1. Introduction

In recent years hybrid (discrete-continuous) dynamical systems—HDS have received increased attention by many researchers with different backgrounds [1–13]. Various modeling and control approaches to control of hybrid systems have been proposed. In our work we have mainly concentrated on mechatronic multicontact systems—MMS such as multi-fingered robotic hands or multi-legged walking machines, a particularly challenging subclass of HDS [14–16]. Differing from conventional research in the area of multicontact mechatronic systems a hybrid control approach is well suited to systematize and formalize modeling, trajectory planning, control, simulation, and implementation issues.

In this article we summarize some of our hybrid control ideas and recent results without presenting all the technical details; these can be found in separate publications listed in the references, e.g. Ref. [14]. In 2. the hybrid state model as a general model for HDS is introduced and hybrid modeling of MMS is illustrated by means of a simple mechatronic example. Hybrid control architectures and hybrid optimal control are briefly discussed in 3. followed by two application examples in 4.

2. Hybrid (Discrete-Continuous) Systems

2.1 Hybrid Dynamical Systems

A conventional continuous dynamical system is described by the velocity vector field \( f(x,u,t) \) depending on the continuous state \( x \), the continuous control input \( u \), and time; the continuous output \( y \) is generated by the output function \( h(x,u,t) \), see Fig. 1(a). In this context, lumped parameter continuous time systems defined by a set of ordinary differential (algebraic) equations are considered. Extensions of these systems, including discontinuous nonlinearities and switching actions lead already in the direction of the class of HDS.

A HDS comprises, in addition to continuous dynamical system aspects, a discrete (symbolic) state \( q \in \mathbb{N}^i \), a discrete (symbolic) control input \( v \in \mathbb{N}^i \), a discrete (symbolic) control output \( y_q \), discrete event generating functions \( s_i \), and discrete dynamics \( \phi_i \),

(1) $M \ddot{x} + Kx = 0$

(2) $m_1 \ddot{x}_1 = u_1$

where \( u_1 \) is the continuous control input force acting on \( m_1 \).

If mass \( m_1 \) moves to the right, the two masses will eventually collide. Considering a not negligible relative velocity before collision, the velocities of the masses after collision at time \( t_s \) follow as

(3) \[
\dot{x}_1^+(t_s) = \frac{m_1 - \rho M}{m_1 + M} \dot{x}_1(t_s) + \frac{(1 + \rho)M}{m_1 + M} \ddot{x}(t_s)
\]

(4) \[
\dot{x}^+(t_s) = \frac{(1 + \rho)M}{m_1 + M} \ddot{x}(t_s) + \frac{M - \rho m_1}{m_1 + M} \ddot{x}(t_s),
\]

where \( 0 < \rho < 1 \) is the coefficient of restitution.

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denotes dynamic behavior, making the two masses move together after collision, Fig. 2(b), with the following combined equation of motion

\[ M \dddot{x} + K x = N \]
\[ m_1 \dddot{x}_1 = u_1 - N \]
\[ x_1 = x \]
\[ \dddot{x}_1 = \dddot{x} \]
\[ \dddot{x}_1 = \dddot{x} \quad \text{if } N > 0, \]

where the position, velocity and positive accelerations of both masses are the same as long as the internal force \( N > 0 \). If \( N < 0 \) the masses break contact and move separately again according to eqs. (1), (2).

Now, we assign the discrete state \( q = 0 \) to the two situations without/with contact, cf. Fig. 2(a)/(b). There exist 3 discontinuity surfaces \( s_1, s_2, s_3 \), for the events of making contact, (in)elastic collision and breaking contact. It is straightforward to derive the complete hybrid model for this mechatronic example system; details are omitted here because of length considerations.

