サンプリング歪を考慮した離散時間繰返し制御系の LTR 設計*

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LTR Design of Discrete-Time Repetitive Controllers under Skewed Sampling*

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1. Introduction

We discuss an efficient output feedback design of discrete-time repetitive controllers [1,2] taking account of inherent limitations of digital control systems. As a plant model, we adopt a discrete-time model obtained by skewed sampling [3]. A generalized version of efficient pole assignment method proposed by Hara and Mita [4] is used with a Loop Transfer Recovery (LTR) technique [5]. We show that the design method provides a clear perspective on achievable feedback properties as well as the computational advantage.

2. Preliminaries

Consider a discrete-time plant model obtained by skewed sampling [3];

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w_u(k), \\ y(k+1) &= Cx(k) + Du(k) + w_y(k), \end{aligned} \quad (1)$$

where $x(k) \in R^n$ is a state vector, $u(k) \in R^m$ is a control vector, $y(k) \in R^m$ is a output vector, and the vectors $w_u(k)$ and $w_y(k)$ are zero mean white noise processes with the covariance matrix

$$\text{Cov} \begin{bmatrix} w_u(k) \\ w_y(k) \end{bmatrix} = \begin{bmatrix} Q_u & Q_{uy} \\ Q'_{uy} & Q_y \end{bmatrix}. \quad (A, B, C, D) \quad (2)$$

In addition, we assume that is minimal and that the matrix D is non-singular.

As a reference input, we consider a periodic reference signal

$$r(k) = r(k+N) \equiv r_k, \quad k = 0, 1, \dots, N-1, \quad (3)$$

where N is an integer. The z -transform of $r(k)$ is given by

$$r(z) = (z^N - 1)^{-1} (r_0 z^N + r_1 z^{N-1} + \dots + r_{N-1} z) \quad (4)$$

To guarantee the robust tracking, we assume that

$$G(z) \equiv C(zI - A)^{-1}B + D \quad (5)$$

does not have zeros satisfying $z^N = 1$. It is shown in [6] that this condition is equivalent to the non-singularity of the matrix

$$M_N \equiv \begin{bmatrix} A^N & A^{N-1}B & \dots & AB & B \\ CA^{N-1} & CA^{N-2}B & \dots & CB & D \\ CA^{N-2} & CA^{N-3}B & \dots & D & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ C & D & 0 & \dots & 0 \end{bmatrix}. \quad (6)$$

3. Target State Feedback Design

First, we consider a design of a state feedback repetitive controller as a target of the LTR. Noting that the internal model for the periodic reference signal (3) is given by $(z^N - 1)^{-1}I$, we can choose the structure of the state feedback repetitive controller as

$$\begin{aligned} u(k) &= s(k) - Lx(k), \\ s(k) &= s(k-N) + T(z)[r(k) - u(k)], \end{aligned} \quad (7)$$

where $s(k)$ is a state of the internal model and

$$T(z) \equiv T_{N-1}z^{-(N-1)} + T_{N-2}z^{-(N-2)} + \dots + T_0. \quad (8)$$

For practical applications, N is at least several tens: The order of the controller (7) is extremely high compared with conventional con-

* 原稿受付 2000 年 12 月 20 日

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Key Words: discrete-time systems, repetitive controllers, skewed sampling, loop transfer recovery.

trollers. Hara and Mita [4] proposed a novel pole assignment method which does not require a solution of a high order pole assignment problem. But they did not consider the sampling skew. The following lemma gives a generalization to the model (1). The proof is omitted since the generalization is almost straightforward.

【Lemma 1】 Let F be an arbitrary matrix that makes the matrix $A - BF$ stable and select an arbitrary N -th order stable polynomial

$$\lambda(z) \equiv z^N + \lambda_{N-1}z^{N-1} + \cdots + \lambda_1z + \lambda_0, \quad (9)$$

with real coefficients. Determine the matrices L and T_i in (7) and (8) by the linear matrix equation

$$\begin{bmatrix} L & T_0 & \cdots & T_{N-1} \end{bmatrix} M_N = \begin{bmatrix} J & I + J_0 & J_0 & \cdots & J_{N-1} \end{bmatrix}, \quad (10)$$

where M_N is the matrix defined by (6) and

$$J \equiv F(A^N + \lambda_{N-1}A^{N-1} + \cdots + \lambda_0I), \quad (11)$$

$$J_i \equiv F(A^{N-i-1} + \lambda_{N-1}A^{N-i-2} + \cdots + \lambda_{i+1}I)B + \lambda_iI, \quad i = 0, 1, \dots, N-1. \quad (12)$$

Then the eigenvalues of the closed loop system consisting of the plant (1) and the repetitive controller (7) are those of $A - BF$ and m multiples of the zeros of $\lambda(z)$. $\square$

The following result reveals that the above pole assignment method provides transparent closed loop properties.

