ABSTRACT

The present paper concerns the unsteady laminar incompressible flow between two parallel disks with a fluid source at the center of the disk(s). Both the flow rate of the source and the gap width between the disks are varied arbitrarily with time and independently of each other. Such a flow often appears in a nozzle-flapper valve or in a statical pressure bearing such as the piston shoes of a swash type axial piston pump. Series solutions to the Navier-Stokes equations are obtained, on the basis of an asymptotic expansion in the radial direction and a new theory of "multi-fold series expansion" for the time variable. The solutions describe precisely the non-linear interaction between the two co-existing flows. Experiments were carried out for the case where both the gap width and source flow were varied sinusoidally. The solutions agree well with the experimental results over a wide range of the flow conditions.

KEY WORDS

Unsteady Flow, Parallel Disks, Time-Varying Gap Width, Time-Varying Source Flow, Multi-fold Expansion

NOMENCLATURE

- \( a \): nondimensional amplitude of the gap width
- \( b \): nondimensional amplitude of the flow rate of the source
- \( h \): gap width
- \( h_0 \): reference gap width
- \( p \): pressure
- \( p_e \): pressure outside of the disks
- \( q \): flow rate of the source
- \( q_m \): reference flow rate of the source
- \( \text{Re}_m \): Reynolds number related to the source flow [Eq. (24)]
- \( \text{Re} \): Reynolds number related to the angular frequency [Eq. (23)]
- \( r \): radial coordinate (see Fig. 1)
- \( r_o \): outer radius of the disk
- \( r_0 \): inner radius of the disk
- \( u \): velocity component in the \( r \)-direction
- \( v \): velocity component in the \( \theta \)-direction
- \( y \): coordinate in the direction vertical to the disk (see Fig. 1)
- \( \alpha \): first characteristic function related to the variation of the gap width
- \( \gamma \): second characteristic function related to the variation of the source flow
- \( \delta \): phase lag of the source flow to the gap width
- \( \xi \): independent variable defined by Eq. (10)
- \( \eta \): independent variable defined by Eq. (3)
- \( \kappa \): = \( r_0/r_o \)
- \( \nu \): kinematic viscosity of the fluid
- \( \rho \): mass density of the fluid

INTRODUCTION

The present study concerns the unsteady laminar incompressible flow between two parallel disks with a fluid source at the center of the disk(s). Both the flow rate of the source and the gap width of the disks are varied arbitrarily with time and independently of each other. Such a flow can often be seen in a statical bearing such as the piston shoe of an axial piston pump with swash plate, in a nozzle flapper valve and so on.

The flow is classified into the following three types, depending on the internal impedance \( Z_s \) of the flow source:

1) The time-variation of the flow rate of the source is pre-determined (\( Z_s \) is infinite).
2) The time variation of the pressure of the source is pre-determined (\( Z_s \) is zero).
3) Both the flow rate and the pressure of the source are determined as a result of analysis (\( Z_s \) is neither infinite nor zero).

Although the flow in a statical bearing or that in a nozzle flapper valve has a...
finite internal impedance of the source in practice, the authors will begin their study with type 1, including a constant source flow as the first step because this type seems easier to solve than the other cases.

Some studies have been reported on either the flow with time-varying gap width only (1) - (3) or that with a central fluid source only (6) - (12). The flow we deal with, however, includes two types of flow, one due to the gap-width variation and the other due to source-flow variation, and both flow interact non-linearly with each other. Accordingly, it is easy to imagine that, if each type of flow were solved separately and superposed simply with each other, we could not obtain a good result because of the non-linear interaction.

In the present research, a precise theoretical analysis is developed based on the Navier-Stokes equations under the assumptions which will be described in the following section, by applying partly the theory of "multifold series expansion"(13)-(17). Experimental investigation was also made to verify the validity and applicability of the theory.

TheorY

Governing Equations

To begin with, we make the following assumptions:

1. The outer radius \( r_e \) of the disks is sufficiently large compared with the inner radius \( r_0 \) of the central fluid source pipe \( (r_e \gg r_0) \).
2. The gap width \( h \) is sufficiently small compared with the disk sizes \( r_e \) and \( r_0 \) \((h \ll r_0)\).
3. The Reynolds numbers \( R_q \) and \( R_w \) of the two coexisting flows are not too large, though they are large enough so that the non-linear convective inertia effect cannot be neglected.
4. The flow rate of the source is either kept constant or varied arbitrarily with time.

