Further Investigation of a Gaussian Sum Filter Based on Stochastic Equivalent Linearization*

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We proposed a stochastic (or statistical) Equivalent linearization - Gaussian Sum Filter (EqGS Filter) for discrete time nonlinear Systems. Subsequently, in this paper, we investigate and show the further results related to the EqGS Filter. Especially we discuss a method to apply Gauss-Hermite quadrature rules for evaluation of the conditional expected values of the quantities required to design the EqGS filter. Finally, we show the estimation results of AR modeling comparison with the extended Kalman and the equivalent linearization filters.

1. Introduction

Recently, many attentions again have been focused on nonlinear filtering theory[1,2] and its approximated filters[3–5], such as the unscented Kalman filter[6], the particle (or Monte Carlo) filter[7,8], and the Gaussian filter[9,10]. We had proposed a stochastic (or statistical) Equivalent linearization - Gaussian Sum Filter (abbreviated as EqGS Filter) for discrete time nonlinear Systems in[11,12]. In this paper, we show the further results related to the EqGS Filter, especially we discuss a method of evaluation of the conditional expected values required to design the EqGS filter, by applying Gauss-Hermite quadrature rules[13–15]

Let us consider the following discrete-time nonlinear state space models.

$$x_{t+1} = f_t(x_t) + w_t \quad (1)$$

$$y_t = h_t(x_t) + v_t \quad (2)$$

where x_t is a n -dimensional state vector and y_t is m -dimensional observation vector. w_t and v_t are n -dimensional and m -dimensional Gaussian white noises, respectively. $f_t(x_t)$ and $h_t(x_t)$ are n -dimensional and m -dimensional vector valued nonlinear functions. The mean vectors and covariance matrices of w_t and v_t are given by

$$E[w_t] = 0, \quad E[v_t] = 0 \quad (3)$$

$$E\left\{\begin{bmatrix} w_t \\ v_t \end{bmatrix} \begin{bmatrix} w_s^T & v_s^T \end{bmatrix}\right\} = \begin{bmatrix} Q_t & 0 \\ 0 & R_t \end{bmatrix} \delta_{t-s} \quad (4)$$

and the initial state x_0 is independent of w_t and v_t . Also due to whiteness of noises, we have the relations:

$$E\{w_t x_s^T\} = 0 \quad (t \geq s), \quad E\{v_t x_s^T\} = 0 \quad (t, s \geq 0). \quad (5)$$

2. The Stochastic Equivalent Linearization

The stochastic equivalent linearization technique[17] was applied to obtain the suboptimal linear estimates for continuous time systems by assuming the Gaussian conditional probability density functions (CPDFs). We had extended the equivalent linearization technique[4,5,18] to discrete time systems under the assumption of Gaussian sum CPDFs: $p(x_t|Y^{t-1})$, $p(x_t|Y^t)$, where $Y^t = \{y_0, \dots, y_t\}$. Now we show to derive the EqGS filter for discrete time nonlinear systems according to[11,12].

First, we assume that the CPDFs are given by the following Gaussian sum distribution:

$$\begin{aligned} p(x_t|Y^{t-1}) &= \sum_{i=1}^r \alpha_{t|t-1}^{(i)} (2\pi)^{-n/2} |P_{t|t-1}^{(i)}|^{-1/2} \\ &\quad \times \exp\left\{-\frac{1}{2}(x_t - \mu_{t|t-1}^{(i)})^T (P_{t|t-1}^{(i)})^{-1} (x_t - \mu_{t|t-1}^{(i)})\right\} \\ &\equiv \sum_{i=1}^r \alpha_{t|t-1}^{(i)} N(x_t : \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}), \\ p(x_t|Y^t) &= \sum_{i=1}^r \alpha_{t|t}^{(i)} (2\pi)^{-n/2} |P_{t|t}^{(i)}|^{-1/2} \\ &\quad \times \exp\left\{-\frac{1}{2}(x_t - \mu_{t|t}^{(i)})^T (P_{t|t}^{(i)})^{-1} (x_t - \mu_{t|t}^{(i)})\right\} \\ &\equiv \sum_{i=1}^r \alpha_{t|t}^{(i)} N(x_t : \mu_{t|t}^{(i)}, P_{t|t}^{(i)}). \end{aligned}$$

Namely, these CPDFs are Gaussian sum distribution

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with means: $\mu_{t|t-1}^{(i)}, \mu_{t|t}^{(i)}$, $i = 1, \dots, r$ and covariances: $P_{t|t-1}^{(i)}, P_{t|t}^{(i)}$; $i = 1, \dots, r$, where $\alpha_{t|t-1}^{(i)}$ and $\alpha_{t|t}^{(i)}$ denote weighting factors of distributions with the property, $\sum_{i=1}^r \alpha_{t|*}^{(i)} = 1$; $\alpha_{t|*}^{(i)} \geq 0$.

Then we consider the linear approximation to the nonlinear functions: $f_t(x_t)$ and $h_t(x_t)$ in (1) and (2), respectively, as

$$f_t(x_t) = a_t + \sum_{i=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)}) + e_t^f, \quad (6)$$

$$h_t(x_t) = c_t + \sum_{i=1}^r D_t^{(i)}(x_t - \mu_{t|t-1}^{(i)}) + e_t^h, \quad (7)$$

where a_t and c_t are n -dimensional and m -dimensional vectors, respectively, and $B_t^{(i)}$ and $D_t^{(i)}$ are $n \times n$ and $m \times n$ matrices. Further, e_t^f and e_t^h denote the linear approximation errors. Similar to the stochastic equivalent linearization[17], we assume that the linear approximations are unbiased, namely

$$E\{f_t(x_t)|Y^t\} = E\{a_t + \sum_{i=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)})|Y^t\} \quad (8)$$

$$E\{h_t(x_t)|Y^{t-1}\} = E\{c_t + \sum_{i=1}^r D_t^{(i)}(x_t - \mu_{t|t-1}^{(i)})|Y^{t-1}\}. \quad (9)$$

Furthermore, we consider the following optimization problem for minimizing the square error of linear approximation. Then a_t , $B_t^{(i)}$, c_t and $D_t^{(i)}$ in (6) and (7) are obtained by minimizing the conditional expectation of the square errors norms of e_t^f and e_t^h :

$$\begin{aligned} E\{\|e_t^f\|^2|Y^t\} &\equiv J_f(a_t, \{B_t^{(i)}\}) \\ &\equiv E\{\|f_t(x_t) - a_t - \sum_{i=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)})\|^2|Y^t\} \quad (10) \\ E\{\|e_t^h\|^2|Y^{t-1}\} &\equiv J_h(c_t, \{D_t^{(i)}\}) \\ &\equiv E\{\|h_t(x_t) - c_t - \sum_{i=1}^r D_t^{(i)}(x_t - \mu_{t|t-1}^{(i)})\|^2|Y^{t-1}\} \quad (11) \end{aligned}$$

