Stability Analyses of the Stochastic Delayed Infectious Models with Reinfection*

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At present, the unprecedented cholera outbreak occurs in Yemen and various kinds of infectious diseases are still threat to us in the high-developed medical technology society. Hence, the stratagem to control the spread of the infectious diseases becomes imperative. In the vector-borne diseases such as malaria and dengue fever, there exists time delay caused by an incubation period in the virus development in the vectors on the transmission of disease. It should be noted that there is possibility of getting reinfeected in the infectious disease such as malaria. Moreover, in the realistic spread of the infectious disease, environmental change and individual difference cause some kinds of random fluctuations in the infection, the recovery rates and the vaccination effect. Taking these facts into consideration, we propose two types of the stochastic delayed infectious models with reinfection. Since the spread of infection has reference to the stability of the disease-free steady state (DFS) of the stochastic infectious models, we analyze the stability of the DFS by using the stochastic Lyapunov theorem. By calculating the Lyapunov exponent, we study the influence of the random noise in the infectious model on each population behavior by numerical simulations.

1. Introduction

The infectious disease prevention and the control is one of the highest priority public sanitation issues in the modern society [1]. A threat from the various kinds of infectious diseases including three major infectious diseases, HIV, tuberculosis and malaria has been growing. Mathematical models have become important tools in analyzing the infectious spread and in constructing the vaccination program [2–6]. In the vector-borne infectious diseases such as malaria [7] and dengue fever, there exists time delay caused by an incubation period in the virus development in the vectors (mosquitoes) on the transmission of disease. Moreover, in the real spread of the infectious disease, changes in the environment and the weather cause some kinds of random fluctuations in the infection, the recovery rates and the vaccination effect. In consideration of these facts, and noting that there is possibility of getting reinfeected in the infectious disease such as malaria, we propose two types of the stochastic delayed infectious models with reinfection. One is the stochastic SIRS (susceptible-infected-recovered-susceptible) model and the other is the stochastic SIRVS (susceptible-infected-recovered-vaccinated-susceptible) model. These two models are different in that the latter is considered the vaccine protection wane and the vaccination population, but such an effect and the vaccination population are not included in the former model.

In Section 2, firstly, one of conventional infectious models, the deterministic delayed SIRS model, is explained. For more practical analysis of the spread of infectious diseases, introducing the random fluctuation in the recovery rate, we propose the stochastic delayed SIRS model. Then, by considering the randomness in the vaccination effect, we consider the stochastic delayed SIRVS model.

Since the behavior of each population in the infectious model has a strong relation with the stability of the disease-free steady state (DFS), we study the stability of the DFS. The DFS means the equilibrium solution with zero infected individuals. Firstly, we explain some concepts of the stochastic stability of the stochastic delayed system in Section 3. In Section 4, we study the stability of the DFS of the stochastic delayed infectious models proposed in Section 2 using the stochastic Lyapunov theorem, and we derive the sufficient conditions for the DFS to be stable. In Section 5, we verify the validity of the Theorems derived in Section 4 by numerical simulations. Moreover, by calculating the Lyapunov exponents of the stochastic SIRS and SIRVS models, time delay, Lyapunov exponent, stochastic stability, simulations.
tic infectious models derived in Section 2, we clarify that the random noise has the stabilization effect of the DFS through the numerical simulations.

2. Stochastic Delayed Infectious Models with Reinfection

In this section, we propose the two types of the stochastic delayed infectious models with reinfection.

2.1 Stochastic Delayed SIRS Model

Letting the population density of the susceptible, the infected and the recovered at time $t$ be $S(t), I(t)$ and $R(t)$, we consider the interaction between each population as shown in Fig. 1.

\[ \dot{S}(t) = \mu S(t) - uS(t)I(t-h) - \beta S(t)I(t-h) + \nu R(t) \]
\[ \dot{I}(t) = \beta S(t)I(t-h) - (\mu + \gamma) I(t), \]
\[ \dot{R}(t) = uS(t) + \gamma I(t) - (\mu + \nu) R(t). \]