A system of stationary cylindrical coordinates \((r, \theta, y)\) is introduced as shown in Fig. 1. The Navier-Stokes equations and the equation of continuity for an axisymmetric and radial flow of an incompressible viscous fluid without body forces are,

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial y} &= \frac{1}{\rho} \frac{\partial p}{\partial r} \\
&+ \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial y} &= \frac{1}{\rho} \frac{\partial p}{\partial \theta} \\
&+ \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{u}{r^2} \right) \frac{\partial^2 v}{\partial y^2} \\
\frac{\partial (r u)}{\partial r} + \frac{\partial (r v)}{\partial \theta} &= 0 \\
\frac{\partial (r u)}{\partial y} &= 0 \\
u &= 0, \quad \frac{\partial h}{\partial t} = h(t) \\
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\text{at } y = 0, \quad v &= 0, \quad \frac{\partial h}{\partial t} = h(t) \\
\end{align*}
\]

Asymptotic Expansion

First, an original form of asymptotic series expansion is derived with respect to \( r \), in the region where \( r \) is sufficiently large. From many previous researches it is already known that the single flow \( u_1 \) caused by the gap-width variation without flow source can be expressed as \( u_1 = \frac{r f(y, t)}{r^2} \), while the single flow \( u_2 \) due to the constant source flow without gap-width variation as the following form:

\[
u_2 = \frac{a_1(y)}{r} + \frac{a_3(y)}{r^3} + \frac{a_5(y)}{r^5} + \ldots
\]

In the present problem, these two flows coexist. We may, therefore, naturally have an idea of combining the above two expressions. The former expression, \( u_1 = \frac{r f(y, t)}{r^2} \), is proportional to \( r \), and the latter asymptotic series expansion has a leading term of order \( r^{-2} \) and a common term ratio proportional to \( r^{-2} \). Formally, the former expression just corresponds, accidentally and fortunately, to the next lower-order extension of the latter asymptotic series expansion. Thus, these two expressions can be combined without any conflict with the Navier-Stokes equations. Since the flow is unsteady in this problem, the coefficients \( a_1, a_3, a_5, \ldots \) naturally become functions of \( y \) and \( t \).

A new variable is introduced in place of \( y \) defined by

\[
\eta \equiv \frac{y}{h(t)}
\]

Arranging under dimensional-analytic consideration, we may formulate the following asymptotic series expansions:

\[
\begin{align*}
u &= \frac{1}{2} \frac{r f(y, t)}{h^2} \frac{\partial f(y, t)}{\partial \eta} + \frac{q_1}{r} \frac{\partial^2 f(y, t)}{\partial \eta^2} + \frac{q_2}{r^3} \frac{\partial^3 f(y, t)}{\partial \eta^3} + \ldots \\
\end{align*}
\]

\[
\begin{align*}
u &= \frac{r}{h} \frac{\partial f(y, t)}{\partial \eta} + 0 + \frac{2q_1 h}{r^3} G(y, t) + \ldots \\
\end{align*}
\]

\[
\begin{align*}
u &= \frac{K(t)}{h^2} + \frac{1}{h^3} P_0(y, t) + \frac{1}{h^4} P_1(y, t) + \ldots \\
&+ \frac{r_0}{h^2} \log \left( \frac{r}{r_0} \right) P_2(y, t) + \frac{1}{h^3} P_3(y, t) + \ldots
\end{align*}
\]

where \( K(t) \) is a constant of integration with respect to \( r \) and \( y \) to be determined later, and

\[
\begin{align*}
q = 2\pi a_1 = 2\pi a_3 \left( \frac{1}{2} \frac{r f(y, t)}{h^2} \frac{\partial f(y, t)}{\partial \eta} \right)
\end{align*}
\]

The first term on the right-hand side

Fig. 1 Flow geometry and coordinate system
of Eq. (4a) corresponds to the flow caused by the gap-width variation, while the second and the subsequent terms are due to the source flow. Equation (4d) is obtained by integrating the right-hand side of Eq. (4a) without the first term over the flow cross sectional area. Equations (4a) and (4b) satisfy the equation of continuity, Eq. (1c).

