未知の柔軟対象物に対する接触作業のための マニピュレータのゲインスケジューリング制御*

呉 建青**・羅 志偉***・山北 昌毅****・伊藤 宏司*****

Gain Scheduled Control of Manipulators for Contact Tasks on Uncertain Flexible Objects*

Jianqing WU**, Zhiwei LUO***, Masaki YAMAKITA**** and Koji ITO*****

This paper studies position/force hybrid control of a robot manipulator to interact with its uncertain flexible object. Because of its flexibility, the object dynamics will influence the robot's control system, and since it is a distributed parameter system, the object dynamics as seen from the robot's end-effector will change when the robot moves on its different locations. In this paper, the variation of the object dynamics as seen from the robot end-effector is formulated as an uncertain linear parameter-varying system (ULPV). Gain scheduled control is developed for controlling this kind of uncertain system. It is found, although the robot's position control loop is not influenced by the contact force, the robot's moving velocity influences the solution of the force control system's Riccati equation, which was used in constructing the system's force control input. Therefore, the position control loop should be designed with respect to the force control loop. Computer simulations show the effectiveness of our control approach.

1. Introduction

This paper studies a robot manipulator's contact tasks on its uncertain flexible objects. As shown in Fig. 1, consider how to recover a protective coat on the antenna of an artificial satellite. To perform this task successfully, the robot is required to move on the surface of the object while maintaining a constant contact force along the normal direction of the object. If the contact force is too small, the robot is not able to stick

the coat onto the surface. Conversely, if the force is too large, the robot may destroy both the object and the coat. Because of its flexibility, the object dynamics will influence the robot's control system, and since it is a distributed parameter system, the object dynamics as seen from the robot's end-effector will change when the robot moves on its different locations. In order to specify position/force hybrid control for the robot, it is necessary to take into account about the object dynamics in designing the robot's force controller.

Fig. 1 A robot manipulator recovers a protective coat on the antenna of an artificial satellite

By now, impedance control^{5),6)} and position/force hybrid control⁷⁾ have been proposed

* 原稿受付 1996年6月26日

** 豊橋技術科学大学 情報工学系 Department of Information and Computer Sciences, Toyohashi University of Technology; 1-1 Hibarigaoka Tempaku-cho, Toyohashi 441, JAPAN

*** バイオミメティック制御センター 理化学研究所 Bio-Mimetic Control Research Center, RIKEN; 8-31, Rokuban 3-chome, Atsuta-ku, Nagoya 456, JAPAN

**** 東京工業大学 工学部 Faculty of Engineering, Tokyo Institute of Technology; 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152, JAPAN

***** 東京工業大学 総合理工学部 Computational Intelligence and Systems Science, Tokyo Institute of Technology; 4295, Nagatsuta, Midori-ku, Yokohama 226, JAPAN

Key Words: flexible object, robot, gain scheduled control, uncertain linear parameter-varying system.

for a robot to interact with its environment. Impedance control uses force feedback to adjust the robot's mechanical impedance as seen from its environment. It does not control the contact force directly. The force is controlled implicitly by setting the robot's equilibrium position. Therefore, the force can only be controlled in the steady state without any environmental uncertainties. On the other hand, hybrid control divides the robot's control space into position control subspace and force control subspace. The position control is realized along the free motion subspace and the contact force is controlled explicitly using force feedback. Unfortunately, there are few studies which considered the uncertain dynamics and variation of the environment in hybrid control.

In our early studies of Refs. 1)~3), we first determined the robot's position response using optimal state feedback control⁸⁾, and then treated the force control loop as an unknown linear time-varying(ULTV) system. We studied the simple adaptive control of this ULTV force loop^{9),10)}. Simple adaptive control guarantees the ULTV system's Lyapunov stability, however it is not clear how the robot's position response influences the force control. In Ref. 4), we treated the robot's force control loop as a linear parameter-varying(LPV) system, varying with respect to the robot's contact position. We assumed that the system parameters are all known, and then applied Watanabe et al.'s gain scheduled control algorithm¹²⁾ for this known LPV system. We thus find that the robot's moving velocity influences the solution of the force control system's Riccati equation, which was used in constructing the system's force control input. Unfortunately, these proposed gain scheduled control algorithms^{11),12)} did not handle the system uncertainties.

In many practical cases of a robot's contact task, however, the object's dynamics are unknown or have uncertainties. Therefore, in this paper, we will first develop the gain scheduled control for the uncertain linear parameter-varying(ULPV) system, and then apply it to design the robot's force control input. To do this, we will study the theories of Refs. 12)~15), which were proposed for controlling a general uncertain linear time-varying(ULTV) system. Computer simulations show the effectiveness of our control approach.

2. Problem Formulation

Consider the robot's contact task on its flexible object as shown in Fig. 2(a), for example, to

recover a protective coat on the antenna of an artificial satellite. It is understood that, if the contact force is too small, the robot is not able to stick the coat onto the surface. Conversely, if the force is too large, the robot may destroy both the object and the coat. Therefore, in order to perform the required task successfully, it is necessary to control the contact force exactly as well as the robot's end-effector position. Without losing generality, we simply assume that

- (1) the robot's end-effector is equipped with a roller bearing, there is no friction between the robot and the object at the object's surface.
- (2) the robot does not move so fast on the object, and the flexible object only makes small deformation when the robot imposing a contact force on it.

Fig. 2 Robot's contact task on a flexible cantilever

2.1 Approximation of Object Dynamics

The dynamic equation of the cantilever is

$$\begin{aligned} \frac{\partial^2 y_o}{\partial t^2} + 2\zeta \sqrt{\frac{EI}{\mu A}} \frac{\partial^3 y_o}{\partial x_o^2 \partial t} + \frac{EI}{\mu A} \frac{\partial^4 y_o}{\partial x_o^4} \\ = \frac{1}{\mu A} \delta(x_o - \rho) f(t) \end{aligned} \quad (1)$$

where $y_o(x_o, t)$ is a vertical displacement of the cantilever as shown in Fig. 2(b), EI denotes bending stiffness, μA is mass per unit length and ζ is damping coefficient. $\delta(x_o)$ is a delta function, and ρ is the location of the contact force $f(t)$. The boundary conditions at the fixed end are

$$y_o(0, t) = 0, \quad \frac{\partial y_o(0, t)}{\partial x_o} = 0,$$

and the boundary conditions at the free end are

$$\frac{\partial y_o^2(l, t)}{\partial x_o^2} = 0, \quad \frac{\partial y_o^3(l, t)}{\partial x_o^3} = 0,$$

where l is the length of the beam.

