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Paper

Optimal Vaccination Strategy under Saturated Treatment using the Stochastic SIR Model*

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This paper is concerned with the control strategy by vaccination of the infectious disease spread in the populations consisting of the susceptible, the infected and the recovered (SIR). In the realistic spread of the infectious disease, changes in the environment and the weather cause some kinds of random fluctuations in the infection and the recovery rates, etc. Moreover, medical facilities have generally the maximal capacity for treatment of diseases. Taking these facts into consideration, we propose the stochastic infectious model with vaccination and saturated treatment, and we consider the stochastic optimal vaccination problem for the SIR model with saturated treatment using the stochastic maximum principle and the four-step scheme. We construct a feasible optimal vaccination system. By numerical simulations, we validate the efficacy of the optimal vaccination strategy.

1. Introduction

Although advances in sophisticated medical technology in health systems are constantly being made, the infectious disease prevention and the control is still one of the highest priority public sanitation issues [1]. A threat from infectious diseases such as tuberculosis, hepatitis C and malaria has been growing. As the medical program to diminish such a threat, the vaccination seems to be the most likely method to control and prevent the infectious disease spread. Mathematical models have become important tools in analyzing the infectious spread and in constructing the vaccination program [2-10]. In the past, various kinds of the infectious disease models with vaccination have been proposed. Since medical facilities have generally the maximal capacity for treatment of diseases, it is plausible to consider the saturated treatment rate. Moreover, in the real spread of the infectious disease, changes in the environment and the weather cause some kinds of random fluctuations in the infection and the recovery rates and others. Based on these facts, we propose the stochastic infectious model with vaccination and saturated treatment. The optimal vaccination coverage is found by solving the stochastic optimal control problem for the proposed infectious model. In this research, the optimal vaccination coverage is derived using the stochastic maximum principle [11] and the four-step scheme [12].

In Section **2**, firstly, one of the conventional infectious models, SIR (Susceptible, Infected and Recovered) model with vaccination and treatment, is explained. Taking the capacity of medical institutions into consideration, we consider a treatment rate with saturation. For more practical analysis of the spread of infectious diseases, the stochastic SIR model with vaccination and saturated treatment is proposed. In the stochastic model, the random fluctuation in the infection rate is considered. Since the behavior of the stochastic infectious model has a strong relation with the number of the steady state of the corresponding deterministic model, we study the change of number of the deterministic steady state, and we explain the backward bifurcation relating to the sudden increase in the infected population in Section **3**.

In Section **4**, we consider the optimal vaccination problem for the stochastic infectious model proposed in Section **2**. There are two major methods to obtain the stochastic optimal control: one is the stochastic dynamic programming (SDP) [11] and the other is the stochastic maximum principle. First, we explain the SDP in Section **4.1**. In the SDP, we must solve Hamilton-Jacobi-Bellman (HJB) equation which is a fully nonlinear equation, so that the mathematical treatment of the HJB equation is generally difficult and is also hard to numerically solve it. In Section **4.2**, we show that in the SMP, we must solve the forward-backward stochastic differential equation (FBSDE) instead of the HJB equation. However, the nonlinearity of the FBSDE is generally weaker than the HJB equation. Hence, we apply the stochastic maximum principle to the considered control problem. Using the four-step scheme to solve the FBSDE, we construct the optimal vaccination strategy. In Sec-

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tion **5**, we verify the efficiency of the proposed optimal vaccination strategy by numerical simulations.

An earlier version of this paper appeared in [13]. The present paper is a more complete version which contains the illustrative numerical example.

2. Stochastic SIR Model with Vaccination and Saturated Treatment

In this section, we first explain the conventional SIR model with vaccination and treatment. Then, we propose the stochastic SIR model with vaccination and a plausible treatment rate.

Letting the population size of the susceptible, the infected and the recovered at time t be $S(t), I(t)$ and $R(t)$, we consider the interaction between each population as shown in Fig. 1.

