雑音を含む AR 過程の ARMA モデリングに関する 新しい結果*

Md. Kamrul HASAN**・谷萩 隆嗣***

New Results on the ARMA Modeling of AR Signal Plus White Noise Process*

Md. Kamrul HASAN** and Takashi YAHAGI***

1. Introduction

Consider the following p th-order autoregressive (AR) process :

$$x(n) = -\sum_{i=1}^p a_i x(n-i) + u(n) \quad (1)$$

where the measurement of the signal $x(n)$ is assumed to be noisy :

$$y(n) = x(n) + v(n) \quad (2)$$

In the above equations $u(n)$ and $v(n)$ are stationary uncorrelated zero-mean Gaussian white noise sequences. The order p and parameters $\{a_i\}$ of the AR process are assumed to be unknown. The input sequence $u(n)$ is assumed to be unobservable. The problem is to estimate the order p and parameters $\{a_i\}$ using $y(n)$ only.

Pagano¹⁾ and Kay²⁾ have shown that an AR process corrupted by additive white Gaussian noise can be exactly represented by an autoregressive moving-average (ARMA) process. As described in Refs. 1) and 2), PSD (Power Spectral Density) of the observed process $y(n)$ is

$$\begin{aligned} P_y(z) &= \frac{\sigma_u^2}{A(z)A(z^{-1})} + \sigma_v^2 \\ &= \frac{\sigma_u^2 + \sigma_v^2 A(z)A(z^{-1})}{A(z)A(z^{-1})} \end{aligned} \quad (3)$$

where $A(z) = 1 + \sum_{i=1}^p a_i z^{-i}$ and σ_v^2 is the variance of the white noise sequence $v(n)$. If we write $\sigma_u^2 B(z)B(z^{-1}) = \sigma_u^2 + \sigma_v^2 A(z)A(z^{-1})$, then $y(n)$

may be modeled as

$$\sum_{i=0}^p a_i y(n-i) = \sum_{j=0}^p b_j w(n-j), \quad a_0 = b_0 = 1 \quad (4)$$

where $B(z) = 1 + \sum_{j=1}^p b_j z^{-j}$, $w(n)$ is white noise with autocorrelation function $R_w(k) = \sigma_w^2 \delta(k)$ and $\delta(k)$ is the Kronecker delta function. If the transfer function $H(z)$ of the ARMA process is assumed to be minimum phase, then it can be expressed as

$$\begin{aligned} H(z) &= \frac{B(z)}{A(z)} = \left[\frac{A(z)}{B(z)} \right]^{-1} \\ &= \frac{1}{1 + c_1 z^{-1} + c_2 z^{-2} + \dots + c_p z^{-p} + \dots} \end{aligned} \quad (5)$$

Therefore, it appears that as the AR model order increases, $y(n)$ can be more and more accurately characterized by an AR process.

If the order of the AR process is known, the parameters of the model of eq. (4) can be estimated by using the methods proposed so far. Pagano has proposed a *nonlinear regression method* to estimate the AR parameters. Li and Dickinson³⁾ have shown that robust estimation of the pure AR process parameters is possible through the use of LF (Lattice Filter). However, for the ARMA process the performance of the estimator deteriorates. The modified Yule-Walker (MYW) equations³⁾ can also be used for the same purpose. This approach is not quite satisfactory, especially for short data records. However, for unknown process order the aforementioned methods cannot be used directly. Here, we first, estimate the process order using extended Hannan and Quinn's criterion (HQC) method and second, propose a more effective method that permits the use of the Kalman filter

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** 千葉大学 大学院 自然科学研究科 Graduate School of Science and Technology, Chiba University; 1-33 Yayoicho, Inage ward, Chiba city, Chiba 263, JAPAN

*** 千葉大学 工学部 Department of Information and Computer Sciences, Chiba University; ditto

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algorithm for estimating the AR parameters through the estimated input sequence $w(n)$ of the ARMA process.

2. Order estimation of AR process

If $v(n) = 0$, the ITC (Information Theoretic Criterion) methods can estimate the order p based on the estimate $\hat{\sigma}_u^2$ of σ_u^2 , obtained by maximizing the Gaussian likelihood. The estimate is to be obtained by minimizing any of the quantities stated below :

$$\text{AIC}(k) = \log(\hat{\sigma}_u^2) + \frac{2k}{N} \quad (6)$$

$$\text{BIC}(k) = \log(\hat{\sigma}_u^2) + \frac{k \log(N)}{N} \quad (7)$$

$$\text{HQC}(k) = \log(\hat{\sigma}_u^2) + 2\alpha \frac{k \log(\log(N))}{N}, \quad \alpha > 1 \quad (8)$$

where N is the data length, AIC and BIC denote Akaike's information criterion and Bayesian information criterion, respectively. However, if $v(n) \neq 0$, the above mentioned criteria cannot be used directly as the computation of $\hat{\sigma}_u^2$ requires the knowledge of measurement noise variance σ_v^2 in advance as shown in eq. (9) :

$$\hat{\sigma}_u^2 = E\{|\hat{A}(z)y(n)|^2\} - [1, \hat{\mathbf{a}}^T] [1, \hat{\mathbf{a}}^T]^T \hat{\sigma}_v^2 \quad (9)$$

where $\hat{\mathbf{a}} = [\hat{a}_1, \hat{a}_2, \dots, \hat{a}_k]^T$ is the estimate of the unknown AR process parameters and $\hat{\sigma}_v^2$ is the estimate of the measurement noise variance σ_v^2 . For noisy measurement, the extended Hannan and Quinn's criterion can be expressed as

$$\begin{aligned} \text{HQC}^c(k) \\ = \log\{E\{|\hat{A}(z)y(n)|^2\} - [1, \hat{\mathbf{a}}^T] [1, \hat{\mathbf{a}}^T]^T \hat{\sigma}_v^2\} \\ + 2\alpha \frac{k \log(\log(N))}{N}, \quad \alpha > 1. \end{aligned} \quad (10)$$

The superscript c on HQC stands for noise compensation. It is therefore necessary to estimate σ_v^2 using $y(n)$ only.

