]
Hence, we have
\[
\zeta = \sum_{i=1}^{M} \xi_i P(u_{0}, \omega_i)
\]
and
\[
U_i(\xi_i) > \alpha \quad \text{for any } i = 1,2,\ldots,M,
\]
which means
\[
\xi_i \in L_{\alpha} U_i \quad \text{for any } i = 1,2,\ldots,M.
\]

(ii) If there exists \( \zeta \) such that
\[
\zeta \in \left\{ x \Big| \min_{i=1,2,\ldots,M} \left\{ (\bar{U}_i)(u_i) \right\} \geq \alpha \right\}
\]
for any \( \alpha \in \mathbb{I} \),
\[
\zeta \in \left\{ x \Big| \sup_{i=1,2,\ldots,M} \left\{ (\bar{U}_i)(u_i) \right\} \geq \alpha \right\}
\]
there is some element \( \xi \) of \( \mathcal{U} \) satisfying \( \zeta = E(\xi) = \sum_{i=1}^{M} \xi_i P(u_{0}, \omega_i) \) and
\[
(\bar{U}_i)(\xi) \geq \alpha \quad \text{for any } i = 1,2,\ldots,M,
\]
which means
\[
\min_{i=1,2,\ldots,M} \left\{ (\bar{U}_i)(u_i) \right\} \geq \alpha,
\]
and hence it follows that
\[
\zeta \in \left\{ x \Big| \sup_{i=1,2,\ldots,M} \left\{ (\bar{U}_i)(u_i) \right\} \geq \alpha \right\}
\]
or equivalently
\[
E[L_{\alpha} \bar{U}(u_{0}, \omega)] \subseteq \left\{ x \Big| \left( s_{\bar{U}}(\xi) \right) \geq \alpha \right\},
\]

(iii) From eq. (27), eq. (30) and eq. (32), we have eq. (25).

Therefore, the following definition of the expectation of an elementary fuzzy random set \( \bar{U}(u_{0}, \omega) \) may be reasonable:

**Definition 2**  Let \( \bar{U}(u_{0}, \omega) \) be an elementary fuzzy random set defined by eq. (15). Then, the expectation of \( \bar{U} \) is given by
\[
E[\bar{U}]_{u_{0}} = \left( \mathbb{R}^{n}, [E[\bar{U}]_{u_{0}}, \alpha_{\bar{U}}(\bar{U})] \right)
\]
with its set representation
\[
[E[\bar{U}]_{u_{0}}]_{\alpha} = \left\{ [E[\bar{U}]_{u_{0}}]_{\alpha} \Big| \alpha \in \mathbb{I} \right\}
\]
and
\[
[E[\bar{U}]_{u_{0}}]_{\alpha} = E[[\bar{U}(u_{0}, \omega)]_{u_{0}}]
\]
\[
= \sum_{i=1}^{M} [\bar{U}_i]_{\alpha} \cdot P(u_{0}, \omega_i),
\]
where \( s_{\bar{U}}(\xi) \) in eq. (33) is the predicate associated with the proposition given by eq. (21).

The following corollary is obtained immediately from eq. (27), eq. (30) and eq. (32) in the proof of Proposition 1.

**Corollary 1**  Let \( \bar{U}(u_{0}, \omega) \) be an elementary fuzzy random set given by eq. (15). Then we have
\[
L_{\alpha} \left( E[\bar{U}]_{u_{0}} \right) \subseteq E[L_{\alpha} \bar{U}(u_{0}, \omega)] \subseteq \left[ E[\bar{U}]_{u_{0}} \right]_{\bar{U}}
\]
\[
= E[[\bar{U}(u_{0}, \omega)]_{u_{0}}] \subseteq E[L_{\bar{U}}(u_{0}, \omega)]
\]
\[
\subseteq L_{\bar{U}} \left( E[\bar{U}]_{u_{0}} \right)
\]
for any \( \alpha \in \mathbb{I} \),
\[
L_{\alpha} \left( E[\bar{U}]_{u_{0}} \right) = \left\{ x \Big| \left( s_{\bar{U}}(\xi) \right) > \alpha \right\}
\]
and
\[
L_{\alpha} \left( E[\bar{U}]_{u_{0}} \right) = \left\{ x \Big| \left( s_{\bar{U}}(\xi) \right) \geq \alpha \right\}
\]
The following proposition is immediately obtained from Theorem 2.21 of Maruyama[12]:

**Proposition 2**  Let \( \bar{U}(u_{0}, \omega) \) be an elementary fuzzy random set given by eq. (15). Then, the expec-
5. Capricious Vague Perceptions of Random Phenomena

There are many crisp phenomena that are perceived capriciously and vaguely as mentioned in Section 3, and also there are many crisp phenomena which are themselves randomly changed. For instance, the quantities like as the temperature, the humidity and the wind strength in Example 1 may fluctuate randomly due to the weather conditions. Furthermore, the indices in Example 2 concerned with economic conditions also should fluctuate randomly. Therefore, in order to consider the capricious vague perceptions of random phenomena, the two types of randomness should be considered, one of which is the randomness due to the capricious person’s feelings mentioned in the previous section, and another of which is the randomness stemmed from the phenomena itself.