Then we compute (10) as follows:

$$\begin{aligned} J_f(a_t, \{B_t^{(i)}\}) &\equiv J_f(a_t, B_t) \\ &= \text{trace}\left[E\left\{[f_t(x_t) - a_t - \sum_{i=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)})] \right. \right. \\ &\quad \times \left. \left. [f_t(x_t) - a_t - \sum_{j=1}^r B_t^{(j)}(x_t - \mu_{t|t}^{(j)})]^T | Y^t \right\}\right] \\ &= \text{trace}\left[E\left\{(f_t(x_t) - a_t)(f_t(x_t) - a_t)^T \right. \right. \\ &\quad + \sum_{i=1}^r \sum_{j=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)})(x_t - \mu_{t|t}^{(j)})^T (B_t^{(j)})^T \\ &\quad - \sum_{j=1}^r (f_t(x_t) - a_t)(x_t - \mu_{t|t}^{(j)})^T (B_t^{(j)})^T \end{aligned}$$

$$\begin{aligned} &\left. - \sum_{i=1}^r B_t^{(i)}(x_t - \mu_{t|t}^{(i)})(f_t(x_t) - a_t)^T | Y^t \right\}] \\ &= \text{trace}\left[a_t a_t^T - E\{f_t(x_t)|Y^t\} a_t^T - a_t E\{f_t^T(x_t)|Y^t\} \right. \\ &\quad + E\{f_t(x_t) f_t^T(x_t)|Y^t\} + \sum_{i=1}^r \sum_{j=1}^r B_t^{(i)} F_t^{(ij)} (B_t^{(j)})^T \\ &\quad - \sum_{j=1}^r \Gamma_t^{(j)} (B_t^{(j)})^T - \sum_{i=1}^r B_t^{(i)} (\Gamma_t^{(i)})^T] \\ &= \text{trace}\left[(a_t - E\{f_t(x_t)|Y^t\})(a_t - E\{f_t(x_t)|Y^t\})^T \right. \\ &\quad + E\{f_t(x_t) f_t^T(x_t)|Y^t\} - E\{f_t(x_t)|Y^t\} E\{f_t^T(x_t)|Y^t\} \\ &\quad \left. + B_t F_t B_t^T - \Gamma_t B_t^T - B_t \Gamma_t^T\right] \quad (12) \end{aligned}$$

where

$$B_t \equiv \begin{bmatrix} B_t^{(1)} & B_t^{(2)} & \dots & B_t^{(r)} \end{bmatrix}; n \times nr \text{ matrix} \quad (13)$$

$$F_t \equiv \begin{bmatrix} F_t^{(11)} & \dots & F_t^{(1r)} \\ \vdots & \ddots & \vdots \\ F_t^{(r1)} & \dots & F_t^{(rr)} \end{bmatrix}; nr \times nr \text{ matrix} \quad (14)$$

$$F_t^{(ij)} \equiv E\{(x_t - \mu_{t|t}^{(i)})(x_t - \mu_{t|t}^{(j)})^T | Y^t\} \quad (15)$$

$$\Gamma_t \equiv \begin{bmatrix} \Gamma_t^{(1)} & \Gamma_t^{(2)} & \dots & \Gamma_t^{(r)} \end{bmatrix}; n \times nr \text{ matrix} \quad (16)$$

$$\Gamma_t^{(i)} \equiv E\{(f_t(x_t) - a_t)(x_t - \mu_{t|t}^{(i)})^T | Y^t\}. \quad (17)$$

From (14) and (15), we have the relation:

$$F_t^T = F_t, \quad \text{since} \quad (F_t^{(ij)})^T = F_t^{(ji)}. \quad (18)$$

Then minimization of $J_f(a_t, B_t)$ in (12) can be achieved by solving the following equations:

$$\frac{\partial J_f(a_t, B_t)}{\partial a_{t,i}} = 0, \quad (19)$$

$$\frac{\partial J_f(a_t, B_t)}{\partial B_{t,ij}} = 0, \quad (20)$$

for $i = 1, \dots, n$; $j = 1, \dots, nr$. Then from (19), we have

$$a_t = E\{f_t(x_t)|Y^t\} \equiv \hat{f}_{t|t}(x_t), \quad (21)$$

Also remarking the relations:

$$\text{trace}[B_t F_t B_t^T] = \sum_k \sum_p \sum_l B_{t,kl} F_{t,lp} B_{t,kp},$$

$$\text{trace}[\Gamma_t B_t^T] = \sum_k \sum_l \Gamma_{t,kl} B_{t,kl},$$

$$\text{trace}[B_t \Gamma_t^T] = \sum_k \sum_l B_{t,kl} \Gamma_{t,kl},$$

then from (20), we have

$$2 \sum_p F_{t,jp} B_{t,ip} - 2 \Gamma_{t,ij} = 0. \quad (22)$$

Namely, we have

$$B_t F_t = \Gamma_t. \quad (23)$$

Also from (8), we have

$$\begin{aligned} \mathbb{E}\{f_t(x_t)|Y^t\} &= a_t + \sum_{i=1}^r B_t^{(i)} [\mathbb{E}\{x_t|Y^t\} - \mu_{t|t}^{(i)}] \\ &= a_t + B_t (\hat{X}_{t|t} - \mu_{t|t}), \end{aligned} \quad (24)$$

where

$$\hat{X}_{t|t} \equiv [\hat{x}_{t|t}^T \cdots \hat{x}_{t|t}^T]^T; nr \times 1 \text{ vector} \quad (25)$$

$$\mu_{t|t} \equiv [(\mu_{t|t}^{(1)})^T \cdots (\mu_{t|t}^{(r)})^T]^T; nr \times 1 \text{ vector.} \quad (26)$$

From (21) and (24), we have $B_t(\hat{X}_{t|t} - \mu_{t|t}) = 0$, so that combining with (23), we have

$$B_t \left[F_t \mid \hat{X}_{t|t} - \mu_{t|t} \right] = \left[\Gamma_t \mid 0_{n \times 1} \right]. \quad (27)$$

Then from (27), we finally have

$$B_t = \tilde{F}_t \tilde{F}_t^T (\tilde{F}_t \tilde{F}_t^T)^{-1} \quad (28)$$

where

$$\tilde{F}_t \equiv \left[F_t \mid \hat{X}_{t|t} - \mu_{t|t} \right], \quad \tilde{F}_t \equiv \left[\Gamma_t \mid 0_{n \times 1} \right]. \quad (29)$$

Also from (17) and (21), we should remark the relations:

$$\begin{aligned} \Gamma_t^{(i)} &= \mathbb{E}\left\{ (f_t(x_t) - a_t) x_t^T | Y^t \right\} \\ &= \mathbb{E}\left\{ f_t(x_t) x_t^T | Y^t \right\} - a_t \hat{x}_{t|t}^T \equiv \Gamma_t^{(0)}. \end{aligned} \quad (30)$$