According to the expansion theorem, the solu-

tion of eq.(1) can be expressed as

$$y_o(x_o, t) = \sum_{i=1}^{\infty} \Phi_i(x_o) \eta_i(t) \quad (2)$$

where Φ_i is eigenfunctions and η_i is modal coordinates. Applying the standard orthogonalization procedure, eq.(1) can be replaced by the following set of ordinary differential equations:

$$\begin{aligned} \ddot{\eta}_i + 2\zeta\omega_i\dot{\eta}_i + \omega_i\eta_i &= \frac{\Phi_i(\rho)}{\mu A} f(t) \\ \omega_i(\rho) &= \left(\frac{i\pi}{l}\right)^2 \left(\frac{EI}{\mu A}\right)^{\frac{1}{2}} \\ \Phi_i(\rho) &= \left(\frac{2}{l}\right)^{\frac{1}{2}} \sin \frac{i\pi\rho}{l} \end{aligned} \quad (3)$$

where subscript i is the mode number, ω_i is the natural frequency^{16),17)}. If we assume that the robot does not move so fast on the object, and the flexible object only makes small deformation when the robot imposing a contact force on it, we can only consider the vertical effective of the 1st. mode, the dynamic equation of the cantilever can be approximated as

$$I(\rho)\ddot{\eta} + d(\rho)\dot{\eta} + k(\rho)\eta = \rho f(t) \quad (4)$$

where

$$I(\rho) = \frac{\mu A \rho}{\Phi^2(\rho)}, d(\rho) = \frac{2\zeta\omega\mu A \rho}{\Phi(\rho)}, k(\rho) = \frac{\omega\mu A \rho}{\Phi(\rho)},$$

and we eliminated the subscript $i=1$.

On the other hand, the relation between y_o and the angle θ of the object's normal direction with respect to y_o is shown in Fig. 3, where O1-C is the tangent line and B-C-C2 is the normal line of the object at the contact point C. Shifting O1-C to cross the point O, we get a new parallel line O-C2. It is easy to show that

Fig. 3 Relation between y_o and θ

$$e = \rho \tan \theta - \frac{y_o}{\cos \theta}. \quad (5)$$

If the object's deformation is very small, then $e \approx 0$, $\tan \theta \approx \theta$ and $\cos \theta \approx 1$. Therefore,

$$y_o \approx \rho \theta. \quad (6)$$

Substituting eqs.(2), (6) into eq.(4), we get the following approximated model of the flexible object at the fixed operating point C

$$I(\rho)\ddot{\theta}(t) + d(\rho)\dot{\theta}(t) + k(\rho)\theta(t) = \rho f(t) + \Delta_m \quad (7)$$

where

$$I(\rho) = \frac{\mu A \rho^2}{\Phi^2(\rho)}, d(\rho) = \frac{2\zeta\omega\mu A \rho^2}{\Phi^2(\rho)}, k(\rho) = \frac{\omega\mu A \rho^2}{\Phi^2(\rho)}.$$

Δ_m represents all model approximation errors which arises from the elimination of the high mode dynamics of the object and the error from y_o to θ , θ is the angle of the contact force f at the contact point C. If there is no friction on the object's surface, it can be measured by the 6-axis force sensor equipped at the robot manipulator's end-effector. The relation between the robot and the object can be represented as shown in Fig. 4.

Fig. 4 Approximated lumped parameter model

Note that, the above parameters $I(\rho)$, $d(\rho)$ and $k(\rho)$ are varying with respect to the operating point position ρ — the robot's end-effector position. It is the first difficulty in the robot's hybrid control on flexible objects.

2.2 Dynamic Model of Robot Manipulator

As shown in Fig. 4, let (x_b, y_b) be the reference frame fixed relative to the robot's basis; (x_o, y_o) be the object frame fixed relative to the fixed side of the beam; and (ρ, θ) be the task frame.

The dynamic equation of the robot manipulator in the reference space is

$$M(x)\ddot{x} + h(x, \dot{x}) + g(x) = F_e - F_n \quad (8)$$

where $\mathbf{x} = [x_b, y_b]^T$ is the robot's end-effector position, $M(\mathbf{x})$ is the robot's inertia matrix, $\mathbf{h}(\mathbf{x}, \dot{\mathbf{x}})$ is the vector of Coriolis and centrifugal forces, $\mathbf{g}(\mathbf{x})$ is the vector of gravitational force. The detailed $M(\mathbf{x})$, $\mathbf{h}(\mathbf{x}, \dot{\mathbf{x}})$ and $\mathbf{g}(\mathbf{x})$ are shown in Appendix A. $\mathbf{F}_e$ is the robot's driving force, and $\mathbf{F}_n$ is the contact force between the robot's end-effector and the object.

Specifying the following nonlinear feedback compensation

$$\mathbf{F}_e = M(\mathbf{x})\dot{\mathbf{x}}_c + \mathbf{h}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{g}(\mathbf{x}) + \mathbf{F}_n + M(\mathbf{x})JM_d^{-1}(\mathbf{u}_e - \mathbf{f}_c) \quad (9)$$

into eq.(8), it follows

$$m_1 \ddot{\rho}(t) = u_1(t) \quad (10)$$

$$m_2 \ddot{\theta}(t) = u_2(t) - \rho f_n(t) \quad (11)$$

$$I(\rho) \ddot{\theta} + d(\rho)\dot{\theta} + k(\rho)\theta = \rho f_n(t) + \Delta_m \quad (12)$$

where $\mathbf{x}_c = [\rho, \theta]^T$, $\mathbf{f}_c = [0, \rho f_n]^T$. $\mathbf{u}_e = [u_1, u_2]^T$ is the new control input vector, f_n is the magnitude of the contact force F_n . $M_d = \text{diag}[m_1, m_2]$ denotes a desired inertia matrix, and

$$J = \begin{bmatrix} -\cos(\theta) & \rho \sin(\theta) \\ -\sin(\theta) & -\rho \cos(\theta) \end{bmatrix}$$

is a Jacobian matrix.

