On the Stochastic Filtering Theory of a Power System Dynamics*

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This paper revisits the Markovian state vector of a Single Machine Infinite Bus (SMIB) system in non-linear filtering framework. In power system dynamics, the non-linear stochastic swing equation was the subject of investigations in the Fokker-Planck setting. The Fokker-Planck setting accounts for the process noise correction term and ignores the observation noise correction term to analyse stochastic systems. In contrast to the Fokker-Planck setting, this paper introduces the notion of the ‘Kushner-Stratonovich setting’, which accounts for the process noise as well as observation noise correction terms in the conditional moment evolution equation. The Kushner-Stratonovich setting is the cornerstone formalism of non-linear filtering problems of stochastic control systems.

1. Introduction

In power system dynamics, non-linear swing equation has central importance [1] that assumes the structure of a second-order non-linear differential equation. The non-linear swing equation is a ramification of short circuits, lighting, small external perturbations. The power system stability dynamics that exploits non-linear swing equations can be explained by adopting the notion of the Lypunov stability of equilibrium point of the nonlinear differential system

\[ x_1 = f(x_1)(Arnold \ [2], \ [3]). \]

After accounting for the stochasticity of the mechanical input to the SMIB system, we arrive at non-linear ‘stochastic’ machine swing equation. A careful observation reveals that the qualitative characteristics of the second-order Phase Locked Loop (PLL) is similar to the second-order non-linear swing equation. The PLL system undergoes unlocking mode under a multiplier noise influence. Under the unlocking state of the PLL, the phase of the voltage controlled oscillator will not follow the reference signal phase. The phase difference and the phase difference derivative will evolve stochastically. Here, we summarize succinctly available results on the second-order non-linear swing equation as well as second-order non-linear stochastic swing equation. In [4], conditions for the Arnold diffusion phenomenon were examined using the vector-Melnikov method from the circuits and systems viewpoint. The conditions for the Arnold diffusion become analytically explicit under the influence of smaller oscillations. The stochastic swing equation is a consequence of stochasticity in the mechanical power of the power system dynamics coupled with short circuits and external perturbations. In theoretical studies of power system dynamics, the qualitative characteristics of the stochastic swing equation were studied in the Fokker-Planck setting [5]. The Fokker-Planck equation is the evolution of conditional probability density for Markov processes. The Fokker-Planck equation is a parabolic linear homogeneous equation of order two in partial differentiation for the transition probability density. The Fokker-Planck operator is a linear operator as well as it has adjoint property. The adjoint of the Fokker-Planck operator is the Kolmogorov backward operator. The Fokker-Planck operator as well as the Kolmogorov backward operator have central importance in stochastic control systems. This paper is inspired from an influential and a philosophical paper [6] authored by David Mumford, see [7] as well. Publications on non-linear swing equations of power systems engineering are relatively very scarce by bringing the notions of circuits, systems and stochastic processes. Relatively few researchers, e.g. [5], have revisited non-linear swing equation in the stochastic framework that sets future directions in power systems engineering. In the theory of Markov-Diffusion process, the evolution of conditional probability density becomes the Fokker-Planck equation. The Fokker-Planck equation has found applications to sketch the non-linear filtering theory as well. For the continuous state-continuous measurement system of stochastic

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systems, the Kushner-Stratonovich equation, a stochastic partial-integro differential equation, is a celebrated method to achieve the filtering. Note that the Kushner-Stratonovich equation assumes the structure of stochastic partial-integro differential equation and allows filtering of stochastic systems. On the other hand, the Fokker-Planck equation assumes the structure of a Partial Differential Equation and allows the prediction. The Fokker-Planck equation is a special case and the Kushner-Stratonovich equation is a general case. For this reason, we wish to revisit the non-linear swing equation by achieving filtering [8] of the swing equation. That can be regarded the noise analysis of non-linear swing equation in the Kushner-Stratonovich setting.

In this paper, first we construct a filtering model of the SMIB system. The filtering model accounts for the stochastic differential equation of the state vector coupled with noisy observation equation. Then, we write the Kushner-Stratonovich equation of the non-linear stochastic differential equation. Finally, we achieve the non-linear filtering equations of the SMIB system, which are the consequence of the Kushner-Stratonovich equation and the exact conditional moment stochastic evolution equation. This paper is an extension of a recently published paper [9] in the following senses: (i) this paper revisits the stochastic swing equation in the Kushner-Stratonovich setting. On the other hand, the Fokker-Planck setting and multi-dimensional stochastic differential rules were the major ingredients to analyse the stochasticity of the SMIB system [9]. (ii) This paper accounts for the contributions of the process noise correction term as well as the observation noise correction term in the non-linear filtering of the SMIB system. On the other hand, the observation noise correction term is ignored in [9].