Setting as $N(t) = S(t) + I(t) + R(t)$, since it follows from (1) to (3) that

\[ \dot{N}(t) = \mu(1 - N(t)), \]
we have $N(t) \equiv 1$ under the condition $N(0) = 1$. Hence, we obtain

\[ R(t) = 1 - S(t) - I(t). \]

The initial conditions of (1) to (3) are given by

\[ S(0) = S_0 > 0, \quad I(s) = I_0(s) > 0, (-h \leq s \leq 0), \]
\[ R(0) = 1 - S_0 - I_0 \geq 0. \]

We introduce the random fluctuation in the recovery rate $\gamma$. Modeling the random fluctuation in the recovery rate $\gamma$ as the white Gaussian noise $\eta(t)$, we replace $\gamma$ by $\gamma + \sigma \eta(t)$, ($\sigma$ is constant). Using the relation $\eta(t)dt = dw(t)$ between $\eta(t)$ and the Wiener process $w(t)$, Eqs.(1) to (3) imply the stochastic delayed SIRS model:

\[ dS(t) = \{\mu - (u + \mu)S(t) - \beta S(t)I(t-h) + \nu R(t)\}dt, \]
\[ dI(t) = \{\beta S(t)I(t-h) - (\mu + \gamma) I(t)\}dt - \sigma I(t)dw(t), \]
\[ dR(t) = \{uS(t) + \gamma I(t) - (\mu + \nu) R(t)\}dt + \sigma I(t)dw(t). \]
Modeling the random fluctuation in the vaccination effect $\alpha$ as the white Gaussian noise $\eta(t)$, we replace $\alpha$ by $\alpha + \varepsilon\eta(t)$ ($\varepsilon$ is constant) and using the relation $\eta(t)dt = dw(t)$ between $\eta(t)$ and the Wiener process $w(t)$, Eqs.(12) to (15) imply the stochastic delayed SIRS model:

$$dS(t) = (\mu - (u + \mu)S(t) - \beta S(t)I(t - h) + \theta V(t) + \nu R(t))dt,$$

$$dI(t) = (\beta S(t) + \alpha V(t))I(t - h) - (\mu + \gamma)I(t)dt + \varepsilon\beta V(t)I(t - h)dw(t),$$

$$dR(t) = (\gamma I(t) - (\mu + \nu)R(t))dt,$$

$$dV(t) = (uS(t) - (\mu + \theta) V(t) - \varepsilon\beta V(t)I(t - h))dt - \varepsilon\beta V(t)I(t - h)dw(t).$$

The initial conditions of (18) to (21) are given by (17).

In this section, we define the stochastic stability [9] of zero solution of (23) with the initial value $\phi$, we define the stochastic stability [9] of zero solution of (23) is called asymptotically mean square stable if it is mean square stable and there exists $\delta > 0$ such that

$$||\varphi||_1 < \delta \rightarrow \lim_{t \to \infty} E\{||x(t;\varphi)||^2\} = 0.$$  \hspace{1cm} (28)

(Definition 3) (Stable in Probability): The zero solution of (23) is called stable in probability if for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists $\delta(\varepsilon) \equiv \delta(\varepsilon_1, \varepsilon_2)$ such that

$$P\{||\varphi|| < \delta(\varepsilon)\} = 1 \rightarrow P\{\sup_{t \geq 0} ||x(t;\varphi)|| > \varepsilon_1\} < \varepsilon_2.$$ \hspace{1cm} (29)

where $||\cdot||$ is defined by

$$||x|| = \sup_{-h \leq s \leq 0} |x(s)|.$$ \hspace{1cm} (30)

For simplicity of descriptions, we hereafter referred to $x(t;\varphi)$ as $x(t)$ unless it causes confusion.