Eqs.(10), (11) and (12) represent the simplified robot-object dynamics. Although the position loop(eq.(10)) is independent from the contact force f_n , the force loop (eqs.(11) and (12)) is dependent on the contact position ρ . In addition, the parameters $I(\rho)$, $d(\rho)$ and $k(\rho)$ are uncertain and variable, it is difficult to design the force control loop.

2.3 Position/Force Hybrid Control

Due to that the position control loop is independent from the contact force f_n , we can design the position control loop previously. As proposed in Refs. 1), 2), the position loop's control input is designed as

$$\mathbf{u}_\rho(t) = -R_\rho^{-1}B_\rho^T P \mathbf{x}_{\rho e}(t) \quad (13)$$

which optimizes the following performance index⁸⁾.

$$J_\rho = \int_0^\infty [\mathbf{x}_{\rho e}^T Q_\rho \mathbf{x}_{\rho e} + \mathbf{u}_\rho^T R_\rho \mathbf{u}_\rho] dt \quad (14)$$

where $\mathbf{x}_{\rho e} = \mathbf{x}_{\rho d} - \mathbf{x}_\rho$, $\mathbf{x}_\rho = [\rho, \dot{\rho}]^T$, $\mathbf{x}_{\rho d} = [\rho_d, \dot{\rho}_d]^T$, $u_\rho = u_1$. Matrix P is the solution of the following Riccati equation.

$$PA_\rho + A_\rho^T P - PB_\rho^T R_\rho B_\rho P + Q_\rho = 0 \quad (15)$$

where

$$A_\rho = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_\rho = \begin{bmatrix} 0 \\ 1 \\ m_1 \end{bmatrix}.$$

Because the object only makes small deformation, then

$$\rho \approx \sqrt{x_o^2 + y_o^2}, \quad \dot{\rho} \approx \frac{(x_o \dot{x}_o + y_o \dot{y}_o)}{\rho}.$$

If we assume that the origin O of the object's frame is known from the robot's reference frame, then ρ and $\dot{\rho}$ can be obtained from the robot's position sensors.

The block diagram of the position control loop is shown in Fig. 5.

Fig. 5 Optimal state feedback control of the position loop

On the other hand, in eqs.(3) and (4), it is understood that, usually the length l and the mass per unit length μA of the object can be known. However, the bending stiffness EI and the damping coefficient ζ may not be known or may have uncertainties. Therefore, in the following studies, we reasonably assume that the object's parameters are of the form

$$\begin{aligned} I &= I_0(\rho) \\ d &= d_0(\rho) + \delta d(\rho) \\ k &= k_0(\rho) + \delta k(\rho) \end{aligned} \quad (16)$$

where I_0 , d_0 and k_0 are known real-valued functions, δd and δk are real-valued functions representing the uncertainties of the flexible object.

We rewrite the force control loop of eqs.(11) and (12) in state-space as

$$\dot{\mathbf{x}}_\theta(t) = [A_\theta(\rho(t)) + \Delta A_\theta(\rho(t))] \mathbf{x}_\theta(t) + \mathbf{b}_{\theta 1}(\rho(t)) \Delta_m + \mathbf{b}_{\theta 2}(\rho(t)) u_f(t) \quad (17)$$

$$y_\theta(t) = [\mathbf{c}_\theta(\rho(t)) + \Delta \mathbf{c}_\theta(\rho(t))] \mathbf{x}_\theta(t) + d_{\theta 1}(\rho(t)) \Delta_m + d_{\theta 2}(\rho(t)) u_f(t) \quad (18)$$

where

$$\mathbf{x}_\theta(t) = [\theta \ \dot{\theta}]^T, \quad u_f(t) = \frac{u_2}{\rho}, \quad y_\theta(t) = f_n$$

$$A_\theta(\rho(t)) = \begin{bmatrix} 0 & 1 \\ -k_0 & -d_0 \\ m_2 + I_0 & m_2 + I_0 \end{bmatrix}$$

$$\Delta A_\theta(\rho(t)) = \begin{bmatrix} 0 & 1 \\ -\delta k & -\delta d \\ m_2 + I_0 & m_2 + I_0 \end{bmatrix}$$

$$\mathbf{b}_{\theta 1}(\rho(t)) = \begin{bmatrix} 0 \\ \frac{1}{m_2 + I_0} \end{bmatrix}^T$$

$$\mathbf{b}_{\theta 2}(\rho(t)) = \begin{bmatrix} 0 \\ \frac{\rho}{m_2 + I_0} \end{bmatrix}^T$$

$$\mathbf{c}_\theta(\rho(t)) = \begin{bmatrix} \frac{m_2 k_0}{\rho(m_2 + I_0)} & \frac{m_2 d_0}{\rho(m_2 + I_0)} \end{bmatrix}$$

$$\Delta \mathbf{c}_\theta(\rho(t)) = \begin{bmatrix} \frac{m_2 \delta k}{\rho(m_2 + I_0)} & \frac{m_2 \delta d}{\rho(m_2 + I_0)} \end{bmatrix}$$

$$d_{\theta 1}(\rho(t)) = \frac{-m_2}{\rho(m_2 + I_0)}$$

$$d_{\theta 2}(\rho(t)) = \frac{I_0}{(m_2 + I_0)}$$

This is an uncertain linear parameter-varying (ULPV) system, varying with respect to the scheduling parameter ρ .

3. Gain Scheduled Control

In this section, we discuss how to control above uncertain linear parameter-varying force control loop. We also study how does the variation of the scheduling parameter ρ (the robot's position) influences the force control design. Gain scheduled control theory usually consider the control of a known linear parameter-varying system¹¹⁾. Recently, Watanabe et al. also took into account about the variation of the scheduling parameter¹²⁾. However, few researches studied the gain scheduled control of uncertain linear parameter-varying system.

In order to control our uncertain force loop and to clarify the relation between the force control and position control design, we develop the gain scheduled control theory to the uncertain linear parameter-varying system. As shown in Fig. 6, our control scheme consists of two parts:

- (1) feedforward compensator, which is designed by the known parts of the system (eq.(17) and (18)).
- (2) feedback compensator using gain scheduled control, which compensates the deviation of the output that arises from the uncertainties of the system.