【Proposition 1】 The sensitivity matrix at the plant input side of the control system consisting of (1) and (7) can be expressed as

$$S(z) = (z^N - 1)\lambda^{-1}(z)S_0(z), \quad (13)$$

where

$$S_0(z) \equiv [I + F(zI - A)^{-1}B]^{-1} \quad (14)$$

is the input sensitivity matrix of the state feedback regulator with the feedback gain matrix F .

【Proof】 The controller transfer function matrix from $[x'(z) \ y'(z)]'$ to $u(z)$ is given by $-[L(1 - z^{-N})T(z)]$ so that the inverse of the sensitivity matrix at the plant input can be written as

$$S^{-1}(z) = (I - z^{-N})^{-1}W(z), \quad (15)$$

where

$$W(z) \equiv z^{-1}T(z)[C(zI - A)^{-1}B + D] + (1 - z^{-N})[I + L(zI - A)^{-1}B]. \quad (16)$$

After some matrix calculations using (10)–(12), we can write $W(z)$ as

$$W(z) = z^{-N}J(zI - A)^{-1}B + z^{-N}J_0 + z^{-(N-1)}J_1 + \cdots + z^{-1}J_{N-1} + I. \quad (17)$$

Define

$$W_i(z) \equiv z^{-i}FA^i(zI - A)^{-1}B + z^{-i}FA^{i-1}B + z^{-(i-1)}FA^{i-2}B + \cdots + z^{-1}FB + I. \quad (18)$$

Using (11) and (12), we can rewrite (17) as

$$W(z) = \sum_{i=0}^N z^{-i}\lambda_{N-i}W_{N-i}(z) \quad (19)$$

with $\lambda_N \equiv 1$. Note that (18) can be simplified as

$$W_i(z) = I + F(zI - A)^{-1}B. \quad (20)$$

Using (20) in (19), we have

$$W(z) = z^{-N}\lambda(z)[I + F(zI - A)^{-1}B]. \quad (21)$$

From (15) and (21), we have the expression (13) for the sensitivity matrix. $\blacksquare$

【Proposition 2】 The transfer function matrix from the reference $r(z)$ to the output $y(z)$ can be expressed as

$$G_{yr}(z) = z^{N-1}\lambda^{-1}(z)[(C - DF)(zI - A + BF)^{-1}B + D]T(z). \quad (22)$$

【Proof】 From (1) and (7), we can express the transfer function matrix from $r(z)$ to $y(z)$ as

$$G_{yr}(z) = V(z)T(z), \quad (23)$$

where

$$V(z) \equiv \{T(z) + z(1 - z^{-N})[I + L(zI - A)^{-1}B]G^{-1}(z)\}^{-1}. \quad (24)$$

It can readily be checked that

$$V(z)W(z) = z^{-1}G(z). \quad (25)$$

where $W(z)$ is defined in (16) and rewritten as (21). From (21), (23) and (25), we have

$$G_{yr}(z) = z^{N-1}\lambda^{-1}(z)[C(zI - A)^{-1}B + D][I + F(zI - A)^{-1}B]^{-1}T(z), \quad (26)$$

which can be rewritten as (22). $\blacksquare$

A large freedom exists for the choice of the polynomial $\lambda(z)$. The simplest choice is $\lambda(z) = z^N$ but this choice completely ignores the freedom.

Fig. 1 Gain characteristic of $(z^N - 1)\lambda^{-1}(z)$

A simple practical choice is $\lambda(z) = z^N - \gamma$ where γ is a positive scalar less than unity. For this choice, the effect of γ on the gain characteristic of $(z^N - 1)\lambda^{-1}(z)$ with $N = 50$ is shown in Fig. 1. The plot and the decomposition (13) provides clear perspective on the achievable sensitivity property. The parameter γ can conveniently be used for the fine shaping of the target sensitivity property in an LTR design.