### 3. Stability of the Stochastic Delayed System

In this section, we review several definitions of the stability of the stochastic system with time delay. Letting $h$ be positive constant which denotes time delay, we define $x(t)$ by

$$x_t(s) = x(t + s), \quad (-h \leq s \leq 0).$$ \hspace{1cm} (22)

Then, the stochastic delayed system is generally described by the functional differential equation:

$$dx(t) = f(t,x_t)dt + g(t,x_t)dw(t).$$ \hspace{1cm} (23)

The initial condition of (23) is given by

$$x(s) = \varphi(s), \quad -h \leq s \leq 0.$$ \hspace{1cm} (24)

Noting that (23) is the functional equation of $x_t(s)$ and setting as $f(t,x_t(s)) = x_t(-h), g(t,x_t(s)) = x_t(-2h)$ in (23), then (23) yields the stochastic equation with two time delays:

$$dx(t) = x(t-h)dt + x(t-2h)dw(t).$$ \hspace{1cm} (25)

Assuming $f(t,0) = g(t,0) = 0$ and $x_t$ is $n$-dimension in (23), we define the stochastic stability [9] of zero solution (23) yields the stochastic equation with two time delays:

$$dx(t) = x(t-h)dt + x(t-2h)dw(t).$$ \hspace{1cm} (26)

where $||\cdot||_1$ is the Euclidean norm, $x(t;\varphi)$ is the solution of (23) with the initial value $\varphi$ and $||\cdot||_1$ is defined by

$$||\varphi||_1 = \sup_{-h \leq s \leq 0} E\{||\varphi(s)||^2\}.$$ \hspace{1cm} (27)

(Definition 2) (Asymptotically Mean Square Stable): The zero solution of (23) is called asymptotically mean square stable if it is mean square stable and there exists $\delta > 0$ such that

$$||\varphi||_1 < \delta \rightarrow \lim_{t \to \infty} E\{||x(t;\varphi)||^2\} = 0.$$ \hspace{1cm} (28)

(Definition 3) (Stable in Probability): The zero solution of (23) is called stable in probability if for any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists $\delta(\varepsilon) \equiv \delta(\varepsilon_1, \varepsilon_2)$ such that

$$P\{||\varphi|| < \delta(\varepsilon)\} = 1 \rightarrow P\{\sup_{t \geq 0} ||x(t;\varphi)|| > \varepsilon_1\} < \varepsilon_2.$$ \hspace{1cm} (29)

where $||\cdot||$ is defined by

$$||x|| = \sup_{-h \leq s \leq 0} |x(s)|.$$ \hspace{1cm} (30)

For simplicity of descriptions, we hereafter referred to $x(t;\varphi)$ as $x(t)$ unless it causes confusion.

### 4. Stability Analysis of the Stochastic Delayed Infectious Model

In this section, we consider the stability of the disease-free steady state (DFS) of two types of the stochastic delayed infectious models proposed in Section 2. The DFS means the equilibrium solution with zero infected individuals. If the DFS is stable, even if the infectious disease breaks, prevalence of disease has eventually ended. Hence, the stability analysis of the DFS is very important in the epidemiology. Moreover, using the stability condition of the DFS, we are able to know the necessary vaccination rate to control prevalence of infectious disease.

#### 4.1 Stability Analysis of the Stochastic Delayed SIRS Model

The stochastic delayed SIRS model (7) to (9) has the DFS such that

$$(S_f, I_f, R_f) = \left(\frac{\mu + \nu}{u + \mu + \nu}, 0, \frac{u}{u + \mu + \nu}\right).$$ \hspace{1cm} (31)

Since the behavior of the recovered $R(t)$ is determined by (5), we consider the stability of the DFS ($S_f, I_f$) in this section.

Setting as

$$x_1(t) = S(t) - S_f, \quad x_2(t) = I(t),$$ \hspace{1cm} (32)

it follows from (10) and (11) that

$$dx_1(t) = \{-(u + \mu + \nu)x_1(t) - \beta x_1(t)x_2(t - h) - \nu x_2(t)\}dt,$$ \hspace{1cm} (33)

$$dx_2(t) = \{\beta x_1(t)x_2(t - h) - \mu x_2(t)\}dt.$$ \hspace{1cm} (34)

The linear parts of (33) and (34) are given by

$$dx_1(t) = \{-(u + \mu + \nu)x_1(t) - \beta S_f x_2(t - h)$$

$$dx_2(t) = \{\beta S_f x_2(t - h) - \nu x_2(t)\}dt,$$ \hspace{1cm} (35)
we define $A$ where

$$0 < \beta < \frac{1}{S_f} \min \left\{ 2u + 2\mu + \nu, \alpha + \gamma - \frac{1}{2} \epsilon^2 \right\}.$$  