### 2.3 Hybrid State Model

**Definition 1** (HSM) A hybrid dynamical system (HDS) is defined by its hybrid state model (HSM) as follows:

\[ \dot{x} = f(x, u, q, t) \quad \text{if } s_i(x, u, q, v, t) \neq 0 \]
\[ x^+(t) = \phi_i(x, u, q, v, t) \quad \text{if } s_i(x, u, q, v, t) = 0 \]
\[ y = h(x, u, q, v, t), \]

where eqs. (7), (8) describe the continuous, discrete dynamic behavior, respectively; the notation \( x^+(t) \) denotes the successor state (limit from the right) of \( x \) at time \( t \). The hybrid output \( y \) is generated by eq. (9). The continuous state vector \( x(t) \in \mathcal{X} \subseteq \mathbb{R}^n \) and the discrete state vector \( q(t) \in \mathcal{Q} \subseteq \mathbb{N}^d \) together form the hybrid state vector

\[ \zeta(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} \in \mathcal{X} \times \mathcal{Q} \subseteq \mathbb{R}^n \times \mathbb{N}^d. \]

The continuous control input \( u(t) \in \mathcal{U} \subseteq \mathbb{R}^m \) belongs to the set \( \mathcal{U} \) of permissible controls. The discrete (symbolic) control input vector is \( v(t) \in \mathcal{V} \subseteq \mathbb{N}^k \). The hybrid output vector

\[ y(t) = \begin{bmatrix} y_x \\ y_q \end{bmatrix} \in \mathcal{Y} \subseteq \mathbb{R}^p \times \mathbb{N}^r \]

combines a \( p \)-dimensional continuous output \( y_x \) and a \( r \)-dimensional discrete (symbolic) output \( y_q \); \( y \) is generated by the hybrid output function

\[ h: \mathcal{X} \times \mathcal{U} \times \mathcal{Q} \times \mathcal{V} \times \mathcal{R} \rightarrow \mathbb{R}^p \times \mathbb{N}^r. \]

The continuous behavior of the HDS is given by the vector field

\[ f: \mathcal{X} \times \mathcal{U} \times \mathcal{Q} \times \mathcal{V} \times \mathcal{R} \rightarrow \mathbb{R}^n. \]

Discontinuous behavior of the HDS is caused by events occurring when the hybrid state intersects discontinuity surfaces

\[ s_i: \mathcal{X} \times \mathcal{U} \times \mathcal{Q} \times \mathcal{V} \times \mathcal{R} \rightarrow \mathbb{R}, \]

for \( i = 1, \ldots, N \). Note, that the discontinuity surfaces may depend on the continuous and/or the discrete control input \( u(t), v(t) \). The hybrid successor state

\[ \zeta^+(t_1) = \begin{bmatrix} x^+(t_1) \\ q^+(t_1) \end{bmatrix}, \]

after discrete events is given by the transition (jump) maps

\[ \phi_i: \mathcal{X} \times \mathcal{U} \times \mathcal{Q} \times \mathcal{V} \times \mathcal{R} \rightarrow \mathcal{X} \times \mathcal{Q}, \]

see also eq. (8). As long as all discontinuity surface functions \( s_i(x, u, q, v, t) \neq 0 \), for \( i = 1, \ldots, N \), the system trajectory evolves continuously according to eq. (7).

The HSM is closely related to the so-called unified model for hybrid systems [17,18]. Main difference between both models is that the HSM uses discontinuity surfaces defined by switching (indicator) functions \( s_i \) instead of jump and jump destination sets in the unified model; advantage of the HSM is the correspondence to conventional variable structure control systems. The unified model as well as the HSM cover a wide class of hybrid dynamical systems; by subsumption also models such as [19–22]. Other hybrid models are discussed e.g. in Refs. [1–13].

### 2.4 Characteristic Dynamic Behavior

The dynamic behavior of HDS can be characterized by discontinuities in the system trajectories. Discontinuities include state resets—SR resulting in state jumps, vector field switches—VFS with discontinu-
uous velocities, and the combination SRVFS thereof. These may be caused by a time event—TE occurring at a certain time or by a state event—SE if the system state reaches a certain value, see Fig. 3. Further events include control events—CE caused by a hybrid control through the discrete input or disturbance events caused by disturbances. Other dynamic effects of HDS include chaotic behavior, see e.g. Refs. [23, 24], or sliding mode, see e.g. Refs. [25, 26]; A typical trajectory of a HDS is shown in Fig. 4; further discussion of dynamic characteristics can be found in Ref. [14].