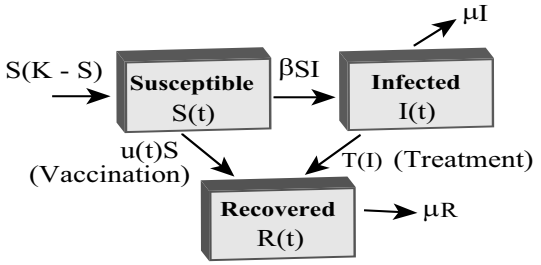

Fig. 1 Interaction between each population

In Fig. 1, the function $u(t) \in (0,1]$ denotes the percentage of susceptible individuals being vaccinated per unit of time, $K(>1)$ the carrying capacity of the susceptible in the absence of the infected, β the infection rate, μ the natural death rate and $T(I)$ the treatment rate.

Figure 1 yields the differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = S(t)(K-S(t)) - \beta S(t)I(t) - u(t)S(t), & (1) \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - \mu I(t) - T(I(t)), & (2) \\ \dfrac{dR(t)}{dt} = T(I(t)) + u(t)S(t) - \mu R(t), & (3) \end{cases}$$

where the initial conditions are given by

$$S(0) = S_0 \geq 0,\ I(0) = I_0 \geq 0,\ R(0) = R_0 \geq 0. \quad (4)$$

Some function forms as to the treatment rate $T(I)$ have been proposed in the past. For example, the treatment rate $T(I)$ is often given by the linear form of the infected population I such as $T(I) = rI$ where r is a constant. However, from the practical viewpoints, this function form is not plausible because the capacity of the medical treatment facilities has limitations. Hence, instead of the linear form, the saturated treatment rate is proposed:

$$T(I) = \frac{rI}{a+I}. \quad (5)$$

The constant a is a half-saturation constant at which the treatment rate is half of the maximum treatment rate r.

In this research, taking the random fluctuation of the infection rate β into consideration, we set as

$$\beta \to \beta + \varepsilon\eta(t), \quad (6)$$

where $\eta(t)$ is a white Gaussian noise and ε is a constant.

Using the relation $\eta(t)dt = dw(t)$ between the white Gaussian noise $\eta(t)$ and a Wiener process $w(t)$, (1) to (3) yield the stochastic vaccination model with saturated treatment:

$$dS(t) = \{S(t)(K-S(t)) - \beta S(t)I(t) - u(t)S(t)\}dt - \varepsilon S(t)I(t)dw(t), \quad (7)$$

$$dI(t) = \left\{\beta S(t)I(t) - \mu I(t) - \frac{rI(t)}{a+I(t)}\right\}dt + \varepsilon S(t)I(t)dw(t), \quad (8)$$

$$dR(t) = \left\{\frac{rI(t)}{a+I(t)} + u(t)S(t) - \mu R(t)\right\}dt. \quad (9)$$

3. Deterministic Steady States

The behavior of the stochastic model (7) to (9) has a strong relation with the number of the steady state of the corresponding deterministic model (1) to (3) with a constant vaccination rate u and (5). Hence, we study the number of the deterministic steady state (S^*, I^*, R^*) defined by the solution of

$$\begin{cases} S^*(K-S^*) - \beta S^* I^* - uS^* = 0, & (10) \\ \beta S^* I^* - \mu I^* - \dfrac{rI^*}{a+I^*} = 0, & (11) \\ \dfrac{rI^*}{a+I^*} + uS^* - \mu R^* = 0. & (12) \end{cases}$$

Since the steady state R^* of the recovered is given by the steady states S^* and I^* of the susceptible and the infected in such a way that

$$R^* = \frac{1}{\mu}\left(\frac{rI^*}{a+I^*} + uS^*\right), \quad (13)$$

we treat only the steady states, S^* and I^*, in the sequel.

It is easily shown that (10) and (11) have two types of the steady states:

(i) Disease free steady state $(S^*, I^*) \equiv (S_f, I_f)$:

Noting $K > u$, we have two disease-free steady states such that

$$(S_f, I_f) = \begin{cases} E_1 \equiv (0,0) \\ E_2 \equiv (\bar{K}, 0), \end{cases} \quad (14)$$

where $\bar{K} = K - u$.