**Notation** Throughout the paper, the notation \( \langle \rangle \) denotes the action of conditional expectation operator \( E \) on the Random Variable. For the brevity of notations, we adopt the notation \( \hat{x}_t = E(x_t)Y_t \) as well, where \( Y_t \) denotes the observations accumulated up to the time.

### 2. A Stochastic Swing Equation

![Fig. 1 A Single Machine Infinite Bus (SMIB) System](image)

Investigations on the ‘non-linear swing equation’ are not restricted to power systems dynamics researchers. That was the subject of research activities of circuits and systems dynamists as well, see an older, but an appealing paper [4]. With these motivations, we wish to analyse non-linear ‘stochastic’ swing equation for the single machine-infinite bus system. Fig. 1 is a schematic diagram of the SMIB system that accounts for the excitation of the synchronous generator, line parameters and terminal voltage. Note that the non-linear stochastic swing equation explains the qualitative characteristics of the SMIB system under the noise influence.

The mathematical method of the paper utilizes the Itô-Stratonovich integral relationship and higher-order estimation equations of stochastic processes. Consider a nonlinear stochastic differential system of the order two, i.e.

\[
\ddot{y}_t = F(t, y_t, \dot{y}_t, \xi_t).
\]

The term \( F(t, y_t, \dot{y}_t, \xi_t) \) has an interpretation as a random forcing term. That can be decomposed into two components, \( f_1(t, y_t, \dot{y}_t) \) and \( f_2(t, y_t, \dot{y}_t)\xi_t \). In the matrix-vector format, the above can be formalized in the SDE setting driven by the white noise \( \xi_t \), i.e.

\[
\ddot{y}_1 = a(t_1, x_1) + b(t_1, x_1)\xi_t,
\]

where the state vector \( x_t = (x_1)_{1\leq i\leq 2} = (x_1, x_2)^T = (y_1, y_2)^T \), \( a(t, x) = (a_1(t, x_1), a_2(t, x_2)) = (y_1, y_2)^T \), \( b(t, x) = (b_1(t, x_1), b_2(t, x_2)) = (0, f(t, y_1, y_2))^T \). The non-linear stochastic swing equation for a single machine-infinite bus system can be stated as

\[
\frac{2H}{\omega^2} \frac{d^2\theta}{dt^2} = P_m(1 + \lambda_\xi) - P_e - \frac{K_D d\delta}{\omega_0} \frac{d\delta}{dt} = P_m(1 + \lambda_\xi) - \frac{E_f v_t}{x_d} \sin\delta,
\]

\[
= P_m(1 + \lambda_\xi) - \frac{E_f v_t}{x_d} \sin\delta,
\]

\[
- \frac{v^2 t}{2} \frac{x_d - x_q}{x_d d\delta} \sin 2\delta - \frac{K_D d\delta}{\omega_0} \frac{d\delta}{dt},
\]

where the term \( H \) is the inertia constant and \( K_D \) is the damping coefficient. The terms \( \omega_0 \) and \( \omega_1 \) are the rated and actual angular velocities of the machine in electrical radian per second in the fixed reference frame, on the other hand, \( \omega_0 = \omega_1 + \omega_2 \) is in the synchronously rotating reference frame. The notation \( v_t \) is the voltage magnitude of the infinite bus, \( E_f \) is the transient emf and \( b_t \) is rotor angle of generator, \( x_d = X_d + x_I \) and \( x_q = X_q + x_I \), where \( X_d \) and \( X_q \) are the direct-axis transient reactance and quadrature-axis transient reactance of the machine respectively and \( X_I \) is the line reactance. The terms \( P_m \) and \( P_e \) are the mechanical input power and electrical power of the salient pole machine. Note that the mechanical input to synchronous generators is influenced by short circuiting, lightning, continuous small random perturbations, fluctuating load conditions and ageing effects of system components. Thus, it is worthwhile to consider the mechanical input as a random variable. For the simplified stochastic analysis, we consider the
white noise correction term to the mechanical input $P_m$. We replace the term $P_m$ with $P_m(1 + \lambda \xi_t)$. The stochastic swing equation is a specific case of equations (1) and the phase space formulation of equation (2), where

$$\begin{align*}
x_t &= (x_1, x_2)^T = (\delta t, \omega A - \omega B)^T = (\delta t, \omega A)^T \\
b(t, x_t) &= (0, \frac{\omega_0 \gamma}{2H})^T, \gamma = \lambda P_m, \\
a(t, x_t) &= (x_2, \frac{\omega_0}{2H} P_m - \frac{K_D}{2H} x_2 - \frac{\omega_0}{2H} E'_1 v_t \sin x_1 \\
&- \frac{\omega_0}{2H} \frac{v_t^2}{2} (\frac{x_d - x_q}{x_d x_q}) \sin 2x_1)^T.
\end{align*}$$

(3)