![Fig. 3 Characteristic discontinuous dynamic behavior of HDS caused by discrete events at a certain time or state; state reset—SR (a), (d); vector field switch—VFS (b), (e); combined state reset and vector field switch (c), (f); cause by time event—TE (a), (b), (c) or state event—SE (d), (e), (f)](image)

3. Control of Hybrid Systems

In our research on hybrid dynamical systems we have been proposing a general hybrid control architecture consisting of three main parts, see Fig. 5: i) the hybrid process model, cf. 2.; ii) the hybrid controller (HC) controlling this process, discussed in the following; iii) the hybrid reference trajectory generator (HRG). The synthesis of reference trajectories as solutions to hybrid optimal control problems is discussed in 3.2. All subsystems again may be hybrid dynamical systems themselves involving continuous as well as discrete dynamics.

![Fig. 5 General hybrid control architecture](image)

3.1 Hybrid Control and Error Compensation

Taking the HSM of Definition 1 as the basis for modeling of HDS and keeping in mind the control architecture described above it is straightforward to generalize classical control concepts, e.g. output following control, to the hybrid case. The resulting hybrid output control (HOC) block diagram with hybrid control signals is depicted in Fig. 6.

![Fig. 6 Hybrid output control](image)
case of a discrete error, however, this may not be easy because the hybrid process may be in a contact state with motion equations other than the continuous part of the hybrid controller currently assumes. One solution to hybrid error compensation is shown in Fig. 7, where a discrete error activates a continuous prefILTER to modify the continuous reference \( y^c_x \to y^d_x \) in such a way that both the discrete as well as the continuous control error eventually vanish. Similar concepts are a discrete prefetcher, more complicated discrete dynamics in the compensation controller, or a combined reference generator adaptation scheme, see Ref. [14] for details.

![Fig. 7 Hybrid error compensation by means of a continuous prefILTER](image)

### 3.2 Hybrid Optimal Control

The hybrid optimal control problem is to find optimal hybrid — i.e., continuous \( u \) and discrete \( v \) — control trajectories such that an integral cost index — typically an integral of a function of the hybrid system state and control input — is minimized subject to the hybrid system dynamics, initial, terminal, and other equality or inequality constraints. A general class of hybrid optimal control problems is considered in the following.

**[Definition 2]** The hybrid optimal control problem is defined as the minimization of the hybrid cost index \( J \)

\[
\min_{u,v} J(u,v) = \Theta + \int_{t_a}^{t_f} \psi(x,u,q,v,t) \, dt,
\]

subject to eqs. (7), (8) and

\[
u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}, \quad v(t) \in \mathcal{V} \subset \mathbb{Z}^{n_v},
\]

\[
x(t) \in \mathcal{X} \subset \mathcal{R}^{n_x}, \quad q(t) \in \mathcal{Q} \subset \mathbb{Z}^{n_q}, \quad \forall t \in [t_a,t_b]
\]

\[
0 \leq h(x,u,q,v,t), \quad t \in [t_a,t_b]
\]

\[
x(t_a) = x_a, \quad q(t_a) = q_a \quad \text{initial conditions},
\]

\[
x(t_f) = x_e, \quad q(t_f) = q_e \quad \text{terminal conditions},
\]

where the initial and final times \( t_a, t_f \) are free or fixed. The Mayer type part \( \Theta \) of the performance index is a general function of the phase transition times (events) \( t_i, i = 0, \ldots, n, \) of the continuous \( x(t_i), x^+(t_i) \) and discrete states \( q(t_i), q^+(t_i) \) just before and just after the transition event written as

\[
\Theta := \Theta[ x(t_0), x^+(t_0), \ldots, x(t_n), x^+(t_n); \ q(t_0), q^+(t_0), \ldots, q(t_n), q^+(t_n); t_0, \ldots, t_n ]
\]

Here, \( t_a = t_0, t_f = t_n \) and the number of phases \( n \) may be given or free. The integrand \( \psi \) is a real-valued function of the continuous/discrete state and control variables and of time.