By using the concept of power spectrum, the measurement noise variance σ_v^2 can be estimated without a *priori* knowledge of the process order. This is shown as follows. The power spectrum $P_y(\omega)$ of output measurement can be easily calculated from $y(n)$, and can be expressed as the sum of the output signal and noise power spectra.

$$P_y(\omega) = P_x(\omega) + P_v(\omega) \quad (11)$$

We assume that $f_{\max}$ is the highest frequency with significant power present in the system output measurement. Its value can be estimated from the system characteristics and/or purposes. By knowing $f_{\max}$ it is possible to decide the frequency

f_s for sampling the system output, where

$$f_s = df_{\max}, \quad d > 2. \quad (12)$$

As $f_{\max} < f_s/2$, $P_x(\omega)$ is negligible in the range $\omega_{\max} < \omega \leq \omega_s/2$, ($\omega = 2\pi f$), and then, information about $v(n)$ can be extracted from $P_y(\omega)$ in this range :

$$P_v(\omega) \cong P_y(\omega), \quad \omega_{\max} < \omega \leq \frac{\omega_s}{2} \quad (13)$$

If the noise is a stationary white process, the relation

$$P_y(\omega) = \sigma_v^2, \quad \omega_{\max} < \omega \leq \frac{\omega_s}{2} \quad (14)$$

holds and the noise variance can be obtained.

In practice, however, as the data length is finite, $P_y(\omega)$ is not constant. Then, an estimate $\hat{\sigma}_v^2$ of σ_v^2 can be evaluated from

$$\hat{\sigma}_v^2 = \bar{P}_y \quad (15)$$

where $\bar{P}_y$ is the average of $P_y(\omega)$ taken in the interval $\omega_{\max} < \omega \leq \omega_s/2$. The method proposed here can be used if and only if the data are sampled with a sampling frequency f_s defined by eq. (12).

Since the proposed variance estimation method is independent of the process order, effective estimation of σ_v^2 is possible by this method.

In the order estimation process the parameters $\{a_i\}$ ($i = 1, 2, \dots, k$) are calculated using the low-order Yule-Walker equations as follows :

$$[\mathbf{R} - \hat{\sigma}_v^2 \mathbf{I}] \hat{\mathbf{a}} = -\mathbf{r} \quad (16)$$

where $\mathbf{R}$ is the $k \times k$ covariance matrix of $y(n)$ defined by $R(m) = E\{y(n)y(n-m)\}$ ($m = 0, 1, \dots, k$) and $\mathbf{r} = [R(1), R(2), \dots, R(k)]^T$. The order of the process can be estimated by using eqs. (10), (15) and (16).

3. New technique for estimating AR parameters

Our method of solving the problem is based on fitting the time series $y(n)$ by an auxiliary AR(l) model with the assumption of white noise input excitation. Then the order l of the auxiliary AR model can easily be estimated by using the ITC theoretic criteria based methods. Here, we use the HQC method⁴⁾ as follows :

$$\begin{aligned} \text{HQC}(k) = \log(\hat{\sigma}_w^2) + 2\alpha \frac{k \log(\log(N))}{N}, \\ \alpha > 1 \end{aligned} \quad (17)$$

where $\hat{\sigma}_w^2$ is the estimate of σ_w^2 . The estimate of l is to be obtained by minimizing the HQC.

The order thus estimated assures that the output of the inverse filter is white noise. Then the robust estimation of the auxiliary AR model parameters

is possible through the use of the lattice filter algorithm proposed by Li and Dickinson. The reason for doing this is that by introducing the auxiliary AR model one can estimate the input sequence. Then, we obtain better estimates of the ARMA (p, p) parameters from the estimated input $w(n)$ and the measurement sequence $y(n)$ with the Kalman filter algorithm.

4. Simulation results

The performance of the proposed method was assessed for the following AR process :

$$\begin{aligned} x(n) = & 2.595x(n-1) - 3.339x(n-2) \\ & + 2.2x(n-3) - 0.731x(n-4) + u(n) \end{aligned} \quad (18)$$

$$y(n) = x(n) + v(n) \quad (19)$$

The distribution of estimated order for 100 different trials are shown in Table 1, where $\text{SNR} = 10 \log(\sigma_x^2/\sigma_v^2)$ [dB]. Without noise compensation, AIC, BIC and HQC all overestimate the process order as expected. In contrast, estimated orders by HQC^c is quite satisfactory.

The poles of the AR process calculated from the estimated AR parameters are shown in Fig. 1. Clearly, the new technique shows better performance over the direct use of the MYW method.

Table 1 Results on AR order estimation
SNR = 20 dB, $N = 4096$, $\alpha = 2$

order	AIC	BIC	HQC	HQC ^c
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	81
5	0	0	0	7
6	0	0	0	2
7	0	2	2	0
8	0	0	0	0
9	25	98	98	0
10	10	0	0	0
above 10	65	0	0	10

• : true values, • : estimates by the MYW,
• : estimates by the proposed method,
SNR = 20 dB, $N = 4096$

Fig. 1 Estimated AR parameters by the MYW and proposed methods

5. Conclusion

In this letter, we have proposed an extension to the HQC method for the order estimation of AR process from noisy measurement. A new technique for estimating the parameters of AR processes using only the noisy output measurement has also been proposed. The results show that for moderate SNR and sufficiently large data samples a significant improvement can be achieved through the use of the new algorithm.

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