Therefore we can approximate

$$\begin{aligned} f_t(x_t) &\approx a_t + \sum_{i=1}^r B_t^{(i)} (x_t - \mu_{t|t}^{(i)}) \\ &= a_{S,t} + B_{S,t} x_t \end{aligned} \quad (31)$$

where

$$a_{S,t} \equiv a_t - \sum_{i=1}^r B_t^{(i)} \mu_{t|t}^{(i)} = a_t - B_t \mu_{t|t} \quad (32)$$

$$B_{S,t} \equiv \sum_{i=1}^r B_t^{(i)}. \quad (33)$$

Similarly, c_t, D_t are obtained by

$$c_t = \mathbb{E}\{h_t(x_t)|Y^{t-1}\} \equiv \hat{h}_{t|t-1}(x_t) \quad (34)$$

$$D_t \left[H_t \mid (\hat{X}_{t|t-1} - \mu_{t|t-1}) \right] = \left[\Phi_t \mid 0_{m \times 1} \right] \quad (35)$$

where

$$D_t = \begin{bmatrix} D_t^{(1)} & D_t^{(2)} & \cdots & D_t^{(r)} \end{bmatrix}; m \times nr \text{ matrix} \quad (36)$$

$$H_t = \begin{bmatrix} H_t^{(11)} & \cdots & H_t^{(1r)} \\ \vdots & \vdots & \vdots \\ H_t^{(r1)} & \cdots & H_t^{(rr)} \end{bmatrix}; nr \times nr \text{ matrix} \quad (37)$$

$$H_t^{(ij)} \equiv \mathbb{E}\left\{ (x_t - \mu_{t|t-1}^{(i)})(x_t - \mu_{t|t-1}^{(j)})^T | Y^{t-1} \right\} \quad (38)$$

$$\Phi_t = [\Phi_{0,t} \ \Phi_{0,t} \ \cdots \ \Phi_{0,t}]; m \times nr \text{ matrix} \quad (39)$$

$$\Phi_{0,t} \equiv \mathbb{E}\left\{ h_t(x_t) x_t^T | Y^{t-1} \right\} - c_t \hat{x}_{t|t-1}^T. \quad (40)$$

Then, similar to (31), we can approximate

$$\begin{aligned} h_t(x_t) &\approx c_t + \sum_{i=1}^r D_t^{(i)} (x_t - \mu_{t|t-1}^{(i)}) \\ &\equiv c_{S,t} + D_{S,t} x_t \end{aligned} \quad (41)$$

$$c_{S,t} \equiv c_t - \sum_{i=1}^r D_t^{(i)} \mu_{t|t-1}^{(i)} \quad (42)$$

$$D_{S,t} \equiv \sum_{i=1}^r D_t^{(i)}: m \times n. \quad (43)$$

Therefore from (31) and (41), the nonlinear state space models in (1), (2) are approximated by the quasi-linear equations:

$$x_{t+1} = B_{S,t} x_t + a_{S,t} + w_t \quad (44)$$

$$y_t = D_{S,t} x_t + c_{S,t} + v_t. \quad (45)$$

3. EqGS Filter

We now present a new and promising nonlinear filter by combining the Gaussian sum filter and the stochastic linearization technique. Similar to the derivation of the Gaussian sum filter, we apply the Bayesian rule to the CPDF $p(x_t|Y^t)$ as follows[19]:

$$\begin{aligned} p(x_t|Y^t) &= \frac{p(x_t, Y^t)}{p(Y^t)} = \frac{p(x_t, Y^{t-1}, y_t)}{p(Y^{t-1}, y_t)} \\ &= \frac{p(y_t|x_t)p(x_t|Y^{t-1})}{p(y_t|Y^{t-1})} = \frac{p(x_t|Y^{t-1})p(y_t|x_t)}{\int p(x_t|Y^{t-1})p(y_t|x_t)dx_t} \\ &= \frac{\sum_{i=1}^r \alpha_{t|t-1}^{(i)} N(x_t: \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) p(y_t - h_t(x_t))}{\int \sum_{i=1}^r \alpha_{t|t-1}^{(i)} N(x_t: \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) p(y_t - h_t(x_t)) dx_t} \\ &\approx \frac{\sum_{i=1}^r \alpha_{t|t-1}^{(i)} N(x_t: \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) N(\xi_t: 0, R_t)}{\int \sum_{i=1}^r \alpha_{t|t-1}^{(i)} N(x_t: \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) N(\xi_t: 0, R_t) dx_t} \end{aligned} \quad (46)$$

where

$$\begin{aligned} y_t - h_t(x_t) &\approx \xi_t \\ &\equiv y_t - D_{S,t} x_t - c_{S,t} = d_t - D_{S,t} x_t, \end{aligned} \quad (47)$$

$$d_t \equiv y_t - c_{S,t}. \quad (48)$$

For evaluating (46), we use the relation:

$$\begin{aligned} &N(x_t: \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) N(\xi_t: 0, R_t) \\ &= \frac{1}{2} \left[(x_t - \mu_{t|t-1}^{(i)})^T (P_{t|t-1}^{(i)})^{-1} (x_t - \mu_{t|t-1}^{(i)}) + \xi_t^T R_t^{-1} \xi_t \right] \\ &= \frac{1}{(\sqrt{2\pi})^{n+m} |P_{t|t-1}^{(i)}|^{1/2} |R_t|^{1/2}}. \end{aligned} \quad (49)$$

Then the power term of the above exponential function is completed by the square as follows. (Here and hereafter the lower subscripts are sometimes omitted, due to the limited space.)

$$(x_t - \mu_{t|t-1}^{(i)})^T (P_{t|t-1}^{(i)})^{-1} (x_t - \mu_{t|t-1}^{(i)}) + \xi_t^T R_t^{-1} \xi_t$$

$$\begin{aligned}
& \equiv (x - \mu^{(i)})^T (P^{(i)})^{-1} (x - \mu^{(i)}) \\
& \quad + (d - D_S x)^T R^{-1} (d - D_S x) \\
& = \left[x - \left((P^{(i)})^{-1} + D_S^T R^{-1} D_S \right)^{-1} \right. \\
& \quad \times \left((P^{(i)})^{-1} \mu^{(i)} + D_S^T R^{-1} d \right) \Big]^T \\
& \quad \times \left((P^{(i)})^{-1} + D_S^T R^{-1} D_S \right) \\
& \quad \times \left[x - \left((P^{(i)})^{-1} + D_S^T R^{-1} D_S \right)^{-1} \right. \\
& \quad \times \left((P^{(i)})^{-1} \mu^{(i)} + D_S^T R^{-1} d \right) \Big] \\
& \quad - \left((P^{(i)})^{-1} \mu^{(i)} + D_S^T R^{-1} d \right)^T \\
& \quad \times \left((P^{(i)})^{-1} + D_S^T R^{-1} D_S \right)^{-1} \\
& \quad \times \left((P^{(i)})^{-1} \mu^{(i)} + D_S^T R^{-1} d \right) \\
& \quad + (\mu^{(i)})^T (P^{(i)})^{-1} \mu^{(i)} + d^T R^{-1} d. \tag{50}
\end{aligned}$$