(ii) Endemic steady state $(S^*, I^*) \equiv (S_e, I_e)$:

The endemic steady state $E_3 \equiv (S_e, I_e)$ is given by the solution of

$$\begin{cases} \bar{K}-S_e-\beta I_e=0, & (15)\\ \beta S_e-\mu-\dfrac{r}{a+I_e}=0. & (16)\end{cases}$$

It follows from (15) and (16) that the steady state S_e is given by the solution in the range of $(0,\bar{K})$ of the quadratic equation:

$$x^2-Ax+B=0, \tag{17}$$

where A and B are defined by

$$A=K-u+\frac{\mu}{\beta}+\beta a>0,\ B=\frac{\mu}{\beta}(\beta a+\bar{K})+r>0. \tag{18}$$

Setting as $f(x)=x^2-Ax+B$, we have

$$f(0)=B>0,\ f(\bar{K})=(\mu a+r)(1-R_u), \tag{19}$$

where R_u is called the vaccination reproduction number and is defined by

$$R_u=\frac{\beta a\bar{K}}{\mu a+r}<\frac{\beta aK}{\mu a+r}\equiv R_0, \tag{20}$$

and where R_0 denotes the basic reproduction number which means the number of secondary cases that a single infected case will cause in a population over the course of its infectious period.

Setting the discriminant of (17) as $\Delta\equiv A^2-4B$, and noting (19), we consider the change of the number of the endemic steady state(S_e,I_e) with $S_e\in(0,\bar{K})$. First, it should be noted that the following relations hold:

$$\begin{cases} f(\bar{K})>0\Longleftrightarrow R_u<1,(f(\bar{K})<0\Longleftrightarrow R_u>1), & (21)\\ A<2\bar{K}\Longleftrightarrow R_u>P_0,(A>2\bar{K}\Longleftrightarrow R_u<P_0), & (22)\\ \Delta>0\Longleftrightarrow R_u>P_1,(\Delta<0\Longleftrightarrow R_u<P_1), & (23)\end{cases}$$

where the second relation (22) guarantees that the x-coordinate of the vertex of (17) exists in $(0,\bar{K})$, $((\bar{K},\infty))$, and P_0 and P_1 are defined by

$$P_0=\frac{\beta a}{\mu a+r}\left(\frac{\mu}{\beta}+\beta a\right), \tag{24}$$

and

$$P_1=\frac{\beta a}{\mu a+r}\left(2\sqrt{r}+\frac{\mu}{\beta}-\beta a\right). \tag{25}$$

(i) One endemic steady state: If either of the following conditions is satisfied, there exists a unique endemic steady state (S_e,I_e),

$$P_0<R_u=P_1<1,\ \text{or}\ \ R_u>1, \tag{26}$$

(ii) Two endemic steady states: If the following is satisfied,

$$\max(P_0,P_1)<R_u<1, \tag{27}$$

there exist two endemic steady states $(S_{ei},I_{ei}),(i=1,2)$ such that

$$0<S_{e1}<S_{e2}<\bar{K}, \tag{28}$$

$$S_{ei}=\frac{A+(-1)^i\sqrt{\Delta}}{2},\ I_{ei}=\frac{\bar{K}-S_{ei}}{\beta}. \tag{29}$$

(iii) No endemic steady state: If either of the following conditions is satisfied, there exists no endemic steady state.

$$R_u<P_1,\ \ \text{or}\ \ R_u<\min(1,P_0). \tag{30}$$

It follows from (26) to (30) that the number of the endemic steady state changes in the (a,u)-plane as shown in Fig. 2 under $K=2,\beta=0.6,r=1.0,\mu=0.15$. In Fig. 2, the regions I, II and III denote the region in which the number of the endemic steady state is zero, two and one, respectively. Figure 2 shows that the two endemic steady states exist in the region with comparatively small values of the vaccination rate u and the half-saturation constant a of the treatment rate.