Now, substituting Eqs. (4a), (4b) and (4c) into Eqs. (1a) and (1b), and setting each sum of coefficients with equal power of \( r \) to zero, we obtain the following relations:

\[ \phi = -(\phi + \frac{h}{\nu}) \phi + \frac{1}{2} \rho \phi + 2 \phi \]

\[ F = \phi + \frac{h}{\nu} F \]

\[ G = \phi + \frac{h}{\nu} G \]

where

\[ \alpha(t) = \frac{h}{\nu} \frac{d h}{d t} \]

\[ \gamma(t) = \frac{h}{\nu} \frac{d q}{d t} \]

As can be seen from Eq. (5e), \( P_1, P_2 \) and \( P_3 \) are functions of \( t \) only. Hence, it is possible to eliminate \( P_1, P_2 \) and \( P_3 \) from Eqs. (5b), (5c) and (5d) by differentiating them with respect to \( \eta \). Finally we obtain the following partial differential equations for \( \phi(\eta, t), F(\eta, t), G(\eta, t), \ldots \):

\[ \phi = -(\phi + \frac{h}{\nu}) \phi + \frac{1}{2} \rho \phi + 2 \phi \]

\[ F = \phi + \frac{h}{\nu} F \]

\[ G = \phi + \frac{h}{\nu} G \]

Boundary conditions Eqs. (2a) and (2b) are rewritten as

\[ \eta = 0: \phi = \phi = F = F = G = G = 0 \]

\[ \eta = 1: \phi = \alpha(t), F = 1, \phi = \phi = F = F = G = G = 0 \]

### Multifold Expansion

First, the independent variable \( t \) is transformed into another variable \( \zeta \) by the following expression:

\[ \zeta = \exp \left( \frac{\nu d t}{h(t)} \right) \]

where \( t_0 \) is a constant.

By the use of this transformation, the differentiation \( (h/\nu)(\partial/\partial t) \) in Eq. (7) can be transformed into a standard form \( f(\partial/\partial \zeta), (\zeta \neq 0) \), of a parabolic type.

Next, using \( \zeta \), we define the following two sets of infinite number of variable parameters, one consisting of the derivatives of the first characteristic function \( \alpha(\zeta) \) and the other the second characteristic function \( \gamma(\zeta) \):

\[ S_1 = \alpha, S_2 = \frac{h}{\nu} \frac{d h}{d t}, S_3 = \frac{h}{\nu} \frac{d q}{d t}, \ldots \]

\[ K_1 = \tau, K_2 = \frac{h}{\nu} \frac{d q}{d t}, K_3 = \frac{h}{\nu} \frac{d q}{d t} + 2 \alpha - \gamma, \ldots \]

Then, in Eq. (7), the differential operation with respect to \( t \) is decomposed into those with respect to variable parameters \( S_1, S_2, \ldots \) and \( K_1, K_2, \ldots \) as

\[ h \frac{\partial}{\partial t} = \frac{\partial}{\partial \zeta} S_1 \frac{\partial}{\partial S_1} + (S_1 + S_2) \frac{\partial}{\partial S_2} + \ldots \]

\[ + K_1 \frac{\partial}{\partial K_1} + (K_1 + K_2) \frac{\partial}{\partial K_2} + \ldots \]

Note that the differentiation of \( \phi \) with respect to \( K_1 \) is not needed because this equation does not include \( \gamma \).

Owing to the transformations \( \alpha = S_1, \gamma = K_1 \) and Eq. (12), \( \alpha(t) \) or \( h(t) \), \( \gamma(t) \) or \( q(t) \) and \( t \) have vanished from Eq. (7). This means that the unknown functions \( \phi \) and \( F, G, \ldots \) have been transformed from functions of \( (\eta, t) \) into those of \( (\eta, S_1, S_2, \ldots) \) and \( (\eta, S_1, S_2, \ldots, K_1, K_2, \ldots) \), respectively.