(Outline of Proof): Define $V_1(t,x_t)$ in such a way that

$$V_1(t,x_t) = x_1^2 + Ax_2^2 + B \int_{t-h}^t x_2(s)^2 ds,$$  

where $A$ and $B$ are given by

$$A = \frac{2(\nu + \beta S_f)}{2\mu + 2\gamma - 2\beta S_f - \epsilon^2}, \quad B = (1 + A)\beta S_f.$$  

Then, it suffices to show that there exist constants $k_i > 0, (i = 0, 1, 2)$ such that

$$k_0 E\{|x(t)|^2\} \leq E\{V_1(t,x_t)\} \leq k_1||x||^2,$$  

$$\frac{\partial V_1(t,x_t)}{\partial t} + \mathcal{L}_1 V_1(t,x_t) \leq -k_2|x(t)|^2,$$  

(41) where the operator $\mathcal{L}_1(\cdot)$ in (41) is a generating operator of (35) and (36) given by

$$\mathcal{L}_1(\cdot) = \left\{ \frac{\partial(\cdot)}{\partial x} \right\} f + \frac{1}{2} \text{tr} \left\{ \left( \frac{\partial(\cdot)}{\partial x} \right) g f \right\}.$$  

and where $f = [f_1, f_2]'$, $g = [g_1, g_2]'$ and $f_i, g_i (i = 1, 2)$ are defined by

$$f_1 = -(\mu + \theta)x_1(t) - \beta(x_1(t) + S_f)x_2(t - h),$$  

$$f_2 = \beta S_f x_2(t - h) - (\mu + \gamma)x_2(t),$$  

$$g_1 = 0, \quad g_2 = -\epsilon x_2(t).$$  

(45) For the detailed proof of Theorem 1, see Appendix A. [Theorem 2] Under the same condition as Theorem 1, the origin of (33) and (34) is stable in probability.

(Outline of Proof): For any $\delta > 0$, considering $x_t$ satisfies

$$P\{ \sup_{-h \leq s \leq 0} |x_t(s)| < \delta \} = 1.$$  

we define $V_2(t,x_t)$ as

$$V_2(t,x_t) = x_1^2(t) + Ax_2^2(t) + C \int_{t-h}^t x_2(s)^2 ds,$$  

where $A$ is defined by (39) and $C$ is given by

$$C = \beta(1 + A)(\delta + S_f).$$  

By proving that the function $V_2(t,x_t)$ is the stochastic Lyapunov function, i.e., which satisfies

$$\frac{\partial V_2(t,x_t)}{\partial t} + \mathcal{L}_2 V_2(t,x_t) \leq 0,$$  

(49) the proof of Theorem 2 is performed.

In (49), the operator $\mathcal{L}_2(\cdot)$ is defined by replacing $f = [f_1, f_2]'$ in (43) and (44) by

$$f_1 = -(\mu + \theta)x_1(t) - \beta(x_1(t) + S_f)x_2(t - h),$$  

$$f_2 = \beta(x_1(t) + S_f)x_2(t - h) - (\mu + \gamma)x_2(t).$$  

(51) The function $g$ in $\mathcal{L}_2(\cdot)$ is given by (45).

The detailed proof of Theorem 2, see Appendix B.

4.2 Stability Analysis of the Stochastic Delayed SIRVS Model

The stochastic delayed SIRVS model (18) to (21) has the disease-free steady state such that

$$(S_f, I_f, R_f, V_f) = \left( \frac{\mu + \theta}{u + \mu + \theta}, 0, 0, \frac{u}{u + \mu + \theta} \right).$$  

(52) Noting that the vaccinated density $V(t)$ is determined by $S(t), I(t)$ and $R(t)$ as follows:

$$V(t) = 1 - S(t) - I(t) - R(t),$$  

we consider the stability of (18) to (20).