Further, for the terms after the second term at the right-hand side in (50), we proceed to complete the square with respect to d , which contains the observation y in (48). Namely, define

$$W^{(i)} \equiv \left[(P^{(i)})^{-1} + D_S^T R^{-1} D_S \right]^{-1}, \tag{51}$$

then, the terms after the second term in (50) are described by (52) which appears at the top of the next page, Therefore, from (49), (50) and (52), we have the relation:

$$\begin{aligned}
& N(x_t : \mu_{t|t-1}^{(i)}, P_{t|t-1}^{(i)}) N(\xi_t : 0, R_t) \\
& = k_t^{(i)} N(x_t : m_{x_t}^{(i)}, \Sigma_{x_t}^{(i)}) N(y_t : m_{y_t}^{(i)}, \Sigma_{y_t}^{(i)}) \tag{53}
\end{aligned}$$

where

$$\begin{aligned}
k_t^{(i)} & \equiv \frac{\left| \left[(P^{(i)})^{-1} + D_S^T R_t^{-1} D_S \right]^{-1} \right|^{1/2} |R_t + D_S P^{(i)} D_S^T|^{1/2}}{\left| P_{t|t-1}^{(i)} \right|^{1/2} |R_t|^{1/2}} \\
m_{x_t}^{(i)} & \equiv \left[D_{S,t}^T R_t^{-1} D_{S,t} + (P_{t|t-1}^{(i)})^{-1} \right]^{-1} \\
& \quad \times \left[(P_{t|t-1}^{(i)})^{-1} \mu_{t|t-1}^{(i)} + D_{S,t}^T R_t^{-1} d_t \right] \\
\Sigma_{x_t}^{(i)} & \equiv \left[(P_{t|t-1}^{(i)})^{-1} + D_{S,t}^T R_t^{-1} D_{S,t} \right]^{-1} \\
m_{y_t}^{(i)} & \equiv c_{S,t} + D_{S,t} \mu_{t|t-1}^{(i)} \\
\Sigma_{y_t}^{(i)} & \equiv D_{S,t} P_{t|t-1}^{(i)} D_{S,t}^T + R_t.
\end{aligned}$$

Here we assume the Gaussian sum form for (46):

$$p(x_t | Y^t) = \sum_{i=1}^r \alpha_{t|t}^{(i)} N(x_t : \mu_{t|t}^{(i)}, P_{t|t}^{(i)}), \tag{54}$$

then, from (46), (53)

$$\begin{aligned}
& \sum_{i=1}^r \alpha_{t|t}^{(i)} N(x_t : \mu_{t|t}^{(i)}, P_{t|t}^{(i)}) \\
& \approx \frac{\sum_{i=1}^r \alpha_{t|t-1}^{(i)} k_t^{(i)} N(x_t : m_{x_t}^{(i)}, \Sigma_{x_t}^{(i)}) N(y_t : m_{y_t}^{(i)}, \Sigma_{y_t}^{(i)})}{\int \sum_{j=1}^r \alpha_{t|t-1}^{(j)} k_t^{(j)} N(x_t : m_{x_t}^{(j)}, \Sigma_{x_t}^{(j)}) N(y_t : m_{y_t}^{(j)}, \Sigma_{y_t}^{(j)}) dx_t} \\
& = \frac{\sum_{i=1}^r \alpha_{t|t-1}^{(i)} k_t^{(i)} N(x_t : m_{x_t}^{(i)}, \Sigma_{x_t}^{(i)}) N(y_t : m_{y_t}^{(i)}, \Sigma_{y_t}^{(i)})}{\sum_{j=1}^r \alpha_{t|t-1}^{(j)} k_t^{(j)} N(y_t : m_{y_t}^{(j)}, \Sigma_{y_t}^{(j)})} \tag{55}
\end{aligned}$$

Then by applying the matrix inversion lemma, we have

$$\begin{aligned}
\mu_{t|t}^{(i)} & = m_{x_t}^{(i)} \\
& = \left[(P_{t|t-1}^{(i)})^{-1} + D_{S,t}^T R^{-1} D_S \right]^{-1} \\
& \quad \times \left[(P_{t|t-1}^{(i)})^{-1} \mu_{t|t-1}^{(i)} + D_{S,t}^T R^{-1} (y - c_S) \right] \\
& = \mu_{t|t-1}^{(i)} + \left[(P_{t|t-1}^{(i)})^{-1} + D_{S,t}^T R^{-1} D_S \right]^{-1} \\
& \quad \times D_{S,t}^T R^{-1} (y - c_S - D_S \mu_{t|t-1}^{(i)}) \\
& = \mu_{t|t-1}^{(i)} + \left[P_{t|t-1}^{(i)} - P_{t|t-1}^{(i)} D_{S,t}^T (D_S P_{t|t-1}^{(i)} D_S^T \right. \\
& \quad \left. + R)^{-1} D_S P_{t|t-1}^{(i)} \right] D_{S,t}^T R^{-1} (y - c_S - D_S \mu_{t|t-1}^{(i)}) \\
& = \mu_{t|t-1}^{(i)} + \left[P_{t|t-1}^{(i)} - K_t^{(i)} D_{S,t} P_{t|t-1}^{(i)} \right] D_{S,t}^T R^{-1} \\
& \quad \times (y - c_{S,t} - D_{S,t} \mu_{t|t-1}^{(i)}). \tag{56}
\end{aligned}$$

Also we have the relation

$$\begin{aligned}
P_{t|t}^{(i)} & = \Sigma_{x_t}^{(i)} \\
& = \left[(P_{t|t-1}^{(i)})^{-1} + D_{S,t}^T R_t^{-1} D_{S,t} \right]^{-1} \\
& = P_{t|t-1}^{(i)} - P_{t|t-1}^{(i)} D_{S,t}^T \\
& \quad \times (D_{S,t} P_{t|t-1}^{(i)} D_{S,t}^T + R_t)^{-1} D_{S,t} P_{t|t-1}^{(i)}.
\end{aligned}$$

Then finally we have the following equations:

$$\mu_{t|t}^{(i)} = \mu_{t|t-1}^{(i)} + \left[P_{t|t-1}^{(i)} - K_t^{(i)} D_{S,t} P_{t|t-1}^{(i)} \right] \times D_{S,t}^T R_t^{-1} (y_t - c_{S,t} - D_{S,t} \mu_{t|t-1}^{(i)}) \tag{57}$$

$$P_{t|t}^{(i)} = P_{t|t-1}^{(i)} - K_t^{(i)} D_{S,t} P_{t|t-1}^{(i)} \tag{58}$$

$$K_t^{(i)} \equiv P_{t|t-1}^{(i)} D_{S,t}^T [D_{S,t} P_{t|t-1}^{(i)} D_{S,t}^T + R_t]^{-1} \tag{59}$$

Also from (55), we have

$$\begin{aligned}
\alpha_{t|t}^{(i)} & \approx \frac{\alpha_{t|t-1}^{(i)} k_t^{(i)} N(y_t : m_{y_t}^{(i)}, \Sigma_{y_t}^{(i)})}{\sum_{j=1}^r \alpha_{t|t-1}^{(j)} k_t^{(j)} N(y_t : m_{y_t}^{(j)}, \Sigma_{y_t}^{(j)})} \\
& \equiv \frac{\alpha_{t|t-1}^{(i)} \beta_t^{(i)}}{\sum_{j=1}^r \alpha_{t|t-1}^{(j)} \beta_t^{(j)}}, \tag{60}
\end{aligned}$$

where

$$\begin{aligned}
\beta_t^{(i)} & \equiv k_t^{(i)} N(y_t : c_{S,t} + D_{S,t} \mu_{t|t-1}^{(i)}, D_{S,t} P_{t|t-1}^{(i)} D_{S,t}^T + R) \\
& = \frac{\left| \left[(P_{t|t-1}^{(i)})^{-1} + D_{S,t}^T R_t^{-1} D_S \right]^{-1} \right|^{1/2}}{(\sqrt{2\pi})^m |P_{t|t-1}^{(i)}|^{1/2} |R_t|^{1/2}} \\
& \quad \times e^{-\frac{1}{2} \left[(y - c_S - D_S \mu)^T (D_S P D_S^T + R)^{-1} (y - c_S - D_S \mu) \right]} \tag{61}
\end{aligned}$$