The input u_f of the force control loop is represented as

$$u_f = u_m + u_k \quad (19)$$

where u_m and u_k are the output of the feedforward and feedback compensator, respectively.

Fig. 6 Two compensator control scheme for the force loop

3.1 Design of Feedforward Compensator

Let $f_d(t)$ be the desired output of the force control loop. Without losing generality, the feedforward compensator is designed as follows.

$$u_m(t) = d_{\theta 2}^+(-c_\theta x_\theta + f_d(t)) \quad (20)$$

where $(\cdot)^+$ is pseudo-inverse matrix. In this paper, $d_{\theta 2}$ is scalar variable, and hence

$$u_m(t) = d_{\theta 2}^{-1}(-c_\theta x_\theta + f_d(t)). \quad (21)$$

3.2 Design of Feedback Compensator

First, we select the input $\mathbf{w}(t)$ and the output $\mathbf{z}(t)$ as

$$\mathbf{w}(t) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \Delta_m \\ f_d \end{bmatrix} \quad (22)$$

$$\mathbf{z}(t) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_f \\ f_d - y_\theta \end{bmatrix} \quad (23)$$

as shown in Fig. 6. Then, the plant shown in Fig. 6 can be described by the following uncertain linear parameter-varying system

$$\dot{\mathbf{x}}_\theta(t) = [A(\rho(t)) + \Delta A(\rho(t))]\mathbf{x}_\theta(t) + B_1(\rho(t))\mathbf{w}(t) + B_2(\rho(t))u_k(t) \quad (24)$$

$$\mathbf{z}(t) = [C(\rho(t)) + \Delta C(\rho(t))]\mathbf{x}_\theta(t) + D_1(\rho(t))\mathbf{w}(t) + D_2(\rho(t))u_k(t) \quad (25)$$

where

$$\begin{aligned}
 A(\rho(t)) &= A_\theta - b_{\theta 2} d_{\theta 2}^{-1} c_\theta \\
 \Delta A(\rho(t)) &= \Delta A_\theta(\rho(t)) \\
 B_1(\rho(t)) &= \begin{bmatrix} b_{\theta 1} & b_{\theta 2} d_{\theta 2}^{-1} \end{bmatrix}, B_2(\rho(t)) = b_{\theta 2} \\
 C(\rho(t)) &= \begin{bmatrix} -d_{\theta 2}^{-1} c_\theta \\ \mathbf{0} \end{bmatrix}, \Delta C(\rho(t)) = \begin{bmatrix} \mathbf{0} \\ -\Delta c_\theta \end{bmatrix} \\
 D_1(\rho(t)) &= \begin{bmatrix} 0 & d_{\theta 2}^{-1} \\ -d_{\theta 1} & 0 \end{bmatrix}, D_2(\rho(t)) = \begin{bmatrix} 1 \\ -d_{\theta 2} \end{bmatrix}
 \end{aligned}$$

$\rho(t)$ is a scheduling parameter, A , B_1 , B_2 , C , D_1 and D_2 are known real matrices, ΔA and ΔC are real-valued matrix functions representing time-varying parameter uncertainties in the state and the output matrices, respectively. This uncertain linear parameter-varying (ULPV) system is denoted as Σ .

We now consider the gain scheduled control of a general ULPV system which in form of eqs.(24) and (25). Let the parameter uncertainties ΔA and ΔC be of the form

$$\Delta A(\rho(t)) \triangleq G_1 F(\rho(t)) E_1 \quad (26)$$

$$\Delta C(\rho(t)) \triangleq G_2 F(\rho(t)) E_2 \quad (27)$$

where, G_1, E_1, G_2 , and E_2 are known real matrices and $F(\rho(t))$ is an unknown bound time-varying matrix function with Lebesgue measurable elements and satisfies

$$0 \leq F(\rho(t)) \leq I. \quad (28)$$

Let u_k be given by the form

$$u_k = K(\rho(t)) x_\theta(t). \quad (29)$$

Here, u_k should be designed such that the feedback law minimizes the effectiveness of w on z under the constraint that the closed-loop system(Fig. 7) is internally stable.

Fig. 7 Feedback compensator for the force control loop

[Definition 1] $\rho(t)$ is regarded as scheduling parameter if it satisfies

$$(1) \quad \rho(t) \in C^1 \quad (30)$$

$$(2) \quad |\dot{\rho}(t)| \leq \nu_{\max}, \forall t \in [0, \infty) \quad (31)$$

[Definition 2] System Σ is said to be quadratically stabilizable via linear state (output) feedback control if there exists a linear controller $u = K(t)x$ ($u = K(t)y$) such that the overall closed-loop system is quadratically stable.

Before describing the **Theorem 1** and **2**, we present the following propositions that will be needed in the proof of the **Theorem 2**.

[Proposition 1] Given any real matrices X , Y and Z with appropriate dimensions and such that Y a symmetric positive definite matrix, then¹³⁾

$$X^T Y X + X^T Z + Z^T X + Z^T Y^{-1} Z \geq 0 \quad (32)$$

[Proposition 2] Given P a symmetric matrix, then¹⁵⁾

$$\begin{aligned}
 \max \{ (\xi^T P D F E \xi)^2 : F^T F \leq I \} \\
 = \xi^T P D D^T P \xi \xi^T E^T E \xi
 \end{aligned} \quad (33)$$

for all $\xi \in \mathbb{R}^n$.

[Proposition 3] Let X , Y and Z be given $k \times k$ symmetric matrices such that $X \geq 0, Y < 0$ and $Z \geq 0$, furthermore, assume that

$$(\xi^T Y \xi)^2 - 4(\xi^T X \xi \xi^T Z \xi) > 0 \quad (34)$$

for all non-zero $\xi \in \mathbb{R}^k$. Then, there exists a positive definite scalar function λ such that¹⁵⁾

$$\lambda^2 X + \lambda Y + Z < 0 \quad (35)$$

[Proposition 4] Let F be a norm bound matrix such that $0 \leq F \leq I$, then

$$F^T F \leq I \quad (36)$$

$$F^T M F \leq M \quad (37)$$

$$E^T F^T M F E \leq E^T M E \quad (38)$$

for any matrix $M \geq 0$ and any matrix E with appropriate dimensions.