Let \((\Omega_1,A_1,P_1)\) be an elementary probability space describing the randomness of capricious persons’ minds defined as \((\Omega,A,P)\) in Definition 1, and let \((\Omega_2,A_2,P_2)\) be a complete probability space, on which a random vector \(u_o\) as a model of random phenomena is defined. Then, the fuzzy random set as a capricious vague perception of the random vector \(u_o\) is defined on \((\Omega,A,P)\) and \((\Omega_2,A_2,P_2)\) and given as follows:

**Definition 3** A fuzzy random set \(\tilde{U}(\omega)\) on \((\Omega,A,P)\) obtained as a capricious vague perception of an ordinary random vector \(u_o(\omega^{(2)})\) is defined by

\[
\tilde{U}(\omega) = \left( R^n, \tilde{U}(\omega), s_{\tilde{U}} \right) \in \mathbb{F}^{b}_{cc}(R^n)
\]  

(40)

with

\[
[\tilde{U}(\omega)]_\alpha = \left\{ [\tilde{U}(\omega)]_\alpha \right\},
\]  

(41)

where \(s_{\tilde{U}}\) is the predicate associated with the proposition such as

\[
s_{\tilde{U}}(u) = \left\{ u \text{ is an original random vector of } u_o \right\}.
\]  

(42)

Then, we can rewrite \(\tilde{U}(\omega)\) in eq. (40) by

\[
\tilde{U}(\omega) = \sum_{i=1}^{M} 1_{\omega_i(1)}(\omega) \tilde{U}_i,
\]  

(43)

where \(\{\tilde{U}_i; i=1,2,\cdots,M\}\) is a collection of fuzzy sets given by eq. (16), and

\[
1_{\omega_i(1)}(\omega) = \begin{cases} 
1 & \text{if } \omega \in \{\omega_i(1)\} \times \Omega_2 \\
0 & \text{otherwise.}
\end{cases}
\]  

(44)

Hence, we have the relation between the predicates such that

\[
s_{\tilde{U}} = \sum_{i=1}^{M} 1_{\omega_i(1)}(\omega)s_{\tilde{U}_i},
\]  

(45)

or equivalently

\[
s_{\tilde{U}}(u) = \sum_{i=1}^{M} 1_{\omega_i(1)}(\omega)s_{\tilde{U}_i}(u).
\]  

(46)

The measurability of \(\tilde{U}(\omega)\) is given through

\[
\tilde{U}^{-1}(\tilde{B}) \in A_1 \otimes A_2 \text{ for any } \tilde{B} \in \mathbb{B}.
\]  

(47)

6. Expectation of \(\tilde{U}(\omega)\)

As described in Section 5, a vague perception of the crisp random vector \(u_o(\omega^{(2)})\) called the random original point is modeled by the fuzzy random set \(\tilde{U}(\omega)\) given by eq. (40).

In order to explore the reasonable defining method for its expectation, consider first the proposition given by

\[
s_{E[\tilde{U}]}(x) = \left\{ (x = E\{u\}) \text{ and } (u = u_o \text{ for some } u \in U_e) \right\},
\]  

(48)

where the admissible class of \(u\) is given by

\[
U_e = \left\{ u \in u(\omega) = \sum_{i=1}^{M} 1_{\omega_i(1)}(\omega)u_i(\omega^{(2)}); \right\}
\]  

(49)

Then, the above proposition may be given by a composite one such as

\[
s_{E[\tilde{U}]}(x) = \bigvee_{\xi \in U_e} \left\{ (x = E\{\xi\}) \land (\xi(\omega) = u_o(\omega^{(2)})) \right\},
\]  

(50)

as on \((\Omega,A,P)\).