Therefore the filtered estimate can be computed by

$$\begin{aligned}
\hat{x}_{t|t} & = E[x_t | Y^t] = \int_{-\infty}^{\infty} x_t p(x_t | Y^t) dx_t \\
& = \int_{-\infty}^{\infty} x_t \sum_{i=1}^r \alpha_{t|t}^{(i)} N(x_t : \mu_{t|t}^{(i)}, P_{t|t}^{(i)}) dx_t \\
& = \sum_{i=1}^r \alpha_{t|t}^{(i)} \mu_{t|t}^{(i)}. \tag{62}
\end{aligned}$$

$$\begin{aligned}
& -[(P^{(i)})^{-1}\mu^{(i)} + D_S^T R^{-1}d]^T [(P^{(i)})^{-1} + D_S^T R^{-1}D_S]^{-1} [(P^{(i)})^{-1}\mu^{(i)} + D_S^T R^{-1}d] + (\mu^{(i)})^T (P^{(i)})^{-1}\mu^{(i)} + d^T R^{-1}d \\
& = -[(\mu^{(i)})^T (P^{(i)})^{-1} + d^T R^{-1}D_S] W^{(i)} [(P^{(i)})^{-1}\mu^{(i)} + D_S^T R^{-1}d] + (\mu^{(i)})^T (P^{(i)})^{-1}\mu^{(i)} + d^T R^{-1}d \\
& = \left[d - (R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1})^{-1} R^{-1} D_S W^{(i)} (P^{(i)})^{-1} \mu^{(i)} \right]^T (R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1}) \\
& \quad \times \left[d - (R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1})^{-1} R^{-1} D_S W^{(i)} (P^{(i)})^{-1} \mu^{(i)} \right] \\
& \quad + (\mu^{(i)})^T \left[(P^{(i)})^{-1} - (P^{(i)})^{-1} W^{(i)} (P^{(i)})^{-1} - (P^{(i)})^{-1} W^{(i)} D_S^T R^{-1} (R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1})^{-1} R^{-1} D_S W^{(i)} (P^{(i)})^{-1} \right] \mu^{(i)} \\
& = (d - D_S \mu^{(i)})^T (R + D_S P^{(i)} D_S^T)^{-1} (d - D_S \mu^{(i)}) = (y - c_S - D_S \mu^{(i)})^T (R + D_S P^{(i)} D_S^T)^{-1} (y - c_S - D_S \mu^{(i)}), \tag{52}
\end{aligned}$$

where we have applied the matrix inversion lemma: $R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1} = (R + D_S P^{(i)} D_S^T)^{-1}$,

and the relations:

$$\begin{aligned}
& (R^{-1} - R^{-1}D_S W^{(i)} D_S^T R^{-1})^{-1} R^{-1} D_S W^{(i)} (P^{(i)})^{-1} = (R + D_S P^{(i)} D_S^T)^{-1} R^{-1} D_S [(P^{(i)})^{-1} + D_S^T R^{-1}D_S]^{-1} (P^{(i)})^{-1} \\
& = (D_S + D_S P^{(i)} D_S^T R^{-1} D_S) (I + P^{(i)} D_S^T R^{-1} D_S)^{-1} = D_S \quad (\text{appear on 3rd and 4th lines in (52)}),
\end{aligned}$$

and

$$\begin{aligned}
& (P^{(i)})^{-1} - (P^{(i)})^{-1} W^{(i)} (P^{(i)})^{-1} - (P^{(i)})^{-1} W^{(i)} D_S^T R^{-1} D_S \\
& = (P^{(i)})^{-1} - (P^{(i)})^{-1} ((P^{(i)})^{-1} + D_S^T R^{-1} D_S)^{-1} (P^{(i)})^{-1} - (P^{(i)})^{-1} ((P^{(i)})^{-1} + D_S^T R^{-1} D_S)^{-1} D_S^T R^{-1} D_S \\
& = (P^{(i)})^{-1} - (P^{(i)})^{-1} [(P^{(i)})^{-1} + D_S^T R^{-1} D_S]^{-1} ((P^{(i)})^{-1} + D_S^T R^{-1} D_S) = O \quad (\text{appear on the 5th line in (52)}).
\end{aligned}$$

Also the filtered error covariance matrix $P_{t|t}$ is given by

$$\begin{aligned}
P_{t|t} & \equiv E\left\{ (x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T \middle| Y^t \right\} \\
& = E\left\{ x_t x_t^T \middle| Y^t \right\} - \hat{x}_{t|t} \hat{x}_{t|t}^T \\
& = \sum_{i=1}^r \alpha_{t|t}^{(i)} (P_{t|t}^{(i)} + \mu_{t|t}^{(i)} (\mu_{t|t}^{(i)})^T) - \hat{x}_{t|t} \hat{x}_{t|t}^T. \tag{63}
\end{aligned}$$

Therefore the quasi-linear filter with Gaussian sum CPDFs is summarized as the following formula of updating:

$$\hat{x}_{t|t} = \sum_{i=1}^r \alpha_{t|t}^{(i)} \mu_{t|t}^{(i)}, \tag{64}$$

with (57)-(61).

Time updating formula

Let us derive the time updating formula as follows. Define

$$\hat{x}_{t+1|t} \equiv \sum_{i=1}^r \alpha_{t+1|t}^{(i)} \mu_{t+1|t}^{(i)}. \tag{65}$$

Also from (44), we have the relations:

$$\begin{aligned}
\hat{x}_{t+1|t} & = B_{S,t} \hat{x}_{t|t} + a_{S,t} \\
& = B_{S,t} \sum_{i=1}^r \alpha_{t|t}^{(i)} \mu_{t|t}^{(i)} + a_{S,t} \\
& = \sum_{i=1}^r \alpha_{t|t}^{(i)} [B_{S,t} \mu_{t|t}^{(i)} + a_{S,t}], \tag{66}
\end{aligned}$$

where we use the relation: $\sum_{i=1}^r \alpha_{t|t}^{(i)} = 1$. Comparing (66) with (65), we have

$$\alpha_{t+1|t}^{(i)} = \alpha_{t|t}^{(i)} \tag{67}$$

$$\mu_{t+1|t}^{(i)} = a_{S,t} + B_{S,t} \mu_{t|t}^{(i)}. \tag{68}$$

Also define

$$P_{t+1|t}^{(i)} \equiv E\left\{ [x_{t+1} - \mu_{t+1|t}^{(i)}] [x_{t+1} - \mu_{t+1|t}^{(i)}]^T \middle| Y^t \right\}, \tag{69}$$

then from (44) and (68), we have

$$x_{t+1} - \mu_{t+1|t}^{(i)} = B_{S,t} x_t + w_t - B_{S,t} \mu_{t|t}^{(i)}. \tag{70}$$

Therefore, from (69) and (70), we have the relation:

$$P_{t+1|t}^{(i)} = B_{S,t} P_{t|t}^{(i)} B_{S,t}^T + Q_t \tag{71}$$

Then, however, we need to obtain a_t , $\{B_t^{(i)}\}$, c_t , $\{D_t^{(i)}\}$ such that we should evaluate the values of

$$\begin{aligned}
& E[f_t(x_t) | Y^t], \quad E[f_t(x_t) x_t^T | Y^t] \\
& E[h_t(x_t) | Y^{t-1}], \quad E[h_t(x_t) x_t^T | Y^{t-1}]
\end{aligned}$$

which appear in (21), (27), (34) and (35).