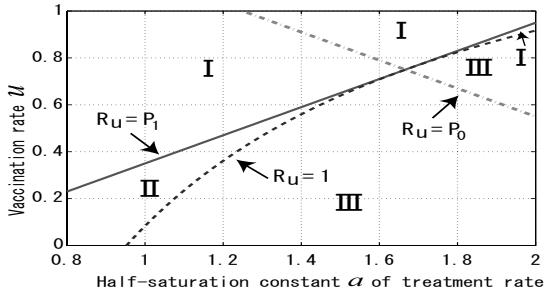

Fig. 2 Change of the number of the endemic steady state in the (a,u)-plane

The existence of the two endemic steady states means the onset of the backward bifurcation [5, 6]. For example, the backward bifurcation as shown in Fig. 3 occurs under the vaccination rate $u=0.2$. The horizontal and the vertical axes in Fig. 3 are the vaccination reproduction number R_u and the value of the endemic steady state I_e.

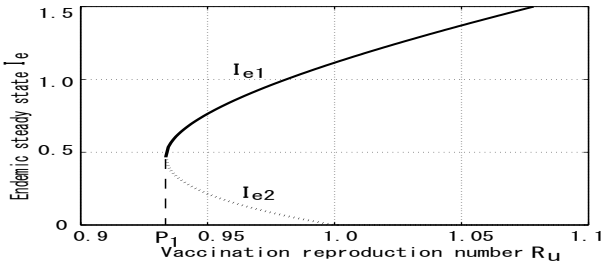

Fig. 3 Backward bifurcation under $u=0.2$

On the other hand, in the region with comparatively large value of vaccination rate u, instead of the backward bifurcation, the forward bifurcation occurs. For example, under $u=0.8$, we can see the forward bifurcation as shown in Fig. 4.

In this way, depending on the parameter values, steady states in the infectious model undergo the for-

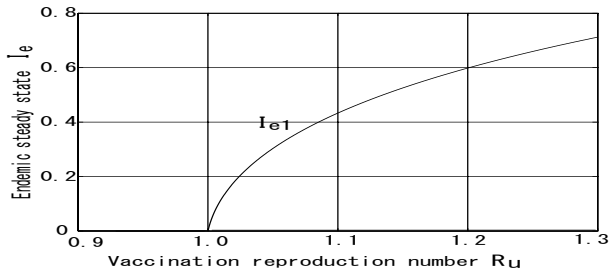

Fig. 4 Forward bifurcation under $u = 0.8$

ward and backward bifurcations. The forward bifurcation is a so-called transcritical bifurcation [6] and the disease-free steady state changes its stability at $R_u = 1$. The backward bifurcation is a so-called saddle-node bifurcation [6] and appearance(or disappearance) of the steady state occurs at $R_u = P_1$).

The control of the spread of the infectious disease has a close relation with the type of bifurcation. In the forward bifurcation, in order to eradicate infectious disease, it is only necessary to make the vaccination reproduction number R_u less than one by vaccination. However, in the backward bifurcation, even if we make the vaccination reproduction number R_u less than one, when R_u is greater than P_1, we cannot eradicate the infectious disease by vaccination. In order to eradicate infectious disease in this case, we must keep R_u less than P_1 by vaccination.

4. Optimal Vaccination Strategy

It follows from (7) to (9) that the susceptible and the infected $S(t)$ and $I(t)$ are independent of the recovered $R(t)$. Hence, we treat two states, $S(t)$ and $I(t)$ in the sequel.

Choosing the vaccination rate $u(t)$ as the control variable, we introduce the cost functional $J(u)$ such that

$$J(u) = E\left\{ h(S(T),I(T)) + \int_0^T \ell(S(t),I(t),u(t))dt \right\} \quad (31)$$

where $h : R^2 \to R^1, \ell : R^3 \to R^1$, h and ℓ are assumed to be continuously differentiable. The concrete form of the cost functional is given in Section **5**.

We study the optimal control problem of finding the optimal vaccination rate $u^* \in \mathcal{U}(0,T)$ such that

$$J(u^*) \le J(u), {}^\forall u \in \mathcal{U}(0,T), \quad (32)$$

where $\mathcal{U}$ is an admissible control set defined by

$$\mathcal{U}(0,T) \equiv \{u | 0 \le u(t) \le C_p, {}^\forall t \in [0,T]\}, \quad (33)$$

and where $C_p \in (0\ 1]$ is a possible maximum vaccination rate.

In this paper, we derive the optimal vaccination strategy using the stochastic maximum principle (SMP). At first, referring to the stochastic dynamic programming (SDP), we explain the background that we apply the SMP to the considered problem.