(Proof) First note that

$$x^T (I - F) x \geq 0 \quad (39)$$

for any $x \neq 0$, we have

$$\begin{aligned}
 x^T (I - F^T) x &= (x^T (I - F^T) x)^T \\
 &= x^T (I - F) x \geq 0
 \end{aligned} \quad (40)$$

for any $x \neq 0$, and hence

$$0 \leq F^T \leq I. \quad (41)$$

Then, it follows

$$F^T F \leq F \leq I. \quad (42)$$

Furthermore, since $M \geq 0$, we have

$$MF \leq M \quad (43)$$

$$F^T M \leq M \quad (44)$$

$$F^T MF \leq MF \leq M. \quad (45)$$

Since $M - F^T MF \geq 0$, it follows

$$\begin{aligned} \mathbf{x}^T E^T (M - F^T MF) E \mathbf{x} \\ = (E \mathbf{x})^T (M - F^T MF) (E \mathbf{x}) \\ = \mathbf{y}^T (M - F^T MF) \mathbf{y} \geq 0 \end{aligned} \quad (46)$$

where $\mathbf{y} = E \mathbf{x}$. Thus, we have

$$\begin{aligned} E^T (M - F^T MF) E \geq 0 \\ E^T F^T M F E \leq E^T M E. \end{aligned} \quad (47)$$

□

For a given known linear parameter-varying system, Watanabe et al. took into account about the variation of the scheduling parameter, and show the following theorem

[Theorem 1] For the given constant $\gamma > 0$, the system II

$$\dot{\mathbf{x}}(t) = A(\rho(t))\mathbf{x}(t) + B(\rho(t))\mathbf{w}(t) \quad (48)$$

$$\mathbf{z}(t) = C(\rho(t))\mathbf{x}(t) + D(\rho(t))\mathbf{w}(t) \quad (49)$$

is quadratically stable with an H_∞ -norm bound γ if there exists a real symmetric matrix $Q(\rho)$ that satisfies the following conditions¹²⁾.

$$(1) \quad Q(\rho) > 0, \quad (50)$$

$$(2) \quad \frac{dQ}{d\rho}(\rho) \leq 0 \quad (51)$$

$$\begin{aligned} (3) \quad & (A + BR^{-1}D^T C)^T Q \\ & + Q(A + BR^{-1}D^T C) + QBR^{-1}B^T Q \\ & + C^T(I + DR^{-1}D^T)C - \nu_{\max} \frac{dQ}{d\rho} < 0 \end{aligned} \quad (52)$$

where R satisfies

$$R \triangleq \gamma^2 I - D^T D > 0. \quad (53)$$

(Proof) See Ref. 7) for detail.

□

Using above theorem and recent results of uncertain linear time-varying system of Refs. 13)~15), we proof the following theorem for a general uncertain linear parameter-varying system which is in form eqs.(24) and (25).

[Theorem 2] The uncertain system Σ is quadratically stabilizable with an H_∞ -norm bound $\gamma > 0$ if there exists a proper state feedback

law $u_k = K(\rho(t))\mathbf{x}$ and a positive definite scalar function λ such that there exists a real symmetric matrix $Q(\rho)$ that satisfies the following conditions.

$$(1) \quad Q(\rho) > 0, \quad (54)$$

$$(2) \quad \frac{dQ}{d\rho}(\rho) \leq 0 \quad (55)$$

$$\begin{aligned} (3) \quad & (A + B_1 R_1^{-1} D_1^T C)^T Q \\ & + Q(A + B_1 R_1^{-1} D_1^T C) \\ & + Q\{B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T + \lambda R_3 R_3^T \\ & - (B_2 + B_1 R_1^{-1} D_1^T D_2) R_2^{-1} (B_2 \\ & + B_1 R_1^{-1} D_1^T D_2)^T\} Q + 3C^T H C \\ & + E_2^T (3G_2^T H G_2 + \frac{1}{\lambda} I) E_2 + E_1^T R_\varepsilon E_1 \\ & - \nu_{\max} \frac{dQ}{d\rho} < 0 \end{aligned} \quad (56)$$

where

$$R_1 \triangleq \gamma^2 I - D_1^T D_1 > 0 \quad (57)$$

$$H = I + D_1 R_1^{-1} D_1^T > 0 \quad (58)$$

$$R_2 = 3D_2^T H D_2 > 0 \quad (59)$$

$$R_3 = B_1 R_1^{-1} D_1^T G_2 \quad (60)$$

$$R_\varepsilon \triangleq I + \frac{1}{\varepsilon} E_1 E_1^T > 0 \quad (61)$$

and $\varepsilon > 0$ is a design parameter. Moreover, a suitable feedback control law is given by

$$u_k(t) = K(\rho(t))\mathbf{x}_\theta(t) \quad (62)$$

$$K(\rho(t)) = -R_2^{-1} (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q. \quad (63)$$

(Proof) Substituting eq.(62) into eqs.(24) and (25), we obtain

$$\dot{\mathbf{x}}(t) = A_c(\rho(t))\mathbf{x}(t) + B_c(\rho(t))\mathbf{w}(t) \quad (64)$$

$$\mathbf{z}(t) = C_c(\rho(t))\mathbf{x}(t) + D_c(\rho(t))\mathbf{w}(t) \quad (65)$$

where

$$A_c = A + G_1 F E_1 + B_2 K, \quad B_c = B_1$$

$$C_c = C + G_2 F E_2 + D_2 K, \quad D_c = D_1.$$

It follows from Theorem 1, that if there exists a real symmetric positive definite matrix Q such that $dQ/d\rho \leq 0$ and

$$\begin{aligned} & (A_c + B_c R_c^{-1} D_c^T C_c)^T Q \\ & + Q(A_c + B_c R_c^{-1} D_c^T C_c) \\ & + Q B_c R_c^{-1} B_c^T Q + C_c^T (I + D_c R_c^{-1} D_c^T) C_c \\ & - \nu_{\max} \frac{dQ}{d\rho} < 0 \end{aligned} \quad (66)$$

where

$$R_c = \gamma^2 I - D_c^T D_c > 0 \quad (67)$$

then, the system Σ is quadratically stabilizable with H_∞ norm-bound $\gamma > 0$.