The minimization of eq. (15) is subject to the initial and terminal conditions (18), (19), admissible values for the continuous/discrete control variables (16), and inequality constraints (17). Obviously, valid hybrid optimal trajectories have to obey the differential equations (7) and the phase transition equations (8) of the discrete aspect. The optimization parameters to be determined are the continuous \( u(t) \) and discrete control input trajectories \( v(t) \) and all, some, or none of the phase transition times.

Fig. 8 shows typical trajectories of a solution to a hybrid optimal control problem. The cost index is shown in Fig. 8(a), where a discontinuous jump in the cost occurs at the transition time \( t_2 \). Likewise, the continuous state trajectory \( x(t) \) shown in Fig. 8(b) may have discontinuities in the state at time \( t_2 \) and in its rate at time \( t_1 \). The discrete state trajectory \( q(t) \) is shown in Fig. 8(c).

![Fig. 8 Typical trajectories of a hybrid system](image)
very many) possible discrete state sequences and to select the best one.

The key challenge when solving hybrid optimal control problems is to reduce the number of discrete state sequence iterations and therewith the number of multi-phase solutions needed. There are various ways to perform the search among all possibilities, e.g., branch-and-bound, heuristics, or other simplifying assumptions, all in search for practical and possibly suboptimal solutions to the hybrid problem, see Ref. [27] for details.

4. Application Examples

4.1 Cooperative Multi-Arm Transport Task

Fig. 9 shows a cooperative multi-arm transport task. The sphere object is initially on the right and is to be transported to the elevated goal position on the left. There is the object to be picked up by transport arm 1, handed over to arm 2, then to arm 3, to be finally placed in the goal position. Each transport arm \( j \) has two rotational joints \( \theta_{j,i} \) driven by control input torques \( u_{j,i} \), \( j = 1,2,3 \), \( i = 1,2 \). The effector of each transport arm can be opened/closed to grasp/release the object by a discrete control input \( v_j \). The transportation task should be performed such that the cost index of quadratic power consumption is minimized, i.e.,

\[
\min_{u_{j,i}(t),v_j(t)} J = \int_0^T \sum_{j=1}^3 \sum_{i=1}^2 (u_{j,i}(t) \dot{\theta}_{j,i}(t))^2 \, dt.
\]

To solve this hybrid optimal control problem we need to determine the optimal hybrid control trajectories \( u_{j,i}^*(t) \), \( v_j^*(t) \), the positions, velocities and times of object handover. An analytic solution for this cooperative transport task is not available to our knowledge.

Fig. 9 Cooperative multi-arm transport task

From an engineering point of view a suboptimal solution procedure — related to dynamic programming and based on the efficient numerical solution method DIRCOL [28,29] for multi-phase optimal control problems — is outlined in the following, see Ref. [30] for details.

In the first step, the coupling of the optimal control problems for each of the transport arms is eliminated by fixing the possible times and states of handover to constant values, possibly on a grid for a handover area. In Fig. 9 the object handover time from arm 1 to 2 is fixed to \( t_1 = 2 \) and only two possible handover positions — on the ground and in the air — are considered. Next, all separated optimal control problems now with fixed boundary conditions are solved numerically. Then a weighted graph with nodes representing the grid nodes of the fixed handover positions, times, and velocities is formed. The vertices of this graph are weighted by the optimal cost of the separately solved optimal control problem solutions. Finally, a minimum path graph search through this weighted graph yields the best suboptimal solution.