First, for simplicity of descriptions, we define the vector:

$$x(t) = [x_1(t)\ x_2(t)]' \equiv [S(t)\ I(t)]'. \quad (34)$$

Then, (7) and (8) can be rewritten by

$$dx(t) = f(x(t),u(t))dt + g(x(t))dw(t), \quad (35)$$

with the initial condition

$$x(0) = x_0 \equiv [S_0\ I_0]', \quad (36)$$

where f and g are vectors with components such that

$$f_1 = x_1(K - x_1) - \beta x_1 x_2 - u x_1, \quad (37)$$

$$f_2 = \beta x_1 x_2 - \mu x_2 - \frac{r x_2}{a + x_2}, \quad (38)$$

$$g_1 = -\varepsilon x_1 x_2,\ g_2 = \varepsilon x_1 x_2. \quad (39)$$

4.1 Stochastic Dynamic Programming

We define the value function V:

$$V(t,x) = \min_{u(s) \in \mathcal{U}(t,T)} E\Big\{ h(x(T)) + \int_t^T \ell(x(s),u(s))ds | x(t) = x \Big\}. \quad (40)$$

It follows from the principle of optimality that the Hamilton-Jacobi-Bellman equation holds:

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u \in \mathcal{U}} \left\{ \ell(x,u) + \left(\frac{\partial V(t,x)}{\partial x} \right)' f(x,u) + \frac{1}{2} \mathrm{tr} \left\{ g(x) g(x)' \frac{\partial}{\partial x} \left(\frac{\partial V(t,x)}{\partial x} \right)' \right\} \right\}, \quad (41)$$

with the terminal condition

$$V(T,x) = h(x). \quad (42)$$

From (41), since u^* is the function of x and V_x, we describe this relation below:

$$u^*(t) = \phi(x, V_x(t,x)). \quad (43)$$

Substituting (43) into (41), we obtain

$$-\frac{\partial V(t,x)}{\partial t} = \ell(x,\phi) + \left(\frac{\partial V(t,x)}{\partial x} \right)' f(x,\phi) + \frac{1}{2} \mathrm{tr} \left\{ g(x) g(x)' \frac{\partial}{\partial x} \left(\frac{\partial V(t,x)}{\partial x} \right)' \right\}. \quad (44)$$

By solving (44) with (42), we can derive the optimal vaccination rate u^* through the relation (43). However, it is generally hard to solve (44) not only analytically but also numerically except the linear quadratic case. The other typical method to solve the optimal control problem is the SMP. For the SMP, we must solve forward-backward stochastic differential equation (FBSDE), however, the powerful method to solve

the FBSDE is already proposed and the method is called the four-step scheme. So, we employ the SMP to construct the optimal vaccination strategy. In the next section we explain the SMP and the four-step scheme.

4.2 Stochastic Maximum Principle

First, we introduce the Hamiltonian $H(x,u,p,q)$ defined by

$$H(x,u,p,q) = p'f(x,u) - \ell(x,u) + q'g(x), \tag{45}$$

where $p = [p_1\ p_2]'$ and $q = [q_1\ q_2]'$ are adjoint vectors.

It follows from the stochastic maximum principle that

$$dx^*(t) = \frac{\partial H(x^*,u^*,p,q)}{\partial p}dt + g(x^*(t))dw(t), \tag{46}$$

$$dp(t) = -\frac{\partial H(x^*,u^*,p,q)}{\partial x}dt + q(t)dw(t), \tag{47}$$

$$H(x^*,u^*,p,q) = \max_{u\in\mathcal{U}} H(x^*,u,p,q), \tag{48}$$

where $x^*(t)$ is an optimal trajectory of $x(t)$.