Substituting A_c, B_c, C_c and D_c into ineq.(66), and using **Proposition 1** and **4**, we obtain (See **Appendix 2** for the detailed proof)

$$\mathbf{x}^T Y \mathbf{x} + 2\mathbf{x}^T Q R_3 F E_2 \mathbf{x} < 0 \quad (68)$$

for all non-zero $\mathbf{x} \in \mathbb{R}^n$. Here,

$$\begin{aligned} Y \triangleq & (A + B_1 R_1^{-1} D_1^T C)^T Q \\ & + Q(A + B_1 R_1^{-1} D_1^T C) \\ & + Q\{B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T \\ & - (B_2 + B_1 R_1^{-1} D_1^T D_2) R_2^{-1} (B_2 \\ & + B_1 R_1^{-1} D_1^T D_2)^T\} Q + 3C^T H C \\ & + E_1^T R_\varepsilon E_1 + 3E_2^T G_2^T H G_2 E_2 \\ & + W_k^T W_k - \nu_{\max} \frac{dQ}{d\rho} < 0 \end{aligned} \quad (69)$$

where

$$W_k = R_2^{\frac{1}{2}} [K + R_2^{-1} (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q]. \quad (70)$$

Hence, it follows

$$\mathbf{x}^T Y \mathbf{x} < -2\max\{\mathbf{x}^T Q R_3 F E_2 \mathbf{x} : F^T F \leq I\} \quad (71)$$

and from **Proposition 2** we have

$$\begin{aligned} (\mathbf{x}^T Y \mathbf{x})^2 & > 4\max\{(\mathbf{x}^T Q R_3 F E_2 \mathbf{x})^2 : F^T F \leq I\} \\ & = 4\{\mathbf{x}^T Q R_3 R_3^T Q \mathbf{x} \mathbf{x}^T E_2^T E_2 \mathbf{x}\}. \end{aligned} \quad (72)$$

By using **Proposition 3**, it follows that there exists a positive definite scalar function λ such that

$$\lambda^2 Q R_3 R_3^T Q + \lambda Y + E_2^T E_2 < 0. \quad (73)$$

Put $W_k = 0$, we then get eq.(63) and ineq(56). $\square$

In our study of a robot's position/force hybrid control, $\nu_{\max}$ in eq.(56) of the above theorem represents the maximum velocity of the robot's position response. It influences the solution $Q(\rho(t))$ of the Riccati inequation, which is necessary in calculating the force feedback controller of eqs.(62) and (63). Therefore, although the position control loop in eq.(10) is not influenced by the contact force, we still need to consider the force in designing the position control loop.

4. Simulation Results

To examine the performance of the approach described above, computer simulation of a two de-

gree of freedom planar robot manipulator sliding on an uncertain cantilever has been done.

The object's parameters are as follows.

$$\begin{aligned} I_0 &= 1.0[\text{Nms}^2/\text{rad}], \quad \rho_0 = 0.5[\text{m}] \\ d_0 &= 2.0[\text{Nms}/\text{rad}], \quad d_1 = 0.4[1/\text{m}] \\ k_0 &= 2.0[\text{Nm}/\text{rad}], \quad k_1 = 0.5[1/\text{m}] \\ \delta d(\rho) &= -d_0(1 - e^{-d_1(\rho - \rho_0)})[\text{Nms}/\text{rad}] \\ \delta k(\rho) &= -k_0(1 - e^{-k_1(\rho - \rho_0)})[\text{Nm}/\text{rad}] \end{aligned}$$

The desired displacement and contact force are specified as

$$\rho_d = \begin{cases} 0.5[\text{m}], & \text{for } t < 1.0[\text{s}] \\ 1.5[\text{m}], & \text{for } t \geq 1.0[\text{s}] \end{cases} \quad (74)$$

$$f_{nd} = \begin{cases} 0.0[\text{N}], & \text{for } t < 1.0[\text{s}] \\ 1.0[\text{N}], & \text{for } 1.0[\text{s}] \leq t \leq 10.0[\text{s}] \\ 0.0[\text{N}], & \text{for } t > 10.0[\text{s}] \end{cases} \quad (75)$$

The design parameters in the position and force controller is chosen as follows:

(1) the position controller

$$Q_\rho = I, \quad R_\rho = I, \quad m_1 = 1.0[\text{kg}]. \quad (76)$$

(2) the force controller

$$\begin{aligned} m_2 &= 1[\text{Nms}^2/\text{rad}] \\ \gamma &= 0.2, \quad \nu_{\max} = 16.0 \\ \epsilon &= 5.5, \quad \delta = 10^{-8}, \quad \lambda = 5.0 \\ Q_0 &= \begin{bmatrix} 40.0 & 30.0 \\ 30.0 & 40.0 \end{bmatrix}. \end{aligned} \quad (77)$$

The results of the simulation are given in Fig. 8 to Fig. 10. In order to show the initial responses clearly, we start our simulation at the time $t=1[\text{s}]$. Fig. 8 and Fig. 9 show the responses of the manipulator end-effector's position ρ and the contact force f_n , respectively. Fig. 10 illustrates the convergence of the gain K . The performances with and without feedforward compensator have been compared in Fig. 9. These results show that the proposed control scheme achieves significantly better performance for the uncertain linear parameter-varying system.

5. Conclusions

This paper studied the position/force hybrid control of a robot manipulator to interact with its uncertain flexible object. The variation of the object dynamics as seen from the robot is formulated as an uncertain linear parameter-varying system (ULPV). Feedforward + Gain scheduled feedback control structure is proposed and the gain sched-

Fig. 8 Robot's position response $\rho(t)$ Fig. 9 Robot's contact force responses $f_n(t)$ Fig. 10 Convergence of control gains k_1 and k_2

uled control theory is developed to control the such kind of the ULPV system. From the theoretical study of Gain scheduled control, it is clear that the maximum velocity of the position response influences the solution $Q(\rho(t))$ of the Riccati inequation, which is necessary in the calculation of the force feedback controller. Therefore, we should design the position control loop with respect to the contact force. Computer simulations show the effectiveness of our control approach.