Furthermore, the statement \(\xi = u_o\) a.s. on \((\Omega,A,P)\)” in eq. (50) is given by

\[
\left( \xi(\omega) = u_o(\omega^{(2)}) \right) \text{ a.s. on } (\Omega,A,P)
\]  

\[
= \bigwedge_{i=1}^{M} \bigwedge_{\omega^{(1)} \in \Omega_2 \setminus N_0} s_{\tilde{U}_i}(\xi),
\]  

(51)

where \(N_0 \in A_2\) is the null set, i.e., \(P(\omega^{(1)} \times N_0) = 0\) for each \(i = 1,2,\cdots,M\); and the proposition \(s_{\tilde{U}_i}(\xi)\) is given by eq. (18). Then, eq. (50) is rewritten by

\[
s_{E[\tilde{U}]}(x) = \bigvee_{\xi \in U_e} \left\{ (x = E\{\xi\}) \land \left( \bigwedge_{i=1}^{M} \bigwedge_{\omega^{(1)} \in \Omega_2 \setminus N_0} s_{\tilde{U}_i}(\xi) \right) \right\},
\]  

(52)

Therefore, applying the concept of the extension prin-
ciple for fuzzy sets proposed by Zadeh[9–11], it may be considered that the truth value of \( s_{\xi|\[\bar{\Omega}\]}(x) \) is given by
\[
l(t(s_{\xi|\[\bar{\Omega}\]}(x)) = \sup_{\xi \in \Omega_{\xi}} \left\{ \min_{i=1,2,\ldots,M} \left\{ \inf_{\omega(2) \in \Omega_{\omega(2)}} \left( \bar{U}_i \right) (\xi) \right\} \left| \neg x = E[\xi] \right\} \right\},
\]
where \( \inf_{\omega(2) \in \Omega_{\omega(2)}} (\bar{U}_i)(\xi) \) is given by the supremum of a satisfying \((\bar{U}_i)(\xi) \geq \alpha \) except for \( \omega(2) \in \Omega_{\omega(2)} \). Then, we can confirm the following property holds:

**Proposition 3** Let \( \bar{U}(\omega) \) be a fuzzy random set given by eq. (40). Then,
\[
\left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \subseteq E[\[\bar{\Omega}\]|\alpha] \subseteq \left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \right\}
\]
for any \( \alpha \in \Omega \), where \( E[\[\bar{\Omega}\]|\alpha] \) is given by
\[
E[\[\bar{\Omega}\]|\alpha] = \int_{\Omega_{\bar{\Omega}}} \left\{ \sum_{i=1}^{M} [\bar{U}_i]|\alpha \right\} P(\omega(1),d\omega(2)).
\]

(Proof) It is clear from eq. (4) that
\[
L_\alpha \bar{U}_i \subseteq [\bar{\Omega}]_\alpha \subseteq L_\alpha \bar{\Omega}
\]
for each \( \alpha \in \Omega \). Then, we have that
\[
\int_{\Omega_{\bar{\Omega}}} \sum_{i=1}^{M} (L_a \bar{U}_i) P(\omega(1),d\omega(2)) \\
\leq \int_{\Omega_{\bar{\Omega}}} \sum_{i=1,2,\ldots,M} \left\{ \inf_{\omega(2) \in \Omega_{\omega(2)}} \left( U_i \right)(\xi) \right\} \left| \neg x = E[\xi] \right\} \left\{ x \right\}
\]

or equivalently
\[
E[L_\alpha \bar{U}] \subseteq E[\[\bar{\Omega}\]|\alpha] \subseteq E[L_\alpha \bar{\Omega}].
\]

(i) If there exists a vector \( \zeta \) such that
\[
\zeta = \left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \right\}
\]
there is some element \( \xi \) of \( \Omega_{\xi} \) satisfying \( \zeta = E(\xi) = \sum_{i=1}^{M} \xi_i \) and
\[
\xi_i \in L_\alpha \bar{U}_i \quad \text{a.s. on} \quad (\Omega_2,A_2,P_2)
\]
for each \( i = 1,2,\ldots,M \). Then, it follows
\[
\zeta = E(\xi) = \int \xi dP = \int \sum_{i=1}^{M} \xi_i P(\omega(1),d\omega(2)) \\
\in \int \sum_{i=1}^{M} (L_\alpha \bar{U}_i) P(\omega(1),d\omega(2)) = E[L_\alpha \bar{U}],
\]
or equivalently we can conclude that
\[
\left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \subseteq E[\[\bar{\Omega}\]|\alpha] \subseteq E[L_\alpha \bar{\Omega}] \quad \text{for} \quad \alpha \in \Omega.
\]

(ii) If there exists \( \xi \) such that
\[
\zeta = \left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \right\}
\]
for each \( i = 1,2,\ldots,M \). The eq. (58) implies that there exists \( \xi \in \Omega_{\xi} \) satisfying \( \zeta = E(\xi) \) and
\[
\min_{i=1,2,\ldots,M} \left\{ \sup_{\omega(2) \in \Omega_{\omega(2)}} (\bar{U}_i)(\xi) \right\} \geq \alpha,
\]
which means that
\[
\zeta \in \left\{ x \left| \sup_{\xi \in \Omega_{\xi}} \left\{ \min_{i=1,2,\ldots,M} \left\{ \sup_{\omega(2) \in \Omega_{\omega(2)}} (\bar{U}_i)(\xi) \right\} \right\} \left| E(\xi) = x \right\} \right\} \geq \alpha
\]
or equivalently
\[
\int L_\alpha \bar{U} dP = E[L_\alpha \bar{\Omega}] \subseteq \left\{ x \left| t(s_{\xi|\[\bar{\Omega}\]}(x)) > \alpha \right\} \right\}
\]
for \( \alpha \in \Omega \).