In this flow phenomenon, all flow characteristics are determined by giving the form of two characteristic functions \( \alpha(t) \) and \( \gamma(t) \). It should be noted that to give the form of \( \alpha(t) \) and \( \gamma(t) \) is equivalent to determine the values of independent variables \( S_1, S_2, \ldots \) and \( K_1, K_2, \ldots \). These variables are composed of the derivatives of \( \alpha(t) \) and \( \gamma(t) \), as can be seen from the Taylor's expansion theorem.

Accordingly, we can solve Eqs. (7) and (8) in a generalized form, without any restriction on the functional forms of \( h(t) \) and \( q(t) \), by expanding the functions \( \phi, F, G, \ldots \) in the following forms of "multifold series" of the independent variable parameters \( S_1, S_2, \ldots, K_1, K_2, \ldots \):

\[ \phi = S_1 \phi_1(\eta) + S_1 \phi_2(\eta) + \ldots \]

\[ + S_1 \phi_3(\eta) + \ldots \]

\[ + S_1 \phi_4(\eta) + \ldots \]

\[ F = S_1 f_1(\eta) + S_1 f_2(\eta) + \ldots \]

\[ + S_1 f_3(\eta) + \ldots \]

\[ + S_1 f_4(\eta) + \ldots \]

\[ G = S_1 g_1(\eta) + S_1 g_2(\eta) + S_1 g_3(\eta) + \ldots \]

\[ + S_1 g_4(\eta) + \ldots \]

\[ + S_1 g_5(\eta) + \ldots \]
In these "multifold series" the \(S_{1}\) and \(K_{1}\)-terms belong to the first order column, the \(S_{2}\), \(S_{3}\), \(S_{1}K_{1}\), \(K_{12}\) and \(K_{2}\)-terms to the second and so forth, and all the series terms belonging to a column are of the same order of magnitude.

Now, substituting Eqs. (12) and (13) into Eqs. (7) and (8), and equating the coefficients with similar forms of variable parameters to zero, we obtain the following ordinary differential equations which determines the functions \(\phi_{1}, \psi_{1}, \ldots, F_{0}, \psi_{0}, G_{0}, \ldots, h_{1}, \ldots, \) and \(j_{1}, \ldots, \):

\[
\begin{align*}
\phi_{1}'' &= 0 \\
\phi_{0}'' &= (-\phi_{1})\phi_{0}'' - 3\phi_{1}'' \\
\phi_{1}' &= -\phi_{0}' - \phi_{1}' \\
&\vdots
\end{align*}
\]

\(F_{0}' = 0\)

\[
\begin{align*}
F_{0}'' &= (\phi_{1} - 1)F_{0}' + (\phi_{0} - 2)F_{0} \\
F_{0}' &= \phi_{1}F_{0}' + \phi_{0}F_{0}' + (\phi_{1} - 1)F_{0}' + (\phi_{0} - 2)F_{0}' \\
F_{0} &= -2F_{0}' + F_{0}' + F_{0}' + F_{0}' \\
&\vdots
\end{align*}
\]

\(G_{0}'' = -2F_{0}'
\]

\[
\begin{align*}
G_{0} &= (\phi_{1} - 1)G_{0}' + (\phi_{0} - 1)G_{0} \\
G_{0}' &= \phi_{1}G_{0}' + \phi_{0}G_{0}' + (\phi_{1} - 1)G_{0}' + (\phi_{0} - 1)G_{0}' \\
G_{0} &= -2G_{0}' + G_{0}' + G_{0}' + G_{0}' \\
&\vdots
\end{align*}
\]

\(h_{0}' = F_{0} ; h_{1}' = h_{0}' ; h_{0}'' = h_{0}' ; h_{0}'' = \phi_{1} + \phi_{0} + j_{1}' \)

\[
\begin{align*}
\phi_{1} &= \psi_{0} ; \psi_{1} = \phi_{0} ; \phi_{1} = h_{0}'' - h_{0}'' + h_{0}' \\
&\vdots
\end{align*}
\]

Boundary conditions are given as follows:

\[
\begin{align*}
\phi_{1}(Y) &= \phi_{0}(Y) = F_{0}(Y) = Y(0) = 0 \quad \text{at} \quad \eta = 0 \\
\phi_{1}(Y) &= \phi_{0}(Y) = F_{0}(Y) = Y(0) = 0 \quad \text{at} \quad \eta = 1
\end{align*}
\]

where \(Y\) represents any function except \(\phi_{0}\) and \(F_{0}\).