(Remark 1) In the case of the system noise w_t having the Gaussian sum distribution[16], namely,

$$p(w_t) \equiv \sum_{i=1}^{q_t} \gamma_t^{(i)} N(w_t; \eta_t^{(i)}, Q_t^{(i)}), \tag{72}$$

we had derived the similar form of the EqGS filter (see[12]).

4. Evaluation of Conditional Expected Values

Our EqGS filter requires to evaluate the expectation of (72) under the assumption of the conditional Gaussian (or, Gaussian sum) PDF. For this purpose, we have already shown a method to evaluate the higher moments by applying the moment generating function[12]. Here we briefly show the method to

evaluate the moments of the vector random variable which have Gaussian distributions.

Namely, let the CPDF for n -dimensional Gaussian random vector $x = [x_1 \ x_2 \ \dots \ x_n]^T$ is described by

$$p(x|Y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \times \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\},$$

where μ and Σ are conditional mean and conditional covariance matrix, respectively, as follows

$$\mu = E[x|Y] \quad (73)$$

$$\Sigma = E[(x-\mu)(x-\mu)^T|Y] \\ = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{bmatrix}. \quad (74)$$

Let $s = [s_1 \ s_2 \ \dots \ s_n]^T$, then it is well known that the moment generating function for the multivariate normal distribution is given by

$$M(s) \equiv E[e^{s^T x}|Y] = \exp\left\{s^T \mu + \frac{1}{2}s^T \Sigma s\right\} \quad (75)$$

For example, we can evaluate the conditional expectation of a fifth moment $E[x_{\nu_1} x_{\nu_2} x_{\nu_3} x_{\nu_4} x_{\nu_5}|Y]$ as follows. Namely, applying the relation,

$$\frac{\partial^5 M(s)}{\partial s_{\nu_1} \partial s_{\nu_2} \partial s_{\nu_3} \partial s_{\nu_4} \partial s_{\nu_5}} \Big|_{s=0} = E[x_{\nu_1} x_{\nu_2} x_{\nu_3} x_{\nu_4} x_{\nu_5}|Y],$$

we have

$$E[x_{\nu_1} x_{\nu_2} \dots x_{\nu_5}|Y^t] \\ = \sigma_{\nu_1 \nu_2} \sigma_{\nu_3 \nu_4} \mu_{\nu_5} + \sigma_{\nu_1 \nu_2} \sigma_{\nu_3 \nu_5} \mu_{\nu_4} + \sigma_{\nu_1 \nu_2} \sigma_{\nu_4 \nu_5} \mu_{\nu_3} \\ + \sigma_{\nu_1 \nu_3} \sigma_{\nu_2 \nu_4} \mu_{\nu_5} + \sigma_{\nu_1 \nu_3} \sigma_{\nu_2 \nu_5} \mu_{\nu_4} + \sigma_{\nu_1 \nu_3} \sigma_{\nu_4 \nu_5} \mu_{\nu_2} \\ + \sigma_{\nu_1 \nu_4} \sigma_{\nu_2 \nu_3} \mu_{\nu_5} + \sigma_{\nu_1 \nu_4} \sigma_{\nu_2 \nu_5} \mu_{\nu_3} + \sigma_{\nu_1 \nu_4} \sigma_{\nu_3 \nu_5} \mu_{\nu_2} \\ + \sigma_{\nu_1 \nu_5} \sigma_{\nu_2 \nu_3} \mu_{\nu_4} + \sigma_{\nu_1 \nu_5} \sigma_{\nu_2 \nu_4} \mu_{\nu_3} + \sigma_{\nu_1 \nu_5} \sigma_{\nu_3 \nu_4} \mu_{\nu_2} \\ + \sigma_{\nu_2 \nu_3} \sigma_{\nu_4 \nu_5} \mu_{\nu_1} + \sigma_{\nu_2 \nu_4} \sigma_{\nu_3 \nu_5} \mu_{\nu_1} + \sigma_{\nu_2 \nu_5} \sigma_{\nu_3 \nu_4} \mu_{\nu_1} \\ + \sigma_{\nu_1 \nu_2} \mu_{\nu_3} \mu_{\nu_4} \mu_{\nu_5} + \sigma_{\nu_1 \nu_3} \mu_{\nu_2} \mu_{\nu_4} \mu_{\nu_5} + \sigma_{\nu_1 \nu_4} \mu_{\nu_2} \mu_{\nu_3} \mu_{\nu_5} \\ + \sigma_{\nu_1 \nu_5} \mu_{\nu_2} \mu_{\nu_3} \mu_{\nu_4} + \sigma_{\nu_2 \nu_3} \mu_{\nu_1} \mu_{\nu_4} \mu_{\nu_5} + \sigma_{\nu_2 \nu_4} \mu_{\nu_1} \mu_{\nu_3} \mu_{\nu_5} \\ + \sigma_{\nu_2 \nu_5} \mu_{\nu_1} \mu_{\nu_3} \mu_{\nu_4} + \sigma_{\nu_3 \nu_4} \mu_{\nu_1} \mu_{\nu_2} \mu_{\nu_5} + \sigma_{\nu_3 \nu_5} \mu_{\nu_1} \mu_{\nu_2} \mu_{\nu_4} \\ + \sigma_{\nu_4 \nu_5} \mu_{\nu_1} \mu_{\nu_2} \mu_{\nu_3} + \mu_{\nu_1} \mu_{\nu_2} \mu_{\nu_3} \mu_{\nu_4} \mu_{\nu_5}.$$

The numerical integration technique consist of approximating the integrand by a polynomial in a region or regions. Namely, a complicated integral of $h(x)$ is factored into a nonnegative weight function $\omega(x)$ and another function $g(x)$:

$$\int_a^b h(x) dx = \int_a^b \omega(x) g(x) dx \approx \sum_{j=1}^N w_j g(z_j). \quad (76)$$

Then we have the following integration rules called as Gaussian quadrature rules.