The consensus on the Itô-Stratonovich dilemma [10] suggests that the Stratonovich calculus is expected for typical, continuous and real physical systems, on the other hand, the Itô calculus is for stochastic finance and theoretical biology. In the Stratonovich setting, equation (1) becomes

$$dx_t = a(t, x_t)dt + b(t, x_t) \circ dB_t,$$

(4)

where \(\circ\) denotes the Stratonovich differential, a linear operator, and \(B_t\) is the Brownian motion and \(dB_t = \xi_t dt\). Thanks to a Theorem of Karatzas and Shreve ([11], p. 56), Patil and Sharma [12], we get

$$\begin{align*}
d(x_1, x_2)^T &= \left( x_2, \frac{\omega_0}{2H} P_m - \frac{K_D}{2H} x_2 - \frac{\omega_0}{2H} E'_1 v_t \sin x_1 \\
&- \frac{\omega_0}{2H} \frac{v_t^2}{2} (\frac{x_d - x_q}{x_d x_q}) \sin 2x_1 \right)^T dt + \left( 0, \frac{\omega_0 \gamma}{2H} \right)^T \circ dB_t \\
&= \left( x_2, \frac{\omega_0}{2H} P_m - \frac{K_D}{2H} x_2 - \frac{\omega_0}{2H} E'_1 v_t \sin x_1 \\
&- \frac{\omega_0}{2H} \frac{v_t^2}{2} (\frac{x_d - x_q}{x_d x_q}) \sin 2x_1 \right)^T dt + \left( 0, \frac{\omega_0 \gamma}{2H} \right)^T dB_t.
\end{align*}$$

(5)

Equation (5) is an immediate consequence of equations (3)–(4) and the Itô-Stratonovich relationship. Since the process noise coefficient \(b(t, x_t) = b(t)\) the Stratonovich stochasticity and Itô stochasticity coincide, see equation (3).

Equation (5) can be interpreted as an SDE diagram, see Fig. 2. The SDE diagram explains the inputs to the stochastic system, system parameters and the stochastic state vector becomes the output vector of the stochastic system.

![Initial datum](image)

$$B_t \xrightarrow{K_D, \theta_0, H, \gamma, x_d, x_q, E'_1, v_t, P_m} x_t = (x_1, x_2)^T$$

Fig. 2 An SMIB system SDE diagram

3. The Stochastic Filtering Theory

This paper is intended to develop higher-order Kushner-Stratonovich filtering for the stochastic system considered here. The filtering method exploits observation equations. Here, we wish to develop ‘filtering’ by taking measurements on the rotor angle and angular velocity of the salient pole synchronous machine. Thus, the vector observation equation becomes

$$dZ_t = b(t, x_t)dt + dB_{\mu},$$

(6)

where the observation vector

$$z_t = (z_1, z_2)^T, h(t, x_t) = (x_1, x_2)^T = (\delta t, \omega A)^T,$$

$$d\eta(t) = \begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix} d\eta_1 d\eta_2 = 0,$$

$$\nu\eta(t) = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} dt = \Psi_\eta dt.$$

The terms \(\eta(t)\) and \(\Psi_\eta\) have interpretations as the observation noise and the intensity of the observation noise respectively. Note that equation (4) in combination with equation (6) becomes ‘the filtering model’ of the SMIB system. R E. Kalman developed filtering for the linear stochastic differential systems [13]. Importantly, the non-linear filtering theory is credited to H J Kushner and R L Stratonovich. The non-linear filtering theory hinges on the filtering density evolution, a non-linear stochastic partial integro-differential equation (Pugachev and Sinitsyn ([15], p. 389), Jazwinski ([16], p. 178)), i.e.

$$dp = \mathcal{L}(p) dt + (h - \tilde{h})^T \Psi^{-1}_\eta(t) (dz_t - hdt)p,$$

(7)

where the filtering density \(p = p(x, t|z_s, t_0 \leq t \leq 1)\) and the operator \(\mathcal{L}\) denotes the Kolmogorov-Fokker-Planck operator (Sharma [17]). H J Kushner developed the above density evolution equation in the Itô setting. R L Stratonovich developed the filtering density evolution using different interpretations and notations. For this reason, the above filtering density evolution is usually known as the Kushner-Stratonovich equation. It is important to note that the term ‘Kushner-Stratonovich equation’ is borrowed from Pugachev and Sinitsyn [15]. A greater historical and mathematical remark is available in Pugachev and Sinitsyn ([15], p. 390) as well. As a result of this, we use the term ‘Kushner-Stratonovich’ throughout the paper. Notably, the ‘Kushner-Stratonovich stochastic method’ for dynamical systems exploits the Kushner-Stratonovich equation, the exact stochastic evolutions of conditional mean and conditional variance. Making the use of the Kushner-Stratonovich equation, the definition of the differential of the conditional expectation of the scalar function of a dimensional state vector, we derive the exact stochastic evolutions of conditional mean and conditional variance, see Jazwinski ([16], pp. 182–184). Here, we will not...
delve into the exact stochastic evolutions of conditional mean and conditional variance. The procedure to arrive at exact stochastic evolutions is lengthy as well as they are quite known in stochastic processes literature. The analytical and numerical solutions to exact stochastic evolutions of conditional mean and conditional variance for the non-linear stochastic differential system are not possible [15]. For this reason, we explore higher-order Kushner-Stratonovich filtering for non-linear stochastic differential systems. The theory and application of a higher-order filtering can be found in [15] in a greater detail. For the brevity of presentations, we state directly the ‘higher-order Kushner-Stratonovich filtering equations’ [15], i.e.