As can be seen from the above equations, \(\phi_{1}, \psi_{1}, \ldots, F_{0}, \psi_{0}, G_{0}, \ldots, h_{1}, \ldots, \) and \(j_{1}, \ldots, \) are all functions of \(\eta\) alone and free from any other factors related to the individual actual flow conditions. Hereafter we will call them "universal coefficient-functions". Consequently, if once these equations have been solved, the results can be applied universally for all kinds of, and arbitrary, variations of the gap width and the flow rate of the source with time.

Although analytical solutions for Eqs. (14) and (15) exist, their derivations are lengthy and laborious, especially for the higher-order functions. In the present research, therefore, we have solved Eqs. (14) and (15) numerically by means of Runge-Kutta-Gill's method. Fourteen places (decimal digits) in significant figure of arithmetic operation were retained throughout the computation, and the step size applied was 0.00025.

The important results are shown in Table 1. In Table 1, the relations \(Z_{\phi}(0) = Z_{\phi}(1)\) and \(Z_{\psi}(0) = Z_{\psi}(1)\) result from the symmetry of the radial-flow velocity profile.

Solution for Pressure and Flow Force

When Eqs. (7) and (8) are solved using the above mentioned method, we can obtain important flow characteristics applicable to practical problems (18), i.e., pressure distribution \(p\) and flow force \(W\) exerted on the disks. Here, only the final, nondimensional expressions for \(p\) and \(W\) are shown.

\[
\begin{align*}
p_{1} &= \frac{(r_{1} - 1)P_{1}}{h_{1} \cdot h_{1}} + \frac{R_{1} \log r_{1}}{h_{1} \cdot h_{1}} + \frac{R_{1}}{h_{1} \cdot h_{1}} \\
W_{1} &= \frac{1}{2} \frac{r_{1} \cdot r_{1} \cdot r_{1}}{h_{1} \cdot h_{1}} \left\{ 1 + \frac{1}{2} \left( \frac{1}{h_{1}} \right) \right\} \times \left\{ R_{1} - \frac{S_{1}}{h_{1}} \right\} \left\{ 1 + \frac{1}{2} \left( \frac{1}{h_{1}} \right) \right\}
\end{align*}
\]

Table 1: Universal numerical values related to the fundamental flow characteristics (the notation \(Z\) represents every universal coefficient-function)