Gaussian Quadrature Rules[13]

Let $\omega(x) \geq 0$ be a weight function defined on $[a, b]$. For $\omega(x)$, it is possible to define a sequence of orthonormal polynomials $q_1(x), q_2(x), \dots$ with respect to

$\omega(x)$ and in which $q_j(x)$ is of exact degree j so that

$$\int_a^b q_j(x) q_k(x) \omega(x) dx = \begin{cases} 1; & j = k \\ 0; & j \neq k \end{cases} \quad (77)$$

The polynomial

$$q_m(x) = k_m \prod_{j=1}^m (x - z_j), k_m > 0$$

has m real roots $a < z_1 < z_2 < \dots < z_m < b$. Then we have the following theorems[13,14].

[Theorem 1]

Let $g(x) \in C^{2m}[a, b]$, then

$$\int_a^b \omega(x) g(x) dx = \sum_{j=1}^m w_j g(z_j) + \frac{g^{(2m)}(\xi)}{(2m)! k_m^2} \quad (78) \\ (a < \xi < b)$$

where

$$w_j = -\frac{k_{m+1}}{k_m} \frac{1}{q_{m+1}(z_j) q'_m(z_j)} \quad (79)$$

Thus the Gauss quadrature rule is exact for all polynomials of degree $\leq 2m - 1$.

[Theorem 2]

Let us consider any set of orthonormal polynomials $\{q_j(x)\}_{j=1}^m$, satisfying a three recurrence relationship:

$$q_j(x) = (a_j x + b_j) q_{j-1}(x) - c_j q_{j-2}(x), \quad (80)$$

$$j = 1, \dots, m$$

$$q_{-1}(x) \equiv 0, \quad q_0(x) \equiv 1, \quad (81)$$

with $a_j > 0$ and $c_j > 0$. Then (80) is described by the matrix equation:

$$x \mathbf{q}(x) = T \mathbf{q}(x) + \frac{1}{a_m} q_m(x) \mathbf{e}_m \quad (82)$$

where T is the tridiagonal matrix;

$$T \equiv \begin{bmatrix} -\frac{b_1}{a_1} & \frac{1}{a_1} & 0 & \dots & \dots & \dots & 0 \\ \frac{c_2}{a_2} & -\frac{b_2}{a_2} & \frac{1}{a_2} & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & & 0 \\ \vdots & & & & \ddots & \ddots & \frac{1}{a_m} \\ 0 & \dots & \dots & 0 & \frac{c_m}{a_m} & -\frac{b_m}{a_m} & \end{bmatrix} \quad (83)$$

and

$$\mathbf{q}(x) \equiv [q_0(x), \dots, q_{m-1}(x)]^T \quad (84)$$

$$\mathbf{e}_m \equiv [0, \dots, 0, 1]^T. \quad (85)$$

Then $\{z_j; j = 1, 2, \dots, m\}$ are eigenvalues of T and $\{w_j; j = 1, 2, \dots, m\}$ equal to $|(q_j)_1|^2$ where $(q_j)_1$ is the first element of j th normalized eigenvector of T .

Namely,

$$z_j \mathbf{q}(z_j) = T \mathbf{q}(z_j) \quad (86)$$

$$\mathbf{q}^T(z_j) \mathbf{q}(z_j) = 1 \quad (87)$$

Let us now consider the Gaussian-type numerical integration of a scalar function $g(x)$. Namely,

$$\int_{-\infty}^{\infty} g(x) \omega(x) dx = \sum_{i=1}^m w_i g(z_i) \quad (88)$$

where

$$\omega(x) = e^{-x^2} \quad (89)$$

The conditional expectation of the other nonlinear functions of x_t in (72) as well as the polynomials can be numerically computed by applying Gauss-Hermite quadrature rules[13,9,10,15].

Gauss-Hermite Quadrature Scheme

It is well known that Hermite polynomials $\{H_j(x); j=1, \dots, m\}$ are given by the following three term recurrence relationship:

$$H_j(x) = 2xH_{j-1} - (j-1)H_{j-2}, \quad (90)$$

$$H_{-1}(x) \equiv 0, \quad H_0(x) \equiv 1, \quad (91)$$

and

$$\int_{-\infty}^{\infty} H_j(x) H_k(x) e^{-x^2} dx = \delta_{j-k} 2^j j! \sqrt{\pi}. \quad (92)$$

Thus the normalized weight $\omega(x)$ is given by

$$\omega(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \quad \text{due to} \quad \int_{-\infty}^{\infty} \omega(x) dx = 1. \quad (93)$$

From (92), the corresponding normalized Hermite polynomial $\{\tilde{H}_j(x); j=1, \dots, m\}$ are also given by

$$\tilde{H}_j(x) = \frac{1}{\sqrt{2^j j!}} H_j(x), \quad (94)$$

or

$$H_j(x) = \sqrt{2^j j!} \tilde{H}_j(x). \quad (95)$$

Substituting (95) into (90), we have the three term recurrence equations regarding to $\{\tilde{H}_j(x); j=1, \dots, m\}$ as follows:

$$\tilde{H}_j(x) = \sqrt{\frac{2}{j}} x \tilde{H}_{j-1} - \sqrt{\frac{j-1}{j}} \tilde{H}_{j-2}, \quad (96)$$

$$\tilde{H}_{-1}(x) \equiv 0, \quad \tilde{H}_0(x) \equiv 1, \quad (97)$$

Thus, in (80)

$$a_j = \sqrt{\frac{2}{j}}, \quad b_j = 0, \quad c_j = \sqrt{\frac{j-1}{j}} \quad (98)$$

such that in (83)

$$\frac{c_{j+1}}{a_{j+1}} = \sqrt{\frac{j}{2}} = \frac{1}{a_j}, \quad i = 1, \dots, m-1. \quad (99)$$

Therefore, T in (83) becomes the symmetric tridiagonal matrix with zero diagonals and $T_{j,j+1} = \sqrt{j/2}, j=1, \dots, m-1$. Then the expected value of $F(t)$ with n -dimensional Gaussian PDF $N(t; \mu, \Sigma)$ [9]:

$$I = \int_{R^n} F(t) \frac{1}{((2\pi)^n \det \Sigma)^{1/2}} \times \exp \left\{ -\frac{1}{2} (t - \mu)^T \Sigma^{-1} (t - \mu) \right\} dt \quad (100)$$

can be evaluated as follows.

Let $\Sigma = S^T S$ and $t = \mu + \sqrt{2} S^T z$, then

$$\begin{aligned} & -\frac{1}{2} (t - \mu)^T \Sigma^{-1} (t - \mu) \\ &= -\frac{1}{2} \sqrt{2} z^T S S^{-1} S^{-T} \sqrt{2} S^{-T} z = -z^T z = -\|z\|^2 \end{aligned}$$

and $dt = (\sqrt{2})^n \det S \, dz$. Therefore, we have

$$\begin{aligned} I &= \int_{R^n} F(\mu + \sqrt{2} S^T z) \\ &\quad \times \frac{1}{(\sqrt{2\pi})^n \det S} e^{-\|z\|^2} (\sqrt{2})^n \det S \, dz \\ &= \int_{R^n} F(\mu + \sqrt{2} S^T z) \frac{1}{(\sqrt{\pi})^n} e^{-\|z\|^2} dz \\ &\approx \sum_{j_1=1}^m \cdots \sum_{j_n=1}^m F(\mu + \sqrt{2} S^T [z_{j_1}, \dots, z_{j_n}]^T) w_{j_1} \cdots w_{j_n} \end{aligned} \quad (101)$$

By calculating the eigenvalues and the corresponding normalized eigenvectors of T in (83) with (98) and (99), we have and $z_1 = 2/\sqrt{2}, z_1 = -2/\sqrt{2}$ and $w_1 = w_2 = 1/2$ for $m=2$; $z_1 = 0, z_2 = \sqrt{3}/2, z_2 = -\sqrt{3}/2$ and $w_1 = 2/3, w_2 = w_3 = 1/6$ for $m=3$.