\[ \begin{align*}
    d\hat{x}_i &= (a_i(t,\hat{x}_1) + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 a_i(t,\hat{x}_1)}{\partial x_p \partial x_q}) dt \\
    &+ \sum_{p} \left( P_{pp} \frac{\partial \hat{x}_i(t,\hat{x}_1)}{\partial x_p} + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 \hat{x}_i(t,\hat{x}_1)}{\partial x_p \partial x_q} \right) \psi_{i-1}^q(t) \\
    &\times (dz_t - (h(t,\hat{x}_1) + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 h(t,\hat{x}_1)}{\partial x_p \partial x_q}) dt), \quad (8a)
\end{align*} \]

\[ \begin{align*}
    dP_{ij} &= \sum_{p} \left( P_{pp} \frac{\partial a_j(t,\hat{x}_1)}{\partial x_p} + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 a_j(t,\hat{x}_1)}{\partial x_p \partial x_q} \right) \\
    &+ \sum_{p} \left( P_{pp} \frac{\partial h(t,\hat{x}_1)}{\partial x_p} + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 h(t,\hat{x}_1)}{\partial x_p \partial x_q} \right) \psi_{i-1}^q(t) \\
    &\times (dz_t - (h(t,\hat{x}_1) + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 h(t,\hat{x}_1)}{\partial x_p \partial x_q}) dt) \\
    &+ \sum_{p,q} \frac{\partial^2 h(t,\hat{x}_1)}{\partial x_p \partial x_q} \psi_{i-1}^q(t) \\
    &\times (dz_t - (h(t,\hat{x}_1) + \frac{1}{2} \sum_{p,q} P_{pq} \frac{\partial^2 h(t,\hat{x}_1)}{\partial x_p \partial x_q}) dt), \quad (8b)
\end{align*} \]

where \( \hat{x}_i = E(x_i(t)|Y_t) \), \( P_{ij} = E((x_i(t) - E(x_i(t)|Y_t))(x_j(t) - E(x_j(t)|Y_t))|Y_t) \) and the operator \( E \) is a conditional expectation operator, which is linear. Note that summing variables \( i,j \) runs over 1 to \( n \), where \( n \) is the size of the state vector. The term \( Y_t \) denotes the observations accumulated up to the time \( t \). A special case of equations is available in [17] as well. Note that the proof of the above coupled filtering equations utilizes the ‘Gaussianity’ for the random state vector. That allows convenient structure of filtering equations, see the usefulness of Gaussianity for non-linear dynamical systems in Park and Scheeres [18] as well. For understanding a formal proof of exact non-linear filtering in a greater rigorous mathematical setting, Liptser and Shiryaev [19] will be useful.

The Liptser-Shiryaev filtering approach hinges on the notion of the stochastic evolution of conditional characteristic function. The filtering density evolution equation is a consequence of the conditional characteristic function evolution equation. The Appendix of the paper discusses the conditional characteristic function evolution equation of the SMIB system. From equations (8a)–(8b) and equations (4)–(6), we get the following Kushner-Stratonovich power system filtering equations, i.e. coupled conditional mean and conditional variance equations:

\[ \begin{align*}
    d\hat{\delta} &= (\hat{\omega}_s - P_{\delta\delta}\sigma^{-2}\delta - P_{\delta\omega_s}\sigma^{-2}\omega_s) dt \\
    &+ P_{\delta\delta}\sigma^{-2}dz_1 + P_{\delta\omega_s}\sigma^{-2}dz_2, \quad (9a)
\end{align*} \]

\[ \begin{align*}
    d\hat{\omega}_s &= (-\frac{K_D}{2H}\hat{\omega}_s + (\frac{P_{\delta\delta}}{2}) - 1) \frac{\omega_0 v_t E'_f}{2H} \sin\hat{\delta} + \frac{\omega_0}{2H} P_m \\
    &+ (P_{\delta\delta} - \frac{1}{2}) \frac{\omega_0 v_t^2(x_d - x_q)}{x_d x_q} \sin2\hat{\delta} \\
    &- P_{\omega_s\omega_s}\sigma^{-2}\omega_s - P_{\omega_s\omega_s}\sigma^{-2}, dt \quad (9b)
\end{align*} \]