<table>
<thead>
<tr>
<th>Function</th>
<th>Value of function</th>
<th>Value of function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z_{\phi}(0))</td>
<td>(-Z_{\phi}(1))</td>
<td>(-Z_{\phi}(1))</td>
</tr>
<tr>
<td>(\phi_{1})</td>
<td>6.00000</td>
<td>-1.20000000000000000000</td>
</tr>
<tr>
<td>(\psi_{1})</td>
<td>2.71429 \times 10^{-1}</td>
<td>3.43286</td>
</tr>
<tr>
<td>(\phi_{2})</td>
<td>9.35293 \times 10^{-2}</td>
<td>-1.19859</td>
</tr>
<tr>
<td>(\psi_{2})</td>
<td>6.00000</td>
<td>-1.20000000000000000000</td>
</tr>
<tr>
<td>(f_{1})</td>
<td>-8.57143 \times 10^{-2}</td>
<td>1.37671</td>
</tr>
<tr>
<td>(f_{2})</td>
<td>-5.90600 \times 10^{-4}</td>
<td>5.93242 \times 10^{-1}</td>
</tr>
<tr>
<td>(f_{3})</td>
<td>4.72300 \times 10^{-4}</td>
<td>-2.29710 \times 10^{-4}</td>
</tr>
<tr>
<td>(G_{1})</td>
<td>-7.14293 \times 10^{-1}</td>
<td>1.52428</td>
</tr>
<tr>
<td>(G_{2})</td>
<td>4.32900 \times 10^{-4}</td>
<td>-1.09380 \times 10^{-4}</td>
</tr>
<tr>
<td>(G_{3})</td>
<td>-1.43936 \times 10^{-4}</td>
<td>-6.28571 \times 10^{-4}</td>
</tr>
<tr>
<td>(h_{1})</td>
<td>2.16597 \times 10^{-4}</td>
<td>1.27623 \times 10^{-4}</td>
</tr>
<tr>
<td>(h_{2})</td>
<td>1.00000 \times 10^{-4}</td>
<td>-1.20000000000000000000</td>
</tr>
<tr>
<td>(h_{3})</td>
<td>-7.42857 \times 10^{-4}</td>
<td>1.28571 \times 10^{-4}</td>
</tr>
<tr>
<td>(h_{4})</td>
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</tr>
<tr>
<td>(h_{5})</td>
<td>1.00000 \times 10^{-4}</td>
<td>-1.20000000000000000000</td>
</tr>
<tr>
<td>(j_{1})</td>
<td>1.25844 \times 10^{-1}</td>
<td>-6.28571 \times 10^{-4}</td>
</tr>
<tr>
<td>(j_{2})</td>
<td>-1.25846 \times 10^{-4}</td>
<td>1.28571 \times 10^{-4}</td>
</tr>
<tr>
<td>(j_{3})</td>
<td>-5.97982 \times 10^{-4}</td>
<td>1.63326 \times 10^{-4}</td>
</tr>
<tr>
<td>(j_{4})</td>
<td>-1.25844 \times 10^{-1}</td>
<td>-6.28571 \times 10^{-4}</td>
</tr>
</tbody>
</table>
where
\[ W_\ast = \frac{h^2 R_\ast}{\rho \nu' r_\ast^2} \]
\[ R_\ast = \frac{h^2 \sigma}{\rho \nu' r_\ast^2} \]
\[ \rho_\ast = \frac{h^2 \sigma}{\rho \nu' r_\ast^2} \]

Applying the boundary conditions, Eq. (8), to Eqs. (5b), (5c) and (5d), and substituting Eqs. (13) and the values in Table 1, the coefficient-functions \( P_1, P_2, P_3 \) are expressed as follows:

\[ P_1(t) = -(\sigma t) \phi_{\infty}(0, 0) = -(\sigma t) \phi_{\infty}(l, 1) \]
\[ = (3.00000a + (-8.35711 \times 10^{-4} \times 1')^2 + 2.23044 \times 10^{-2} \times 1')^2 + \ldots \]
\[ P_2(t) = F_{\infty}(0, 0) = F_{\infty}(l, 1) \]
\[ = (-1.20000 \times 10^a + (1.37413 \times 1 - 1.20000 \times 1) + (5.93510 \times 10^a S^1 - 2.35711 \times 10^a S^1 - 4.35711 \times 10^a S^1 - 1.42537 \times 10^a K^1 + 4.23537 \times 10^a K^1 + \ldots \]
\[ P_3(t) = -(\sigma t) \phi_{\infty}(0, 0) = -(\sigma t) \phi_{\infty}(l, 1) \]
\[ = (-1.71499 \times 10^a + (5.19480 \times 10^a S^1 + 5.71429 \times 10^a K^1 + (3.12490 \times 10^a S^1 - 6.38115 \times 10^a S^1 - 2.71633 \times 10^a S^1 - 1.41629 \times 10^a K^1 + \ldots \]

**EXPERIMENT**

**Facilities and instrumentation**

Figure 2 shows the experimental apparatus used. Water was supplied from a head tank through a flow control valve and two fine wire screens into the test section formed by two parallel disks. The test section was set submerged within a water tank. The water levels of both the head tank and water tank were kept constant by overflow devices; the head difference between both levels was 12 m.

A mechanism converting rotation into reciprocating motion, which was driven by a variable-speed electric motor through a timing belt, was used to oscillate sinusoidally the upper disk of the test section.

The source flow was varied by changing the restriction of a spool valve using a cam attached to the axis of the same electric motor that drove the disk.