5. Numerical Examples of AR(1) Modeling

Let us show AR(1) modeling by applying EqGS filter, and show what kinds of evaluation for conditional expected values are required.

Consider a AR(1) model with a linear measurement equation as follows:

$$x_{t+1} = a x_t + w_t \quad (102)$$

$$y_t = x_t + v_t \quad (103)$$

$$w \sim N(0, Q), \quad v \sim N(0, R).$$

Then let $x_{1,t} \equiv x_t$, $x_{2,t} \equiv a$, we have the state and measurement equations:

$$\begin{aligned} \begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} &= \begin{bmatrix} x_{1,t} x_{2,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_t \\ &= f(x_t) + e_t w_t \end{aligned} \quad (104)$$

and

$$y_t = [1 \ 0] \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + v_t \quad (105)$$

From (21), we have

$$a_t = E\{f(x_t)|Y^t\} = \sum_{i=1}^r \alpha_{t|t}^{(i)} \begin{bmatrix} \mu_{1,t|t}^{(i)} \mu_{2,t|t}^{(i)} + P_{12,t|t}^{(i)} \\ \mu_{1,t|t}^{(i)} \end{bmatrix}$$

Then also from (15)

$$F_t^{(ij)} = P_{t|t} + \hat{x}_{t|t} \hat{x}_{t|t}^T - \hat{x}_{t|t} (\mu_{t|t}^{(j)})^T - \mu_{t|t}^{(i)} \hat{x}_{t|t}^T + \mu_{t|t}^{(i)} (\mu_{t|t}^{(j)})^T. \quad (106)$$

From (17), we have

$$\Gamma_t^{(0)} = E\{f(x_t)x_t^T|Y^t\} - a_t \sum_{k=1}^r \alpha_{t|t}^{(k)} (\mu_{t|t}^{(k)})^T \quad (107)$$

The first term of the right hand side in (107) is given by

$$E\{f(x_t)x_t^T|Y^t\} = E\left\{ \begin{bmatrix} x_{1,t} x_{2,t} \\ x_{2,t} \end{bmatrix} \begin{bmatrix} x_{1,t} & x_{2,t} \end{bmatrix}^T | Y^t \right\}$$

These conditional expected values can be evaluated, for example, the (1,1) element:

$$E\{x_{1,t}^2 x_{2,t} | Y^t\} = \sum_{k=1}^r \alpha_{t|t}^{(k)} \left[2\mu_1^{(k)} P_{12}^{(k)} + \mu_2^{(k)} P_{11}^{(k)} + (\mu_1^{(k)})^T \mu_2^{(k)} \right]$$

On the other hand, the measurement equation is linear in this model, therefore in (57)–(61) we have

$$D_{s,t} = H_t \equiv \begin{bmatrix} 1 & 0 \end{bmatrix}. \quad (108)$$

Then from (62)

$$\hat{x}_{t|t} = E\{x_t|Y^t\} = \sum_{k=1}^r \alpha_{t|t}^{(k)} \mu_{t|t}^{(k)} \quad (109)$$

We show simulation results of EqGS filter by comparing with the extended Kalman filter (EKF) and the equivalent linearization filter (EqF). As is mentioned in Section 2, EqGS filter is derived by extending EqF with the Gaussian sum assumptions, therefore the algorithm of EqF can be given by assuming that the CDPDFs in (6) are Gaussian, i.e. $r=1$.

For simulations, we assumed that $a=0.96$, $Q=1$ and $x_{1,0}=0$. In this case, $\text{Var}(x_{1,t}) = 1/(1-0.96^2) = 12.75$, and we assumed $R=12.75 (\equiv R_0)$. For EqGS filter, the number of Gaussian sum was 3, i.e. $r=3$, and the initial values were assumed to be

$$\hat{x}_{0|-1}^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \hat{x}_{0|-1}^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{x}_{0|-1}^{(3)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (110)$$

$$P_{0|-1}^{(1)} = P_{0|-1}^{(2)} = P_{0|-1}^{(3)} = \text{diag}(12.75, 0.5^2), \quad (111)$$

$$\alpha_{0|-1}^{(1)} = \alpha_{0|-1}^{(2)} = \alpha_{0|-1}^{(3)} = 1/3. \quad (112)$$

From (110)–(112), the initial values for EKF and EqF were given as follows.

Fig. 1 A sample of true state and observation processes

$$\hat{x}_{0|-1} = \sum_{k=1}^3 \alpha_{0|-1}^{(k)} \hat{x}_{0|-1}^{(k)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (113)$$

$$P_{0|-1} = \text{diag}(12.75, 0.917). \quad (114)$$

Fig.1 shows the sample trajectories of the true state and observation processes. And Fig.2 (a), (b) and (c) show the estimation results by EKF, EqF and EqGSF, respectively. To compare the performance of filters, 50 simulations with different trajectories were carried out, and the root mean square errors defined by the following formula were evaluated.

$$E_{i,t} = \sqrt{\frac{1}{50} \sum_{j=1}^{50} (x_{i,t}^{\{j\}} - \hat{x}_{i,t}^{\{j\}})^2}, \quad i=1,2, \quad (115)$$

where $\{j\}$ indicates the j -th run in 50 simulations. Table 1 shows the parameter estimation performance of $\hat{x}_{2,150|150}$. From Table 1 it can be seen that the performance of EqF and EqGSF are superior to EKF.

To evaluate the filter performance in noisy environment, similar simulations were carried out for $R=0.1R_0, 0.5R_0, 1.5R_0, 2R_0$, and $E_{2,150}$ values are plotted in Fig.3. From Fig.3 we can see that EqF and EqGSF show superior estimation results than EKF when the signal to noise ratio is lower (R is larger).

Table 1 $E_{2,150}$ of EKF, EqF and EqGSF

EKF	EqF	EqGSF
0.1086	0.0345	0.0326

6. Conclusions

We have shown the Gaussian Sum filter based on the stochastic Equivalent linearization (: EqGS filter) for discrete time nonlinear systems. Furthermore we discuss two methods to evaluate the conditional expected values of quantities having Gaussian distribution which are required to derive Eq and EqGS filters. Also throughout the numerical experiments, we show that from the aspect of accuracy of estimation, the EqGS filter is superior among three filters when measurement noise become larger. However we should point out that our EqGS filter requires a little heavy computational cost compare to others.

Fig. 2 Estimation results by EKF, EqF and EqGSF

Fig. 3 $E_{2,150}$ with respect to variations in R

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