\[ \begin{align*}
    dP_{\delta\delta} &= (2P_{\delta\omega_s} - P_{\delta\delta}^2\sigma^{-2} - \omega_s\sigma^{-2}) dt, \quad (10a)
\end{align*} \]

\[ \begin{align*}
    dP_{\omega_s\omega_s} &= (-P_{\delta\delta} v_t E'_f \cos\hat{\delta} - P_{\omega_s\omega_s}) \frac{K_D}{2H} \\
    &+ 2P_{\delta\omega_s} \omega_0 v_t \frac{E'_f}{2H} \\
    &- 2P_{\delta\omega_s} \omega_0 v_t \frac{E'_f}{2H} \cos2\hat{\delta} \\
    &- P_{\omega_s\omega_s}\sigma^{-2} - P_{\omega_s\omega_s}\sigma^{-2} dt, \quad (10b)
\end{align*} \]

\[ \begin{align*}
    dP_{\omega_s\delta} &= (2(-P_{\omega_s\omega_s}) \omega_0 v_t E'_f \cos\hat{\delta}) + \frac{\omega_0^2 \gamma^2}{4H^2} \\
    &+ P_{\omega_s\omega_s} \frac{\omega_0 v_t E'_f}{2H} \cos2\hat{\delta} \\
    &+ 4P_{\omega_s\omega_s} \omega_0 v_t \frac{E'_f(x_d - x_q)}{2H} \cos2\hat{\delta} \\
    &- P_{\omega_s\delta}^2 \omega_s^2 - P_{\omega_s\omega_s}\sigma^{-2} dt. \quad (10c)
\end{align*} \]

**Extended Kalman Filtering (EKF)**

The Extended Kalman Filtering (EKF) is a non-linear filtering method as well as a ‘special case’ of the above higher-order coupled filtering equations. Importantly, the EKF accounts for state-independent diffusion coefficients in the conditional variance evolution equation. Secondly, the EKF accounts for the
first-order partials of the system non-linearity and measurement non-linearity. Higher-order non-linearities evaluated at the conditional expectation of the random state are not accounted for. Thus, the EKF equations are

\[
d\hat{x}_i = a_i(t, \hat{x}_i)dt + \left( \sum_{p} P_{ip} \frac{\partial h^T(t, \hat{x}_i)}{\partial x_p} \right) \Psi^{-1}(t)\eta(t),
\]

\[
\times (dz_i - h(t, \hat{x}_i)dt),
\]

\[
dP_{ij} = \left( \sum_{p} P_{ip} \frac{\partial a_{ij}(t, \hat{x}_i)}{\partial x_p} \right) \Psi^{-1}(t) \times \left( \sum_{p} P_{jp} \frac{\partial h(t, \hat{x}_i)}{\partial x_p} \right) dt.
\]

(11a)

(11b)

Making the use of the above coupled EKF equations, i.e. equations (11a)–(11b), for the power system filtering model, i.e. equations (4)–(6), we get the following evolution equations:

\[
d\delta = (\omega_b - P_{bb}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r)dt + P_{bb}\sigma^2dz_1 + P_{b\omega_r}\sigma^2dz_2,
\]

\[
+ P_{b\omega}\sigma^2dz_1 + P_{\omega\omega_r}\sigma^2dz_2,
\]

(12a)

\[
d\omega_r = \left( -\frac{K_D}{2H} \omega_r - \frac{v_1E_f'}{2H} \sin \delta + \frac{\omega_b}{2H} P_m \right)
\]

\[-\frac{1}{22H} \frac{1}{x_d x_q} \sin 2\delta
\]

\[-P_{b\omega_r}\sigma^2\omega_r - P_{\omega\omega_r}\sigma^2\omega_r) dt
\]

+ P_{b\omega}\sigma^2dz_1 + P_{\omega\omega_r}\sigma^2dz_2,
\]

(12b)

\[
dP_{bb} = (2P_{bb} - P_{bb}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r) dt,
\]

\[
dP_{b\omega_r} = (-P_{bb} \frac{\omega_b}{2H} \frac{v_1E_f'}{x_d} \cos \delta - P_{b\omega_r} - \frac{K_D}{2H})
\]

\[-\frac{1}{22H} \frac{1}{x_d x_q} \sin 2\delta
\]

+ P_{b\omega}\sigma^2dz_1 - P_{b\omega_r} - P_{b\omega_r}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r dt,
\]

(13a)

\[
dP_{b\omega_r} = (2(-P_{bb} \frac{\omega_b}{2H} \frac{v_1E_f'}{x_d} \cos \delta - P_{b\omega_r} - \frac{K_D}{2H})
\]

\[-\frac{1}{22H} \frac{1}{x_d x_q} \sin 2\delta
\]

+ P_{b\omega}\sigma^2dz_1 - P_{b\omega_r} - P_{b\omega_r}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r dt,
\]