The two circular disks, which formed the test section, were made of 15 mm-thick transparent plastic plates with 300 mm outer diameter (2R). The upper disk had in its central portion a fluid source pipe with 30 mm inner diameter (2r). Consequently, in the present experiment, \( \kappa \) was set to 0.1, which fulfills the first assumption 1) in the previous section. Sharp edge at the central opening in the upper disk was rounded off in 2 mm radius to keep oil flow separation from there. The water tank, in which the test section was set submerged, was a water depth of 670 mm and 700 mm x 700 mm bottom sizes and sufficiently larger than the test section, so that the flow within the test section was almost completely unaffected by the existence of the water tank walls.

The time-varying gap width between the two disks, the pressures and the flow force acting on the lower disk surface, were measured respectively by a displacement transducer (capacitor type), pressure transducers (water-proofed, strain-gage type; a capacity of 0.02 MPa) and a load cell (water-proofed, strain-gage type; a capacity of 490 N). The outputs from these transducers were amplified and fed to a low pass filter to cut off high frequency noises from the electric motor and other disturbing sources; then they were recorded by a data logger and fed to a micro-computer in order to be carried out data processing. The load cell was supported and protected by a precisely-made sliding guide, which was sufficiently low resistant for axial forces to be measured but extremely high rigid for any other possible lateral forces.

The fluid pressures were measured on the lower disk surface at five positions of r = 20, 30, 50, 80 and 120 mm, the respective positions being on different radii at angular intervals of 60 deg.

In this experiment, the gap width and the source flow were varied sinusoidally according to the following equations:

\[ h(t) = h(x(t) + \sin t) \]
\[ \sigma(t) = \int_{0}^{t} [1 + b \sin(\omega t + \delta)] \]

![Fig. 2 Schematic diagram of the experimental setup](image)
The angular frequency of $h(t)$ was set equal to that of $q(t)$, the phase difference $\delta$ between them was varied by changing the angular position of the cam to the motor shaft. The angular frequency $\omega$ was varied by changing the motor speed. The non-dimensional amplitude $b$ of the flow was varied by adjusting the opening of the flow control valve set in series with the spool valve. Consequently, it was not possible to vary $b$ and $q$ independently of each other. However, this little hindered the present purpose of experiment, i.e., verification of the validity of the theory.

The frequency $\omega/(2\pi)$ was varied between 0.1 and 3 Hz, $\delta$ between $-180$ and $+90$ deg, $a$ between 0.25 and 0.57, $b$ between 0.26 and 0.56, $R_m$ between 1.0 and 2.0 mm, and $q_m$ between 90 and 270 cm$^3$/s, respectively.

The value of $q(t)$ was calculated from the area (varied with a cam) of the spool valve restriction and the differential pressure (measured by a semi-conductor type differential pressure transducer) immediately upstream and downstream of the spool valve, assuming quasi-steady flow.

Experimental Results and Comparison with the Theory

Non-linear interaction. As already stated, simple superposition of the solution of the flow with time-varying gap width and that of the source flow will not result in a good result because of non-linear interaction. The interaction may grow stronger as the following two kinds of the Reynolds numbers corresponding to the two coexisting flows increase together:

\begin{align}
R_+ &= h\omega/\nu \\
R_- &= q_\alpha/(2\pi \nu r_1)
\end{align}

under the condition that the two flows are kept in a balance of comparable effects. From the expression for mean radial flow velocity in the gap

\begin{equation}
\bar{u} = \frac{1}{2} \frac{r}{h} \frac{dh}{dt} + \frac{a}{hr} \frac{d^2}{dh^2} S_1 \approx R_+
\end{equation}

it follows that the two flows are of comparable effects when

\begin{equation}
\frac{1}{2} \frac{r}{h} \frac{dh}{dt} \approx \frac{a}{hr} \frac{d^2}{dh^2} S_1 \approx R_-
\end{equation}