The initial and terminal conditions of (46) and (47) are given by

$$x^*(0) = x_0, \tag{49}$$

$$p(T) = -\frac{\partial h(x^*(T))}{\partial x}. \tag{50}$$

In order to find the optimal vaccination rate u^*, we must solve the stochastic two-point boundary value problem (46) and (47) with conditions (49) and (50). In other words, we need to solve the forward-backward stochastic differential equation (FBSDE). Generally, we can not solve the FBSDE because the future state of the stochastic system is unknown. It should be noted that in the case of the linear quadratic problem, we can solve the FBSDE using the decoupling technique [14]. However, since our problem is nonlinear, the decoupling approach fails to derive the optimal vaccination rate. Then, in this paper, using the four-step scheme [12], we solve the FBSDE in the next section. Since (48) implies that the optimal control $u^*(t)$ is a function of $p(t)$,$q(t)$ and $x^*(t)$, we have

$$u^*(t) = \phi(x^*(t),p(t),q(t)), \tag{51}$$

where ϕ is determined by (48).

Hence, (46) and (47) can be rewritten by

$$dx^*(t) = \frac{\partial H(x^*,\phi,p,q)}{\partial p}dt + g(x^*(t))dw(t), \tag{52}$$

$$dp(t) = -\frac{\partial H(x^*,\phi,p,q)}{\partial x}dt + q(t)dw(t). \tag{53}$$

4.3 The Four-step Scheme

Assume that $p(t)$ and $x^*(t)$ are related by

$$p(t) = \theta(t,x^*(t)), \tag{54}$$

where θ is some vector-valued function with components $\theta_i(t,x^*(t))$ $(i=1,2)$ to be determined.

In the sequel, the asterisks of $x(t)$ and $u(t)$ are omitted for simplicity of descriptions.

Using the Itô's lemma [15] to $\theta_i(t,x(t))$, we have for $i=1,2$,

$$\begin{aligned} d\theta_i(t,x) = \Bigg\{ &\frac{\partial\theta_i(t,x)}{\partial t} + \left(\frac{\partial\theta_i(t,x)}{\partial x}\right)' f(x,u) \\ &+ \frac{1}{2}\mathrm{tr}\left[\frac{\partial}{\partial x}\left(\frac{\partial\theta_i}{\partial x}\right)' g(x)g(x)'\right]\Bigg\}dt \\ &+ \left(\frac{\partial\theta_i(t,x)}{\partial x}\right)' g(x)dw(t). \end{aligned} \tag{55}$$

Noting that $p(t) = \theta(t,x(t))$ and $u = \phi(x,p,q) = \phi(x,\theta,q)$ and by the term-wise comparison between (47) and (55), we have

$$\begin{aligned} &\frac{\partial\theta_i(t,x)}{\partial t} + \left(\frac{\partial\theta_i(t,x)}{\partial x}\right)' f(x,\phi,\theta,q) + \frac{\partial H(x,\phi,\theta,q)}{\partial x_i} \\ &\quad + \frac{1}{2}\mathrm{tr}\left[\frac{\partial}{\partial x}\left(\frac{\partial\theta_i(t,x)}{\partial x}\right)' g(x)g(x)'\right] = 0. \end{aligned} \tag{56}$$

$$q_i(t) = \left(\frac{\partial\theta_i(t,x)}{\partial x}\right)' g(x), \tag{57}$$

with the terminal condition

$$\theta_i(T,x) = -\frac{\partial h(x)}{\partial x_i}, \quad (i=1,2). \tag{58}$$

Since (56) is a deterministic partial differential equation, we can solve (56) with (57) under the terminal condition (58). Hence, using the solution $\theta(t,x(t))$, we can obtain the adjoint state $p(t)$ by (54). The optimal vaccination system is realized by (46), (47) and (51).

5. Numerical Example

In this section, we validate the efficacy of the optimal vaccination strategy proposed in Section **4** by numerical simulations.

Considering the standard cost functional in the research field of the infection control [7-10], we introduce the cost functional such that

$$J(u) = E\left\{k'x(T) + \int_0^T (m'x(t) + ru(t)^2)dt\right\}, \tag{59}$$

where $k = [k_1\ k_2]'$ and $m = [m_1\ m_2]'$ are constant vectors with positive elements.

The reason why the cost functional of (59) is used is that the square of the vaccination rate u reflects the severity of the side effects of the vaccination and each population is nonnegative. Moreover, it should be noted that even if we consider the quadratic cost

functional, we cannot apply the decoupling technique [14] due to the nonlinearity of the system dynamics.