The best solution to the transport task is to pick up the object by arm 1, hand it over to arm 2 in the air, again hand it over to arm 3 in the air, at the fixed positions and times as shown in Fig. 9. The suboptimal solution to the hybrid optimal control problem is then implemented in the hybrid reference generator, cf. Fig. 6.

4.2 Multi-Fingered Robotic Hand

A grasp of an object by \( n \) fingers with not more than 6 DOF each and assuming point contact, see Fig. 10(a), can be described in Cartesian workspace coordinates with respect to an inertial reference frame, by defining the continuous state \( x = [x^{tip},p,\dot{x}^{tip},\dot{p}] \), where \( x^{tip} \in \mathbb{R}^{3n} \) contains the fingertip positions of the grasping hand and \( p \in \mathbb{R}^6 \) are generalized coordinates of the object pose. The joint torques of the finger actuators form the continuous input into the system,
i.e. \( u(t) = \tau \); there is no discrete input \( v \). The continuous dynamics of the overall system \( \ddot{x} = f(x,q,\tau) \) is given by Newton–Euler equations depending on the actual contact situation reflected in the discrete grasp state \( q \), \( q_i = \{1,2,3\}, i = 1,\ldots,n \). For each finger \( i \) we define, see Fig. 10(b),

\[
q_i = \begin{cases} 
1, & \text{if finger } i \text{ contacts stably} \\
2, & \text{if finger } i \text{ does not contact} \\
3, & \text{if finger } i \text{ slides on the object}
\end{cases} \quad (20)
\]

The hybrid finger contact model can be represented graphically as a hybrid automaton, see Fig. 11. The transition between the different contact states of finger \( i \) are triggered by the following discontinuity surfaces

\[
s_1 = 0, \text{ if } \dot{x}_{\text{tip}} \in \partial \mathcal{O} \land ||\dot{x}_{\text{tip}} - \dot{v}_{\text{obj}}||_\perp \geq \epsilon \\
s_2 = 0, \text{ if } \dot{x}_{\text{tip}} \in \partial \mathcal{O} \land ||\dot{x}_{\text{tip}} - \dot{v}_{\text{obj}}||_\perp < \epsilon \\
s_3 = 0, \text{ if } c_{i,1} < 0 \\
s_4 = 0, \text{ if } ||\dot{x}_{\text{tip}} - \dot{v}_{\text{obj}}||_2 = 0 \\
s_5 = 0, \text{ if } \mu c_{i,1} < \sqrt{c_{i,2}^2 + c_{i,3}^2} \\
s_1,\ldots,s_5 = 1 \text{ otherwise}
\]

![Diagram of contact states](image)

![Diagram of regrasping HRG](image)

Fig. 11 Hybrid contact states of finger \( i \) (a) and hybrid reference generator (HRG) for a regrasping task (b)

where \( c_i \) denotes the contact wrench of finger \( i \) (cf. Fig. 10), \( \mu \) the static friction coefficient, \( \partial \mathcal{O} \) the surface of the grasped object and \( \dot{v}_{\text{obj}} \) the velocity of a reference point on the object; \( ||\cdot||_\perp \) denotes a norm on the normal component of the relative velocity between fingertip and object with respect to \( \partial \mathcal{O} \), \( ||\cdot||_2 \) the tangential component, respectively, with a small \( \epsilon > 0 \). The contact wrenches are calculated from the continuous system state, the joint torques and external forces acting on the grasped object. If \( s_i = 0 \) the jump transition maps \( \phi_i \) determine the new contact state

\[
[x_{\text{tip}}^i, p_i, (\dot{x}_{\text{tip}})^i, \ldots, p^+_i, q_1, q_2^+, \ldots, q_n^+] = \phi_i ,
\]

\( i = 1,\ldots,5 \). This velocity reset of the fingers and object can be calculated by simplified models from Poisson’s restitution theory or more rigorous impulse based methods [31].

The continuous output consists of fingertip positions and their velocities \( y_a = [x_{\text{tip}}, \dot{x}_{\text{tip}}] \) calculated from measured joint angles through forward kinematics of the hand. The discrete output \( y_c \) contains the contact state information \( q \) of the fingers measured by contact sensors.