(13b)

\[
dP_{b\omega_r} = \left( 2\left( P_{bb} - P_{bb}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r \right) dt
\]

\[-\frac{1}{22H} \frac{1}{x_d x_q} \sin 2\delta
\]

+ P_{b\omega}\sigma^2dz_1 - P_{b\omega_r} - P_{b\omega_r}\sigma^2 - P_{b\omega_r}\sigma^2\omega_r dt.
\]

(13c)

\[
\text{Remark 1:} \text{ Here, we briefly explain the qualitative characteristics of two non-linear filters, a higher-order non-linear filter with cubic non-linearity as well as the celebrated EKF.}
\]

\[
The \text{right hand side of equation (7), Kushner equation, is}
\]

\[
\mathcal{L}(p)dt + (h - \hat{h})^T \Psi^{-1}(t)(dz_1 - \hat{h}dt)p.
\]

\[
\text{The term } \mathcal{L} \text{ denotes the Fokker-Planck operator that hinges on the system parameters as well as the process noise coefficient term. The second term}
\]

\[
(h - \hat{h})^T \Psi^{-1}(t)(dz_1 - \hat{h}dt)p
\]

\[
\text{denotes the observation noise correction term in the filtering of non-linear stochastic systems. Here, we take further two cases: first, the observation noise intensity } \Psi_{\eta} \text{ is smaller, the term } \Psi_{\eta}^{-1} \text{ grows and the filtered state trajectories become unbounded. Secondly, the observation noise intensity } \Psi_{\eta} \text{ is larger, the term } \Psi_{\eta}^{-1} \text{ decays and the filtered state trajectories become bounded. Thus, the observation noise intensity decides the efficacy of non-linear filtering techniques for analysing the stochasticity of dynamic systems.}
\]

\[
\text{Remark 2: The higher-order non-linear filters preserve greater qualitative characteristics of non-linearities as well as account for the stochastic correction term in the conditional variance stochastic evolution equation. On the other hand, the EKF does not account for higher-order non-linearities completely. Secondly, the conditional variance evolution equation of the EKF does not account for the observation correction term. The EKF conditional variance evolution equation becomes ordinary differential equation. Thus, the higher-order non-linear filtering is superior to the EKF for the filtering of greater order non-linearities.}
\]

\section{4. Numerical Simulations}

\[
\text{For the numerical simulation of the stochastic problem of concern here, we consider the following first system of initial conditions and system parameters [9]:}
\]

\[
v_t = 1.0[\text{pu}], E_f' = 1.375[\text{pu}], K_D = 11.304[\text{pu}], \sqrt{\Psi_{\eta}} = \sigma,
\]

\[
H = 3.1[\text{MWs/MVA}], X_d' = 0.33[\text{pu}], X_l = 0.2[\text{pu}],
\]

\[
\omega_0 = 2\pi60 = 376.8[\text{rad/sec}], P_m = 1[\text{pu}], \gamma = 0.2, \sigma = 1000,
\]

\[
X_q' = 0.66[\text{pu}], X_l = 0.2[\text{pu}], x_q = 0.86[\text{pu}], x_d = 0.53[\text{pu}],
\]

\[
\langle \delta(0) \rangle = 1[\text{red}], \langle \omega_r(0) \rangle = 2[\text{red/sec}], P_{\delta\delta}(0) = 1[\text{red}^2],
\]

\[
P_{\delta\omega_r}(0)[\text{red}^2/\text{sec}], P_{\omega_r\omega_r}(0) = 1[\text{red}^2/\text{sec}^2].
\]

\[
\text{Here, we choose the sampling interval 0.01 second for the time interval, i.e. } dt = 0.01[\text{sec}].
\]

\[
\text{Figs. 3, 4 demonstrate numerical simulations of the unperturbed and filtered trajectories of the rotor angle and angular velocity of machine. The solid line}
\]

\[
\text{of Figs. 3, 4 denotes the unperturbed trajectories of the rotor angle and angular velocity. The dotted line}
\]

\[
\text{denotes higher-order ‘filtered’ state trajectories of the rotor angle and angular velocity. These two solid and dotted line trajectories utilize the first set}
\]

\[
\text{of system parameters and initial conditions with a relatively smaller damping parameter, } K_D = 11.304[\text{pu}].
\]
Figs. 5, 6 demonstrate numerical simulations of the EKF and higher-order non-linear filtering trajectories of the rotor angle and angular velocity. Note that the solid line of Figs. 5, 6 exploits the EKF equations and the dotted line is a consequence of higher-order filtering equations. Now, we state the following second set of data:

\[ \begin{align*}
  v_1 &= 1.0 [\text{pu}], E' &= 1.375 [\text{pu}], \sqrt{\Phi_\eta} = \sigma, \\
  H &= 3.1 [\text{MWs/MVA}], X_i &= 0.2 [\text{pu}], X_l &= 0.2 [\text{pu}], \\
  \omega_0 &= 2\pi 60 = 376.8 [\text{rad/sec}], \gamma = 0.2, \sigma = 1000, \\
  X_q &= 0.66 [\text{pu}], \Delta x &= 0.86 [\text{pu}], \Delta d &= 0.53 [\text{pu}], \\
  \langle \delta(0) \rangle &= 1 [\text{red}], \langle \Delta \omega_s(0) \rangle = 2 [\text{red/sec}], P_{\delta\delta}(0) = 1 [\text{red}^2], \\
  P_{\delta\omega}(0)[\text{red}^2/\text{sec}], P_{\omega_s\omega_s}(0) &= 1 [\text{red}^2/\text{sec}^2].
\end{align*} \]

A careful observation shows that the second set of data is different from the first set of data in the sense that the second set accounts for the larger damping parameter \( K_D = 56.52 [\text{pu}] \) in lieu of the first set. Figs. 7, 8 reveal that the difference between the state trajectories, which are the consequence of deterministic swing equation and stochastic swing equation, are larger. The difference between the two trajectories are attributed to the larger damping parameter. This suggests the fact that stochasticity considerations are recommended for the machine swing equation describing the large damping machine dynamics.

Numerical simulations of stochastic differential equations are greatly explained in Kloeden and Platten [20]. The numerical simulations demonstrated in Figs. 7–10 correspond to the case, where larger damping system parameter is exploited. It is worth to mention the following two points:

(i) Under larger damped and lower damped system parameters, the both non-linear filters of the paper, higher-order filtering equations (9)–(10) and EKF equations (12)–(13), will work, where the observation noise intensity is larger.

(ii) Under larger damped and lower damped system parameters, the EKF of the paper will not be effective with the lower noise intensity. On the other hand, the higher-order filter will work because of non-linear corrections as well as stochastic correction terms.

The numerical simulations demonstrated in Figs. 5, 6 and Figs. 9, 10 suggest the superiority of non-linear filtering equations (9)–(10) in comparison to...
EKF equations (12)–(13). This paper recommends the non-filtering equations stated in (9)–(10) for analyzing the stochasticity of the SMIB system in the conditional variance sense. Note that evolution of conditional variance, the non-linear filtering equation (8b), accounts for the observation noise correction term coupled with higher-order measurement non-linearities. On the other hand, evolution of conditional variance, EKF equation (11b), ignores the observation noise correction term coupled with higher-order measurement non-linearities.

5. Conclusion

The main achievement of this paper is to sketch non-linear filtering of the SMIB system. Generally, in engineering literature, the non-linear filtering is restricted to the EKF setting. The non-linear filtering of the paper, i.e., the Kushner-Stratonovich filtering, is beyond the EKF. The Kushner-Stratonovich filtering has ability to preserve greater qualitative characteristics of non-linear stochastic systems as well as non-linear observation equations.

This paper is the first contribution in the power systems dynamics literature in the following sense: (i) this paper 'tells' the application of non-linear filtering beyond Extended Kalman Filtering to the SMIB system. That involves formal, systematic, unified and well-based mathematical method to achieve non-linear filtering of the stochastic system of the paper.

This paper is different from our recent previous contribution [9] and the available results become a special case of the results of this paper. This paper 'respects' the process noise term as well as observation noise term of the filtering model of the SMIB system by accounting their contributions to the non-linear filtering equations of the SMIB system. On the other hand, the observation noise correction term is ignored in [9].

Another theoretical contribution of this paper is to reveal connections between stochastic differential equations, partial differential equations, Ordinary Differential Equations and non-linear swing equation of the power systems dynamics. In this regard, the paper [21] will be useful.

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Appendix

Calculation of the Conditional Characteristic Function of the SMIB System

The notion of characteristic function has found applications to sketch the proof of the Fokker-Planck equation as well as non-linear filtering density evolution equation. Since this paper is about the non-linear filtering of the stochastic SMIB system, it is worthwhile to write the conditional characteristic function evolution equation in the ‘non-linear filtering sense’. Note that the conditional characteristic function in non-linear filtering sense satisfies a stochastic partial integro differential equation. That accounts for observation correction terms. Thus, the stochastic evolution \( d(\varphi(x_1)) \) of conditional moment becomes

\[
\mbox{d} \langle \varphi(x_1) \rangle = \left( \alpha \mathbf{T} \langle x_1 \rangle \right) + \frac{1}{2} \left( \mathbf{T} \left( \langle \mathbf{b} \mathbf{T} \rangle \langle x_1 \rangle \right) \right) d\mathbf{t} + \left( \langle \mathbf{b} \mathbf{T} \rangle - \langle \mathbf{T} \rangle \right) \varphi_{-1} \mathbf{T} d\mathbf{z}.
\]