Then, the Hamiltonian $H(x,u,p,q)$ in (45) becomes

$$H(x,u,p,q) = p'f(x,u) - (m'x + ru^2) + q'g(x), \quad (60)$$

where f and g are defined by (37) to (39).

Equations (33), (48) and (59) yield that

$$u^*(t) = \max\left\{\min\left\{-\frac{p_1(t)}{2r}x_1^*(t), C_p\right\}, 0\right\}. \quad (61)$$

It follows from the stochastic maximum principle ((46), (47), (49) and (50)) that

$$dx_1^* = \{x_1^*(K - x_1^*) - \beta x_1^* x_2^* - u^* x_1^*\}dt - \varepsilon x_1^* x_2^* dw, \quad (62)$$

$$dx_2^* = \{\beta x_1^* x_2^* - \mu x_2^* - \frac{rx_2^*}{a + x_2^*}\}dt + \varepsilon x_1^* x_2^* dw, \quad (63)$$

$$dp_1 = \{(\beta x_2^* + 2x_1^* - K)p_1 - \beta x_2^* p_2 + u^* + m_1 \quad (64)$$

$$+ \varepsilon x_2^*(q_2 - q_1)\}dt + q_1 dw, \quad (65)$$

$$dp_2 = \{\beta p_1 x_1^* - \beta p_2 x_1^* + \mu p_2 + m_2 + \frac{ra}{(a + x_2^*)^2} \quad (66)$$

$$+ \varepsilon x_1^*(q_1 - q_2)\}dt + q_2 dw, \quad (67)$$

with the initial and terminal conditions

$$x_1^*(0) = S_0,\ x_2^*(0) = I_0, \quad (68)$$

$$p_1(T) = -k_1,\ \ p_2(T) = -k_2. \quad (69)$$

In the sequel, the asterisks of $x(t)$ and $u(t)$ are omitted for simplicity of descriptions.

Setting as $p_i(t) = \theta_i(t,x(t))$, the four-step scheme (56)-(58) yields that

$$\frac{\partial\theta_1}{\partial t} + \frac{\partial\theta_1}{\partial x_1}\{x_1(K - x_1) - \beta x_1 x_2 - u x_1\}$$

$$+ \frac{\partial\theta_1}{\partial x_2}(\beta x_1 x_2 - \mu x_2 - \frac{rx_2}{a + x_2})$$

$$+ p_1(K - 2x_1 - \beta x_2 - u) + p_2\beta x_2 - m_1 + \varepsilon x_2(q_2 - q_1)$$

$$+ \frac{1}{2}\varepsilon^2 x_1^2 x_2^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^2\theta_1$$

$$+ \varepsilon^2 x_1 x_2^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)(\theta_1 - \theta_2) = 0, \quad (70)$$

$$\frac{\partial\theta_2}{\partial t} + \frac{\partial\theta_2}{\partial x_1}\{x_1(K - x_1) - \beta x_1 x_2 - u x_1\}$$

$$+ \frac{\partial\theta_2}{\partial x_2}(\beta x_1 x_2 - \mu x_2 - \frac{rx_2}{a + x_2}) - p_1\beta x_1$$

$$+ p_2\left(\beta x_1 - \mu - \frac{ra}{(a + x_2)^2}\right) - m_2 + \varepsilon x_1(q_2 - q_1)$$

$$+ \frac{1}{2}\varepsilon^2 x_1^2 x_2^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^2\theta_2$$

$$+ \varepsilon^2 x_1^2 x_2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)(\theta_1 - \theta_2) = 0, \quad (71)$$

where the terminal conditions are given by

$$\theta_1(T,x(T)) = -k_1,\ \theta_2(T,x(T)) = -k_2. \quad (72)$$

Noting that $p_i = \theta_i$ and (61), the optimal vaccination rate $u(t)$ is given by

$$u(t) = \max\left\{\min\left\{-\frac{\theta_1(t,x(t))}{2r}x_1(t), C_p\right\}, 0\right\}. \quad (73)$$

Consequently, solving (70) and (71) with (73) under the terminal condition (72) and by setting as $p_i(t) = \theta_i(t,x(t))$, we can realize the optimal vaccination system by (61) to (67).