We set as

$$x_1(t) = S(t) - S_f, \quad x_2(t) = I(t), \quad x_3(t) = R(t).$$  

(54) Using (53) and $V_f = 1 - S_f$ in (18) to (20), we have

$$dx_1(t) = -(\mu + \theta)x_1(t) - \beta(x_1(t) + S_f)x_2(t - h) - \theta x_2(t) + (\nu - \theta)x_3(t) \ \text{dt}$$  

$$dx_2(t) = (\beta S_f x_1(t) + \nu x_2(t) + \alpha \beta V_f - x_1(t)$$  

$$- x_2(t) - x_3(t))x_2(t - h) - (\mu + \gamma)x_2(t) \ \text{dt}$$  

$$+ \epsilon \beta (V_f - x_1(t) - x_2(t)) x_2(t - h) \ \text{dt},$$  

(56)$$dx_3(t) = \{\gamma x_2(t) - (\mu + \nu)x_3(t) \} \ \text{dt}.$$  

(57) The linear parts of (55) to (57) become

$$dx_1(t) = -(\mu + \theta)x_1(t) - \beta S_f x_2(t - h)$$  

$$- \theta x_2(t) + (\nu - \theta)x_3(t) \ \text{dt},$$  

(58)$$dx_2(t) = (\beta S_f x_2(t - h) + \alpha \beta V_f x_2(t - h)$$  

$$- (\mu + \gamma)x_2(t)) \ \text{dt} + \epsilon \beta (V_f - x_1(t) - x_2(t) - x_3(t)) x_2(t - h) \ \text{dt},$$  

(59)$$dx_3(t) = \{\gamma x_2(t) - (\mu + \nu)x_3(t) \} \ \text{dt}.$$  

(60) [Theorem 3] Under the following condition, the origin of (58) to (60) is asymptotically mean square stable.

(i) $2\mu + \nu - \gamma - \theta > 0,$

(ii) $0 < \beta < \min \left\{ \frac{2\mu + 2u - \nu}{S_f}, \beta^* \right\},$  

(62) where $\beta^*$ is the positive solution $x = \beta^* \in (0, 1]$ of

$$f(x) = x^2 V_f^2 x^2 + 2(S_f + \alpha V_f) x - 2(\mu + \gamma) = 0.$$  

(63)
steady state (DFS) becomes stable from Theorems 3 and 4. It should be noted that the denominator of $D$ defined by (65) is positive for $0<\beta<\beta^*$. 

**[Theorem 4]** Under the same condition as Theorem 3, the origin of (55) to (57) is stable in probability.

$$V_3(t,x_t) = x_1^2 + Dx_2^2 + x_3^2 + E \int_{t-h}^{t} x_2(s)^2 ds,$$  \hfill (64)  

where $D$ and $E$ are given by

$$D = \frac{2(\theta + \gamma + \beta S_f)}{2(\mu + \gamma) - 2(S_f + \alpha V_f)\beta - \varepsilon \beta^2 V_f^2},$$  \hfill (65)  

$$E = \beta \{(1 + D)S_f + DV_f(\alpha + \varepsilon^2 \beta V_f)\}. \hfill (66)$$

5. Numerical Simulations

5.1 Stability Analysis by Theorems

In this section, we verify the validity of Theorems derived in Section 4 by numerical simulations. In particular, we perform the numerical simulations for the SIRVS model given by Fig. 2 and study the usefulness of Theorems 3 and 4 because the SIRS model in Fig. 1 is reduced from the SIRVS model in the following way. Firstly, setting the vaccinated and the recovered populations $V(t)$ and $R(t)$ as the recovered population $R(t)$ again, and secondly, taking $\theta = \alpha = 0$ in Fig. 2, the SIRS model is derived.

We set the parameter values as $u = 0.05$, $\mu = 0.05$, $\theta = 0.02$, $\gamma = 0.05$, $\nu = 0.1$, $\alpha = 0.1$, $\varepsilon = 0.2$. Then, the condition (61) is satisfied and it follows from (62) that the stable region of the infection rate $\beta$ becomes $0<\beta<0.1599$. Under the time delay of the infection $h = 20$ and the initial conditions $S(0) = 0.5$, $I(s) = 0.2$, $(-20 \leq s \leq 0)$, $R(0) = 0.2$ in the stochastic SIRS model (18) to (21), we perform the numerical simulations.