In this paper, by assuming there is no friction between the robot and the object, we obtained the object's angle θ from the force sensor and used it to control the force. However, in many practical cases, there exist frictions. In that case, the contact force will never coincide with the normal direction of the object. Then, other kinds of sensors (such as a camera) will be necessary to measure the object's deformations. In addition, by assuming that the robot moves very slowly on the object's surface and the object only makes small deformation, we approximated the object's model and eliminated its higher order modes. However, as was pointed in Ref. 18), the object's higher order modes may result in the system's spillover instability. Recently, Matsuno et al. studied this spillover instability problem. They proposed a robust control approach which design a low-pass controller to solve this problem. As a further research, we should also take into account of this problem and study the frequency characteristics of our gain scheduled controller from both the aspect of the theory and experiment.

Acknowledgments

This work was supported through the scientific research fund from the Ministry of Education, Science and Culture of Japan (#07750534, #07243105, #06452253).

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Appendix

1. Two links robot's parameter matrices

Let $\mathbf{q} = [q_1, q_2]^T$ denote the robot's angle vector (Fig. A1). The dynamic equation of the robot can be described as

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}}(t) + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{p}(\mathbf{q}) = \boldsymbol{\tau}_e(t) \quad (\text{A1})$$

where $\boldsymbol{\tau}_e(t) = [\tau_{e1}, \tau_{e2}]^T$ is the vector of driving torque at the joints. $\mathbf{A}(\mathbf{q})$ is a inertia matrix, $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})$ is a vector of Coriolis and centrifugal forces, $\mathbf{p}(\mathbf{q})$ is a vector of gravitational force in joint space. They are of the form

Fig. A1 A two links robot manipulator

$$\begin{aligned} \mathbf{A}(\mathbf{q}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{p}(\mathbf{q}) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \end{aligned} \quad (\text{A2})$$

where

$$\begin{aligned} a_{11} &= m_1 l_1^2 + m_2 (a_1^2 + l_2^2) + 2m_2 a_1 l_2 \cos(q_2) \\ a_{12} &= m_2 l_2^2 + m_2 a_1 l_2 \cos(q_2) \\ a_{21} &= m_2 l_2^2 + m_2 a_1 l_2 \cos(q_2) \\ a_{22} &= m_2 l_2^2 \\ b_1 &= -m_2 a_1 l_2 \sin(q_2) \dot{q}_2^2 \\ &\quad - 2m_2 a_1 l_2 \sin(q_2) \dot{q}_1 \dot{q}_2 \\ b_2 &= m_2 a_1 l_2 \sin(q_2) \dot{q}_1^2 \\ p_1 &= (m_1 g l_1 + m_2 g a_1) \cos(q_1) \\ &\quad + m_2 g l_2 \cos(q_1 + q_2) \\ p_2 &= m_2 g l_2 \cos(q_1 + q_2). \end{aligned}$$

Let $\mathbf{x} = [x_b, y_b]^T$ be the manipulator's end-effector position at the reference frame Σ_b . The geometric relation between $\mathbf{x}$ and $\mathbf{q}$ is

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_b \\ y_b \end{bmatrix} \\ &= \begin{bmatrix} a_1 \cos(q_1) + a_2 \cos(q_1 + q_2) \\ a_1 \sin(q_1) + a_2 \sin(q_1 + q_2) \end{bmatrix}. \end{aligned} \quad (\text{A3})$$

By differentiating eq.(A3) with respect to time t , we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{J}_q \dot{\mathbf{q}}(t) \quad (\text{A4})$$

$$\ddot{\mathbf{x}}(t) = \dot{\mathbf{J}}_q \dot{\mathbf{q}}(t) + \mathbf{J}_q \ddot{\mathbf{q}}(t) \quad (\text{A5})$$

where the Jacobian matrix $\mathbf{J}_q$ is of the form

$$\mathbf{J}_q = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \quad (\text{A6})$$

where

$$\begin{aligned} j_{11} &= -a_1 \sin(q_1) - a_2 \sin(q_1 + q_2) \\ j_{12} &= -a_2 \sin(q_1 + q_2) \\ j_{21} &= a_1 \cos(q_1) + a_2 \cos(q_1 + q_2) \end{aligned}$$

$$j_{22} = a_2 \cos(q_1 + q_2).$$

Substituting eqs.(A4), (A5) into (A1), we get the manipulator's dynamic equation in $\sum_b$ as

$$M(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{h}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{g}(\mathbf{x}) = \mathbf{F}_e \quad (\text{A7})$$

where

$$\begin{aligned} M(\mathbf{x}) &= J_q^{-T} A(\mathbf{q}) J_q^{-1} \\ \mathbf{h}(\mathbf{x}, \dot{\mathbf{x}}) &= J_q^{-T}(\mathbf{q}) \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) - J_q^{-T} A(\mathbf{q}) J_q^{-1} \dot{J}_q \dot{\mathbf{q}} \\ \mathbf{g}(\mathbf{x}) &= J_q^{-T} \mathbf{p}(\mathbf{q}). \end{aligned}$$

2. Proof of the ineq.(68) of Theorem2

In order to proof ineq.(68), we substituted A_c, B_c, C_c and D_c into ineq.(66), and use **Proposition 1** and **4**, then

$$\begin{aligned} &(A_c + B_c R_1^{-1} D_c^T C_c)^T Q \\ &+ Q(A_c + B_c R_1^{-1} D_c^T C_c) \\ &= (A + B_1 R_1^{-1} D_1^T C)^T Q \\ &+ Q(A + B_1 R_1^{-1} D_1^T C) - W_1^T W_1 \\ &+ Q G_1 R_\varepsilon^{-1} G_1^T Q + E_1^T R_\varepsilon E_1 - \Delta P_1 \\ &+ K^T (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q \\ &+ Q(B_2 + B_1 R_1^{-1} D_1^T D_2) K \\ &+ Q B_1 R_1^{-1} D_1^T G_2 F E_2 \\ &+ (B_1 R_1^{-1} D_1^T G_2 F E_2)^T Q \end{aligned} \quad (\text{A8})$$

where

$$\begin{aligned} E_1^T F^T R_\varepsilon^{-1} F E_1 &= E_1^T R_\varepsilon^{-1} E_1 - \Delta P_1 \\ \Delta P_1 &\geq 0 \\ W_1 &= R_\varepsilon^{\frac{1}{2}} (G_1^T Q - R_\varepsilon^{-1} F E_1) \\ R_\varepsilon &\triangleq I + \frac{1}{\varepsilon} E_1 E_1^T > 0. \end{aligned}$$