(iii) From eq. (55), eq. (57) and eq. (59), we have eq. (52).
Therefore, the following definition of the expectation of a fuzzy random set \( \bar{U} \) may be reasonable:

**Definition 4** Let \( \bar{U}(\omega) \) be a fuzzy random set given by eq. (40). Then, the expectation of \( \bar{U} \) is given by
\[
E[\[\bar{\Omega}\]] = \left( \mathbb{R}^n, [E[\bar{\Omega}],s_{\bar{\Omega}}] \right)
\]
with
\[
[E[\[\bar{\Omega}\]]] = \left\{ E[[\bar{\Omega}]|\alpha] \right\}_{\alpha \in \Omega}
\]
where \( s_{\bar{\Omega}} \) is the predicate associated with the proposition given by eq. (48); and \( [E[\bar{\Omega}]] \) is the set represen-
tation of $\mathcal{E}[^{\tilde{U}}]$ given through eq. (53).

The following corollary is obtained immediately from eq. (55), eq. (57) and eq. (59) in the proof of Proposition 3.

**Corollary 2** Let $\tilde{U}$ be a fuzzy random set given by eq. (40). Then we have

$$L_\alpha \mathcal{E}[^{\tilde{U}}] \subseteq E[L_\alpha ^{^{\tilde{U}}} \subseteq E[E[^{\tilde{U}}]_\alpha] \subseteq E[L_\alpha ^{^{\tilde{U}}}] \subseteq L_\alpha \mathcal{E}[^{\tilde{U}}]$$

(62)

for any $\alpha \in I$, where

$$L_\alpha \mathcal{E}[^{\tilde{U}}] = \left\{ x \mid t(s_{\mathcal{E}[^{\tilde{U}}]}(x)) > \alpha \right\} \text{ for } \alpha \in I \tag{63}$$

and

$$L_\alpha \mathcal{E}[^{\tilde{U}}] = \left\{ x \mid t(s_{\mathcal{E}[^{\tilde{U}}]}(x)) \geq \alpha \right\} \text{ for } \alpha \in I. \tag{64}$$

### 7. Concluding Remarks

In this paper, the author has proposed a probabilistic approach for representing the vague perception obtained from capricious persons, where considering that the vague perception of a crisp phenomenon fluctuates slightly but randomly due to the state of a capricious person’s mind, a new class of fuzzy random sets and their expectations have been firstly introduced.

Secondly, the introduced class of fuzzy random sets have been generalized for the capricious vague perception of crisp random phenomena. The expectations of the generalized fuzzy random sets have also introduced from the viewpoint of the multivalued logic.

Let here $\mathcal{S}$ be the sub $\sigma$-algebra of $\mathcal{A}$ consisting all cylinder sets of the form $A = \Omega_1 \times A^{(2)}$ with $A^{(2)} \in \mathcal{A}_2$. Then, the conditional expectation of $\tilde{U}$ concerned with $\mathcal{S}$ should be given as follows:

$$E[^{\tilde{U}} | \mathcal{S}] = \left( \mathbb{R}^n, [E[^{\tilde{U}} | \mathcal{S}]], s_{\mathcal{E}[^{\tilde{U}} | \mathcal{S}]} \right)$$

with

$$[E[^{\tilde{U}} | \mathcal{S}]] = \left\{ E[^{\tilde{U}}]_I | \sigma^I \right\} \text{ if } I \in \mathcal{S}.$$

The detailed aspect of the conditional expectation of $\tilde{U}$ will be reported in the near future.

The possible application fields of the fuzzy random sets proposed in this paper may be the system modeling with vague random data including complex social systems like economic systems mentioned in Example 2 in Section 3, management systems and so on, where many vague random data have been used without any evaluation of their vagueness. Furthermore, as a direct application of fuzzy random sets, the preliminary result has already reported for vague data analysis of some questionnaire [13].

### References


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**Author**

Tokuo Fukuda (Member)

Tokuo Fukuda received the B.S. and M.S. degree in engineering from Kyoto Institute of Technology, Kyoto, Japan in 1975 and 1977 respectively and he received the Ph.D. degree in engineering from Osaka University, Osaka, Japan in 1987. He is now a Professor at Otemon Gakuin University, Osaka, Japan. He is a member of IEEE, ISCIE, SICE, SOFT and Japan SIAM.