Setting the maximum vaccination rate $C_p = 0.8$, numerical simulations are performed under the parameter values below:

$$S_0 = 0.5,\ I_0 = 0.3\beta = 0.03,\ \mu = 0.01,\ T = 15,$$
$$\varepsilon = 0.02,\ m_1 = m_2 = 1.0,\ r = 0.6,\ k_1 = k_2 = 1.0.$$

Simulation results are shown in Figs. 5 and 6. Figures 5 and 6 show the time evolution of susceptible and infected under the optimal vaccination strategy and no vaccination. In Fig. 5, we can see that the infected is decreasing and the susceptible is at a low level by the optimal vaccination strategy. On the other hand, since Fig. 6 shows that the infected is increasing and the susceptible is at a high level, a high risk of infection still remains under no vaccination.

Figure 7 shows the time evolution of the optimal vaccination rate $u^*(t)$ under the maximum vaccination rate $C_p = 0.8$. We can know at what rate we will vaccinate by Fig. 7.

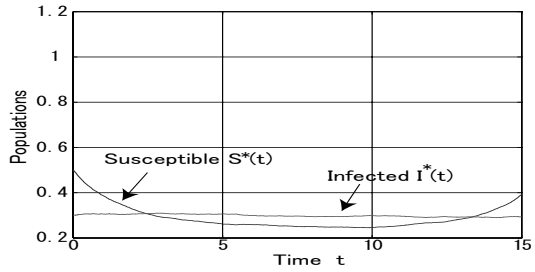

Fig. 5 Behaviors of $S^*(t)$ and $I^*(t)$ under optimal vaccination

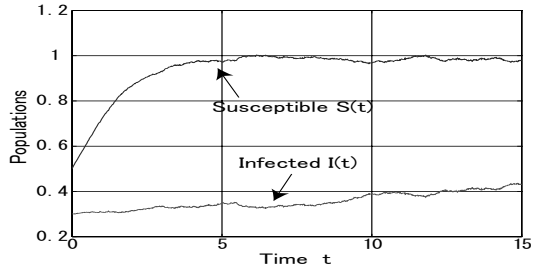

Fig. 6 Behaviors of $S(t)$ and $I(t)$ under no vaccination

6. Conclusions

In this paper, we have proposed the stochastic SIR (Susceptible-Infective-Recovered) model with vacci-

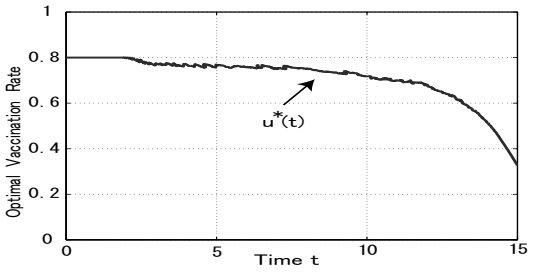

Fig. 7 Optimal vaccination strategy

nation and saturated treatment and have studied the stochastic optimal vaccination problem. Since the basic behavior of the stochastic SIR model relates to the change of the number of the steady state of the corresponding deterministic SIR model, we have shown the dependency of the number of the deterministic steady state on the half-saturation constant a and the vaccination rate u. In the case of the comparatively lower value of a and u, we have found that there exist two endemic steady states. The existence of the two endemic steady states means the onset of the backward bifurcation. In the infectious disease spread with the backward bifurcation, we must make the vaccination reproduction number R_u below the critical value $P_1(<1)$ in order to eradicate the infectious disease, and it is inadequate to lower R_u below one.

In the optimal vaccination problem, we have derived the optimal vaccination system with the use of the stochastic maximum principle. In order to realize the optimal vaccination system, we must solve a nonlinear FBSDE. Since we cannot solve directly the nonlinear FBSDE due to the lack of information of the future state of the stochastic system, we employed the four-step scheme to solve the nonlinear FBSDE.

We have shown the method to construct the optimal vaccination strategy for the stochastic SIR model with saturated treatment, and we have verified the efficiency of the proposed optimal vaccination strategy by numerical simulations.

Since the bifurcation analysis in this paper is performed in the deterministic framework, the study of the stochastic bifurcation analysis [16] of the stochastic infectious model is the future problem.

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