(Case 1) $\beta = 0.15$: In this case, the disease-free steady state (DFS) becomes stable from Theorems 3 and 4. We show the sample paths of the susceptible $S(t)$, the infected $I(t)$, the recovered $R(t)$ and the vaccinated $V(t)$ in Fig. 3. It follows from (52) that the DFS becomes $(S_f,I_f,R_f,V_f) \geq (0.58,0.0,0.0,0.42)$. Figure 3 shows that each population converges to the DFS, so the DFS is stable.

(Case 2) $\beta=0.18$: In this case, the stability of the DFS does not guaranteed from Theorems 3 and 4. We show the sample paths of the susceptible $S(t)$, the infected $I(t)$, the recovered $R(t)$ and the vaccinated $V(t)$ in Fig. 4. In Fig. 4, the infected $I(t)$ does not converge to zero, so the DFS is unstable.

5.2 Stability Analysis by Lyapunov Exponents

Next, in order to consider the effect of the noise on the stability of the DFS, we calculate the maximum Lyapunov exponent $\ell$ of the stochastic SIRVS model (18) to (21). Figure 5 shows the dependency of the maximum Lyapunov exponent $\ell$ on the the infection rate $\beta$ and the noise strength $\varepsilon$ under the values of $u = 0.05, \mu = 0.05, \theta = 0.02, \gamma = 0.05, \nu = 0.1, \alpha = 0.1, h = 20$. We denote the infection rate $\beta$ at which the maximum Lyapunov exponent $\ell$ becomes zero as $\beta(a)$ under the noise strength $\varepsilon = a$, i.e., $\ell(\beta(a)) = 0$. Then, it follows from Fig. 5 that $\ell(\beta(a))$ is a monotonically increasing function of $a$. This fact means that at the infection rate $\beta$ in the region of $\beta(0) < \beta < \beta(a)$, the DFS is unstable under the no noise, however, it becomes stable under the noise with the strength $\varepsilon = a$. Hence, the random noise plays a role to stabilize the DFS.

Under the value of the infection rate $\beta = 0.162$,
Fig. 5  Dependency of the Lyapunov exponent $\ell$ on the infection rate $\beta$ and the noise strength $\varepsilon$

whereas the Lyapunov exponent $\ell$ is positive under the no noise ($\varepsilon = 0$), $\ell$ is negative under the noise ($\varepsilon = 2$) from Figs. 5 and 6. Figure 6 is the magnified figure of Fig. 5 in the range of $0.150 < \beta < 0.170$. Figures 7 and 8 denote sample paths of the infected $I(t)$ under the no noise and the noise with the strength $\varepsilon = 2$, respectively. Figure 7 shows that the DFS is unstable under the no noise because $I(t)$ does not converge to zero, however, Fig. 8 means that the DFS is stabilized by the noise since $I(t)$ converges to zero. In other words, Figs. 7 and 8 show the stabilization effect of the random noise in the DFS.

6. Conclusions

In this paper, we proposed the stochastic SIRS and the SIRVS infectious models. Moreover, introducing the time delay of the infection, and the random fluctuations in the recovery rate and the vaccination effect, we propose the stochastic delayed SIRS and SIRVS models with reinfection.

By using the stochastic Lyapunov function, we derived the sufficient conditions for the disease-free steady state (DFS) to be stable. We have verified the effectiveness of stability theorems proposed in this paper by numerical simulations. The stability theorems obtained in this paper give the necessary vaccination rate for controlling the infectious disease spreading.

In [10], it is shown that the multiplicative noise has the stabilizing effect by calculating the extremal point of the stationary probability density function. On the other hand, we have shown the stabilization effect of the noise by numerical calculations of the Lyapunov exponent of the stochastic SIRVS model. In addition, since we have a similar result to Fig. 5 for the stochastic SIRS model, it is conceivable that the random noise has generally the stabilization effect.

Although time delay commonly affects the stability, it is theoretically proved in [8] that time delay in the deterministic infectious model has little influence on the stability. In the stochastic model in this paper, a similar result to [8] is numerically shown.