4. Loop Transfer Recovery

For the output feedback case, we design an observer-based controller by the LTR procedure based on the Riccati equation formalism [5] for a Kalman filter

$$\hat{x}(k+1) = F\hat{x}(k) + Gu(k) + K\{y(k+1) - [C\hat{x}(k) + Du(k)]\}. \quad (27)$$

The filter gain matrix K in (27) is given by

$$K = (APC' + Q_{uy})(CPC' + Q_y)^{-1}, \quad (28)$$

where P is a non-negative definite solution of the Riccati equation

$$P = APA' - (APC' + Q_{uy})(CPC' + Q_y)^{-1}(CPA' + Q'_{uy}) + Q_u. \quad (29)$$

The output feedback controller is given by

$$\begin{aligned} u(k) &= s(k) - L\hat{x}(k), \\ s(k) &= s(k-N) + T(z)[r(k) - y(k)]. \end{aligned} \quad (30)$$

For the control system consisting of (1) and (30), it can easily be shown that the sensitivity matrix at the plant input can be expressed as

$$\Sigma(z) = S(z)[I + L(zI - A + KC)^{-1}(B - KD)], \quad (31)$$

where $S(z)$ is the sensitivity matrix (13) for the state feedback case.

Note that the discrete-time model (1) is frequently non-minimum phase. Although the perfect recovery of the target feedback property is impossible, LTR techniques are still useful for non-minimum models. See, for example, the discussion in [5].

The following result, which can easily be proved by use of the expression (31) and the asymptotic property of the optimal filter gain matrix [5], is useful to evaluate the difference between the target and the achievable sensitivity properties for the non-minimum case.

[Proposition 3] Decompose the transfer function matrix (5) as

$$G(z) = G_m(z)G_a(z), \quad (32)$$

where $G_m(z)$ and $G_a(z)$ are minimum phase and all-pass, respectively. Let (A_m, B_m, C_m, D_m) denote a realization of $G_m(z)$. Consider the covariance matrix (2) given by

$$Q_u = BB', \quad Q_{uy} = BD, \quad Q_y = \sigma I + DD', \quad (33)$$

where $\sigma \geq 0$. Then, as σ in (33) tends to zero, the sensitivity matrix $\Sigma(z)$ approaches

$$\Sigma^*(z) = S(z)\{I + L(zI - A)^{-1}[B - B_m G_a(z)]\}, \quad (34)$$

where $S(z)$ is the sensitivity matrix (13) for the state feedback repetitive controller. $\square$

[Remark 1] Apparently, the above result implies the perfect recovery for minimum phase models for which $G_a(z) = I$ and $B_m = B$.

5. Conclusion

The results derived in this note are very useful to design a high order repetitive controllers by standard numerical methods with clear design perspectives. An extension of the results to multi-rate sampling case will be reported elsewhere.

References

- [1] S. Hara, Y. Yamamoto, T. Omata and M. Nakano: Repetitive control system: A new type servo system for periodic exogenous signals; *IEEE Trans. on Automatic Control*, AC-33, pp. 659-668 (1988)
- [2] M. Tomizuka, T. C. Tsao and K. K. Chew: Analysis and synthesis of discrete-time repetitive controllers; *Transaction of the ASME, Journal of Dynamic Systems, Measurement, and Control*,

- Vol. 111, pp. 353–358 (1989)
- [3] D. Williamson: *Digital Control and Implementation*, Prentice-Hall (1990)
 - [4] S. Hara and T. Mita: Stabilization of discrete-time servo systems; *Preprints of the 30th Japan Joint Automatic Conference*, pp. 155–156 (1987)
 - [5] T. Ishihara, N. Chiba and H. Inooka: Loop transfer recovery for discrete-time plants with direct feedthrough terms; *Trans. of the Society of Instrument and Control Engineers*, Vol. 32, No. 4, pp. 247–253 (1997)
 - [6] M. Tetsuka, S. Hara and R. Kondo: Ripple phenomena in digital repetitive control; *Trans. Society of Instrument and Control Engineers*, Vol. 27, No. 8, pp. 915–921 (1991)