It is clear that depending on the current discrete contact state \( q \) a specific control mode for the continuous aspect of the HDS hand–object is required. In the control architecture we propose a variable structure impedance control law combined with an on-line capable grasping force optimization (GFO) algorithm, see Fig. 12 and Refs. [32–34] for details.

One of the typical primitives during dextrous manipulation tasks is regrasping. Assuming a currently stable grasp with all fingers in contact the regrasping of one finger can be accomplished in 4 phases, see Fig. 11(b): 1) REDUCE grasp force; 2) MOVE the finger to the new contact point; 3) INCREASE grasp force; 4) reenter a STABLE grasp configuration. Here, the assumption is that during regrasping the other fingers remain in STABLE contact and the overall grasp configuration remains stable.

Regrasping is initiated by the external signal command \texttt{regrasp} to the hybrid automaton. In the REDUCE-state the grasp force of the regrasping finger is reduced from its initial value \( c^* \) (grasp force during the STABLE-state) and rate \( \gamma > 0 \), where \( b \) is a measure for the force contribution of the finger to the overall grasp. After being unloaded the finger lifts off the object and its tip moves along the specified trajectory \( \dot{x}_{\text{tip}} = f(x_{\text{tip}}, t) \) during the MOVE-state. The signal \texttt{sensor} denotes a contact sensor signal. Finally, the finger is reintegrated into the grasp during the INCREASE-state which is followed again by the STABLE-state.

The proposed hybrid control approach to dextrous manipulation has been validated in experiments with the four–fingered TUM–hand [35]. Here, results of an experiment with a cubic object are presented, see
Fig. 12 Hybrid control architecture

Fig. 13 (a) Sample object: cubic. (b) Actual (solid) and desired position (dashed) of regrasping finger during phases I to VII.

Fig. 13 Regrasping experiment of a cubic object

Fig. 13(a). In phase (I), the fingers 1–4 are moved to the surface at the initial contact positions (-22.5,0,90) [mm], (0,-22.5,90) [mm], (22.5,-5,90) [mm] and (0,22.5,90) [mm] with respect to the palm frame of the hand. Phase (II) represents the closure and holding of the grasp by a 4-fingered impedance controller. In phases (III) and (IV), finger 3 is removed to position (22.5,0,80) [mm] which is above the skew part of the side, while the object is held by 3 impedance controlled fingers. In phase (V), an active contact search is performed. Phase (VI) represents holding the object with a 4-fingered grasp. In order to test for grasp stability of the contact positions for the fingertips 1–3, in phase (VII), finger 4 is removed and the object is held by 3 fingers again. Similar manipulation experiments could not successfully be completed without the here proposed advanced hybrid control architecture.

5. Conclusions

In this article we have given an overview of modeling and control of multicontact mechatronic systems from a hybrid dynamical systems point of view. The hybrid state model and hybrid control concepts have been presented. Two application examples of hybrid optimal control for a multi-arm transport task and hybrid control of multi-fingered hands were discussed.

Main contribution of hybrid control strategies in the area of mechatronic multicontact systems is that modeling, planning, control, simulation, and implementation can be formalized and systematized in a hybrid control framework such as proposed in this article. Rather than concentrating on too many technical details, the article introduces methods from the area of HDS applicable to control of mechatronic systems and therewith tries to stimulate new research directions of mechatronic systems with variable contact situations. It has been illustrated in application examples that novel methods for reference trajectory planning using hybrid optimal control and hybrid compensation controllers provide solutions to complex mechatronic control problems, not solvable with conventional modelling and control approaches thus far.
Hybrid control theory is still in an early stage, but in addition to the issues presented here there are many methods related to numerical simulation, verification, safety analysis, reachability computation, etc. being developed currently, which will open new possibilities for advanced control of multicontact mechatronic and other hybrid dynamical systems.

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