It should be noted that even if the correction term in the stochastic SIRS model is considered, the results for the SIRS model in this paper is effective replacing the recovery rate $\gamma$ by $\gamma - \varepsilon^2$. However, in the stochastic SIRVS model, because of the existence of time delay in the diffusion term, the introduction of the Wang-Zakai correction term in the SIRVS is a future problem.
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References


Appendix

Appendix 1. Proof of Theorem 1

Since (22) and (30) easily yield the relation (40), it suffices for the proof of Theorem 1 to prove (41).

Setting as \( x_i(t) = x_i, (i = 1, 2), \bar{x}_2 = x_2(t - h), \) from (35), (36) and (38), we have

\[
\frac{\partial V_i(t,x_i)}{\partial t} + \mathcal{L}_1 V_i(t,x_i) = Bx_1^2 - B\bar{x}_2^2
+ 2x_1\{-(u + \mu + \nu)x_1 - \beta S_f \bar{x}_2 - \nu_x\}
+ 2Ax_2\{\beta S_f \bar{x}_2 - (\mu + \gamma)x_2\} + A\varepsilon x_2^2,
\]

(A1)

(using the Schwartz’s inequality, we have)

\[
\leq -(2u + 2\mu + \nu - \beta S_f)x_1^2
- \{(2\mu + 2\gamma - 2\beta S_f - \varepsilon^2)A - (\nu + \beta S_f)\}x_2^2
+ \{(1 + A)\beta S_f - B\}x_1^2,
\]

(A2)

(using \( B \) defined by (39), we have)

\[
= -(2u + 2\mu + \nu - \beta S_f)x_1^2
- \{(2\mu + 2\gamma - 2\beta S_f - \varepsilon^2)A - (\nu + \beta S_f)\}x_2^2
(\text{using } A \text{ defined by (39), we have})
\]

\[
= -(2u + 2\mu + \nu - \beta S_f)x_1^2 - (\nu + \beta S_f)x_2^2
\leq -k|x(t)|^2,
\]

(A3)

where \( k = \min \{2u + 2\mu + \nu - \beta S_f, \nu + \beta S_f\} \) and \( k \) is positive from (37).

Appendix 2. Proof of Theorem 2

It follows from (33), (34) and (47) that

\[
\frac{\partial V_2(t,x_1)}{\partial t} + \mathcal{L}_2 V_2(t,x_1) = Cx_2^2 - C\bar{x}_2^2
+ 2x_1\{-(u + \mu + \nu)x_1 - \beta (x_1 + S_f) \bar{x}_2 - \nu x_2\}
+ 2Ax_2\{\beta (x_1 + S_f) \bar{x}_2 - (\mu + \gamma)x_2\} + A\varepsilon x_2^2.
\]

(A4)

Noting that the following relations hold,

\[
2|x_1(x_1 + S_f)\bar{x}_2| \leq (\delta + S_f)\bar{x}_2^2 + (\delta + S_f)\bar{x}_2^2,
\]

(A5)

\[
2|x_2(x_1 + S_f)\bar{x}_2| \leq \delta x_1^2 + S_f x_2^2 + (S_f + \delta) x_2^2.
\]

(A6)

we have

The R.H.S. of (A4) \( \leq \{2u + 2\mu + \nu - \beta S_f\}
- \beta(1 + A)\}x_1^2
- \{(2\mu + 2\gamma - 2\beta S_f - \varepsilon^2)A - (\nu + \beta S_f)\}
- \delta(1 + A)\}x_2^2.
\]

(A7)

(\text{using } (48), \text{we have})

\[
= -(2u + 2\mu + \nu - \beta S_f - (1 + A)\delta\beta)\}x_1^2
- \{(2\mu + 2\gamma - 2\beta S_f - \varepsilon^2)A - (\nu + \beta S_f)\}
- \delta(1 + A)\}x_2^2.
\]

(A8)

(\text{using } A \text{ defined by (39), we have})

\[
= -(2u + 2\mu + \nu - \beta S_f - (1 + A)\delta\beta)\}x_1^2
- \{(2\mu + 2\gamma - 2\beta S_f - \varepsilon^2)A - (\nu + \beta S_f)\}
- \delta(1 + A)\}x_2^2.
\]

(A9)

Hence, for a sufficiently small \( \delta > 0 \), we have (49).

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