The mean value of $R_+$ is considered to be $1/2$ because this radial position divides the disk surface into two (inner and outer) parts of equal areas. The quantity $S_1$ is of the order of $aR_\omega$ and $R_\alpha/\nu$ of unity. Consequently, Eq. (25) becomes

\begin{equation}
\frac{1}{4} (aR_\omega) \approx R_-.
\end{equation}

The calculated results for the flow force $W_\alpha$ are shown in Figs. 3, where the source flow is kept constant. In the figures, the results of the approximate superposition theory which neglects the non-linear interaction are shown as the "simple theory". As can be seen from these figures, the difference between the present and simple theories becomes most remarkable in Fig. 3(b), i.e., when $(aR_\omega)/4 \approx R_-$. This fact justifies the conjecture described above. Figure 3(d) shows a comparison of an experimental result with both the simple and present theories for the case of $(aR_\omega)/4 \approx R_-$. As can be seen, while the simple solution which does not take into account the interaction produces a remarkable error, the present solution which takes it into account agrees very well with the experimental result. This fact just demonstrates the validity of the present theory.

The case where the source flow is kept constant. In this case $b = 0$ in Eq. (21), and the flow characteristics are governed by the three parameters, i.e., $R_\omega$, $R_m$, and $\alpha$. Measurements were carried out for various sets of values of these parameters ($R_\omega$, $R_m$, $\alpha$). Some representative results are shown in Fig. 4 for pressures and in Fig. 5 for flow forces. In these figures, the theoretical results of the new series solutions given in the previous section are also displayed.

It can be seen from examination that the present new series solutions agree very well with the experimental results over the range in which the series solutions are rapidly convergent. Namely, in the case of Figs. 4(a), 4(c), 5(a) and 5(b), the solutions are rapidly convergent and really good agreements are attained between the theory and experiments. On the other hand, in the cases of Figs. 4(b), 4(d), and 5(c), the solutions are found to be no longer rapidly convergent and correspondingly some disagreement begins to appear.

Thus, there is naturally a finite range of applicability for the present finitely-truncated series solutions. Therefore, for the convenience of application, we have provided a practical guide which indicates roughly the range of applicability, by comparing the theory with the many experimental results obtained through this experimental approach.
The case where the source flow is varied sinusoidally. The flow characteristics in this case are subjected to $a$, $b$, $\delta$, $R_\infty$ and $R_0$. Some representative results are shown in Figs. 7 and 8.

Theoretical results of the pressure for the case of sinusoidally oscillating gap-width variation

Theoretical and experimental results of the flow forces for the case of sinusoidally oscillating gap-width variations

Range of applicability of the solutions for the case of sinusoidally oscillating gap-width variation: (a) pressure ($a = 0.38$); (b) force ($a = 0.43$)
of the present series solutions are also shown in the same figures. Although it was difficult to select the values of the parameters in round figures because of more parameters than the previous case and the limitation of experimental instrumentation, we tried to pick up the parameters so that the readers may be able to estimate the effect of only one of the parameters, comparing any two graphs of (a) through (f) in Figs. 3 and 4.

As can be seen from these figures, the present theoretical solutions calculated from finitely-truncated series agree very well with the experiments like as the preceding case, and the consistency of the theory is confirmed. The applicable range may be estimated from Fig.6, with \( R \max \) taken in place of \( R \), and in fact this has been confirmed by many experimental results.

CONCLUSIONS

A theoretical analysis and experimental results have been presented for the unsteady laminar incompressible flow between parallel disks with an arbitrarily varying gap width and a central source of arbitrarily varying (or constant) flow rate.

New series solutions to the Navier-Stokes equations are obtained, making use of a technique of asymptotic series expansion in the radial direction and on the basis of a new theory of "multifold series expansion"(17) with respect to the time variable. The solutions can describe precisely the important interaction phenomenon between the two coexisting flows, one caused by the gap-width variation and the other due to the source. This interaction phenomenon is produced through the non-linear convective inertial forces, and its complicated characteristics have not been made clear up to date: the present theory is the first to throw light upon it.

Next, as a typical example, the particular case of sinusoidally oscillating gap-width and source-flow variations was taken up, and the experiments for measurements of the pressures and the flow force acting on the disk surface were carried out. From comparison of the theory with the experiments, the validity of the new theory has been verified. The solutions given by the new theory agree well with the experimental results even in severe cases when the approximate "superposition theory" which neglects the interaction effect produces a remarkable error.

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