The above stochastic evolution of the conditional expectation of the scalar function is the central result of exact non-linear filtering [22]. In the component-wise description, the above evolution is recast as

\[
\mbox{d} \langle \varphi(x_1) \rangle = \left( \sum_i a_i \langle t, x_i \rangle \frac{\partial \varphi(x_1)}{\partial x_i} + \frac{1}{2} \sum_{i<j} \langle \mathbf{b} \mathbf{T} \rangle \mathbf{a}_i \langle t, x_i \rangle \frac{\partial^2 \varphi(x_1)}{\partial x_i \partial x_j} \right) + \left( \sum_{i<j} \langle \mathbf{b} \mathbf{T} \rangle \mathbf{a}_i \langle t, x_i \rangle \frac{\partial^2 \varphi(x_1)}{\partial x_i \partial x_j} \right) \mbox{d} \mathbf{t} + \left( \langle \mathbf{b} \mathbf{T} \rangle - \langle \mathbf{T} \rangle \right) \varphi_{-1} \mathbf{T} d\mathbf{z}.
\]

Suppose \( \varphi(x_1) = e^{\mathbf{t}^\mathbf{T} x_1} \) equation (A1) becomes the conditional characteristic function evolution, i.e.

\[
\mbox{d} \left( e^{\mathbf{t}^\mathbf{T} x_1} \right) = \left( \sum_i a_i \langle t, x_i \rangle e^{\mathbf{t}^\mathbf{T} x_1} + \frac{1}{2} \sum_{i<j} \langle \mathbf{b} \mathbf{T} \rangle \mathbf{a}_i \langle t, x_i \rangle s_i^2 e^{\mathbf{t}^\mathbf{T} x_1} \right) d\mathbf{t} + \left( \sum_{i<j} \langle \mathbf{b} \mathbf{T} \rangle \mathbf{a}_i \langle t, x_i \rangle s_i^2 e^{\mathbf{t}^\mathbf{T} x_1} \right) \mbox{d} \mathbf{z} + \left( \langle \mathbf{b} \mathbf{T} \rangle - \langle \mathbf{T} \rangle \right) \varphi_{-1} \mathbf{T} d\mathbf{z}.
\]
Since the conditional expectation operator \( \langle \cdot \rangle \) is a linear operator, we get
\[
d\langle e^{x_{t+1}} \rangle = \left( \sum a_i(t,x_i)e^{x_{t+1}} \right)
+ \frac{1}{2} \left( \sum b_i(t,x_i)e^{x_{t+1}} \right) dt
+ \sum_{i=1}^{\alpha} \left( \langle e^{x_{t+1} h} \rangle - \langle e^{x_{t+1}} \rangle \right) dh_i.
\]
(A2)

For the stochastic SMIB system of the paper, the stochastic evolution of conditional characteristic function becomes a special case of equation (A2), i.e.

\[
d\langle e^{x_{t+1}} \rangle = d\langle e^{x_{t+1} \delta + x_{2} \omega_{1}} \rangle = \left( \langle x_{2} e^{x_{t+1}} \rangle \right)
+ \left( \frac{\omega_0}{2H} P_{m} - \frac{K_D}{2H} x_2 - \frac{\omega_0}{2H} E_f v_t \sin x_1 \right) dt
+ \left( \frac{\omega_0}{2H} v_t^2 \left( \frac{x_d - x_0}{2} \right) \sin \left( 2 \pi s_1 \right) s_2 e^{x_{t+1} \delta + x_{2} \omega_{1}} \right) dt
+ \frac{1}{2} \left( \frac{\omega_0}{2H} s_2^2 \right) \left( s_2 e^{x_{t+1} \delta + x_{2} \omega_{1}} \right) dt
+ \left( \frac{\omega_0}{2H} P_{m} - \frac{K_D}{2H} \omega_s - \frac{\omega_0}{2H} E_f v_t \sin \delta \right) dt
+ \left( \frac{\omega_0}{2H} v_t^2 \left( \frac{x_d - x_0}{2} \right) \sin \left( 2 \pi s_1 \right) s_2 e^{x_{t+1} \delta + x_{2} \omega_{1}} \right) dt
+ \frac{1}{2} \left( \frac{\omega_0}{2H} s_2^2 \right) \left( s_2 e^{x_{t+1} \delta + x_{2} \omega_{1}} \right) dt
+ \left( \frac{\omega_0}{2H} P_{m} - \frac{K_D}{2H} \omega_s - \frac{\omega_0}{2H} E_f v_t \sin \delta \right) dt
+ \frac{1}{2} \left( \frac{\omega_0}{2H} s_2^2 \right) \left( s_2 e^{x_{t+1} \delta + x_{2} \omega_{1}} \right) dt
+ \left( \frac{\omega_0}{2H} P_{m} - \frac{K_D}{2H} \omega_s - \frac{\omega_0}{2H} E_f v_t \sin \delta \right) dt.
\]

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