Let $H = I + D_1 R_1^{-1} D_1^T > 0$, then we have

$$\begin{aligned} &C_c^T (I + D_c R_1^{-1} D_c^T) C_c \\ &= 3C^T H C + 3E_2^T G_2^T H G_2 E_2 \\ &+ 3K^T D_2^T H D_2 K - 3\Delta P_2 \\ &- W_2^T W_2 - W_3^T W_3 - W_4^T W_4 \end{aligned} \quad (\text{A9})$$

where

$$\begin{aligned} E_2^T F^T G_2^T H G_2 F E_2 &= E_2^T G_2^T H G_2 E_2 - \Delta P_2 \\ \Delta P_2 &\geq 0 \\ W_2 &= H^{\frac{1}{2}} (D_2 K - C) \\ W_3 &= H^{\frac{1}{2}} (G_2 F E_2 - C) \\ W_4 &= H^{\frac{1}{2}} (G_2 F E_2 - D_2 K). \end{aligned}$$

Based on ineq.(66), it follows

$$\begin{aligned} &(A + B_1 R_1^{-1} D_1^T C)^T Q + Q(A + B_1 R_1^{-1} D_1^T C) \\ &+ Q(B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T) Q \\ &+ E_1^T R_\varepsilon E_1 + 3C^T H C + 3E_2^T G_2^T H G_2 E_2 \\ &+ Q B_1 R_1^{-1} D_1^T G_2 F E_2 \\ &+ (B_1 R_1^{-1} D_1^T G_2 F E_2)^T Q - \nu_{\max} \frac{dQ}{d\rho} \\ &+ K^T (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q \\ &+ Q(B_2 + B_1 R_1^{-1} D_1^T D_2) K \\ &+ 3K^T D_2^T H D_2 K - W < 0 \end{aligned} \quad (\text{A10})$$

where

$$\begin{aligned} W &= \Delta P_1 + 3\Delta P_2 + W_1^T W_1 + W_2^T W_2 \\ &+ W_3^T W_3 + W_4^T W_4 \geq 0. \end{aligned}$$

When the following inequation is satisfied, the ineq.(A10) is also satisfied.

$$\begin{aligned} &(A + B_1 R_1^{-1} D_1^T C)^T Q + Q(A + B_1 R_1^{-1} D_1^T C) \\ &+ Q(B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T) Q \\ &+ E_1^T R_\varepsilon E_1 + 3C^T H C + 3E_2^T G_2^T H G_2 E_2 \\ &+ Q B_1 R_1^{-1} D_1^T G_2 F E_2 \\ &+ (B_1 R_1^{-1} D_1^T G_2 F E_2)^T Q - \nu_{\max} \frac{dQ}{d\rho} \\ &+ K^T (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q \\ &+ Q(B_2 + B_1 R_1^{-1} D_1^T D_2) K \\ &+ 3K^T D_2^T H D_2 K < 0. \end{aligned} \quad (\text{A11})$$

Because

$$\begin{aligned} &K^T (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q + Q(B_2 \\ &+ B_1 R_1^{-1} D_1^T D_2) K + 3K^T D_2^T H D_2 K \\ &= -Q(B_2 + B_1 R_1^{-1} D_1^T D_2) R_2^{-1} (B_2 \\ &+ B_1 R_1^{-1} D_1^T D_2)^T Q + W_k^T W_k \end{aligned} \quad (\text{A12})$$

where

$$\begin{aligned} R_2 &= 3D_2^T H D_2 > 0 \\ W_k &= R_2^{\frac{1}{2}} [K + R_2^{-1} (B_2 + B_1 R_1^{-1} D_1^T D_2)^T Q] \end{aligned}$$

then, the ineq.(A11) becomes

$$\begin{aligned} &(A + B_1 R_1^{-1} D_1^T C)^T Q + Q(A + B_1 R_1^{-1} D_1^T C) \\ &+ Q\{B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T - (B_2 \\ &+ B_1 R_1^{-1} D_1^T D_2) R_2^{-1} (B_2 + B_1 R_1^{-1} D_1^T D_2)^T\} Q \\ &+ E_1^T R_\varepsilon E_1 + 3C^T H C + 3E_2^T G_2^T H G_2 E_2 \\ &+ Q B_1 R_1^{-1} D_1^T G_2 F E_2 \\ &+ (B_1 R_1^{-1} D_1^T G_2 F E_2)^T Q \\ &+ W_k^T W_k - \nu_{\max} \frac{dQ}{d\rho} < 0. \end{aligned} \quad (\text{A13})$$

Defining

$$\begin{aligned}
Y \triangleq & (A + B_1 R_1^{-1} D_1^T C)^T Q \\
& + Q(A + B_1 R_1^{-1} D_1^T C) \\
& + Q\{B_1 R_1^{-1} B_1^T + G_1 R_\varepsilon^{-1} G_1^T \\
& - (B_2 + B_1 R_1^{-1} D_1^T D_2) R_2^{-1} (B_2 \\
& + B_1 R_1^{-1} D_1^T D_2)^T\} Q + 3C^T H C \\
& + E_1^T R_\varepsilon E_1 + 3E_2^T G_2^T H G_2 E_2 \\
& + W_k^T W_k - \nu_{\max} \frac{dQ}{d\rho} < 0
\end{aligned}$$

then, it follows

$$\begin{aligned}
Y + Q B_1 R_1^{-1} D_1^T G_2 F E_2 \\
+ (B_1 R_1^{-1} D_1^T G_2 F E_2)^T Q < 0.
\end{aligned} \tag{A14}$$

Hence, we get

$$\mathbf{x}^T Y \mathbf{x} + 2\mathbf{x}^T Q R_3 F E_2 \mathbf{x} < 0. \tag{A15}$$