A Proper Tensorial Form of D’Arcy–Ergun Equation*

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Synopsis
A proper tensorial form of D’Arcy–Ergun equation is derived as

$$-\nabla p = (\rho/R)[f_1 v \mathbf{E} - f_2 \{\nabla R \mathbf{v}^T + \mathbf{v}(\nabla R)^T\}] \cdot \mathbf{v}$$
$$\begin{pmatrix} -\partial p/\partial x \\ -\partial p/\partial y \end{pmatrix} = \begin{pmatrix} c_1 + c_2 |v_x| & 0 \\ 0 & c_1 + c_2 |v_y| \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \dots\dots(6)$$

The validity of widely applied equation which lacks the second term in square brackets is discussed and it is concluded that the second term is of minor importance except in the vicinity of large gradient of hydraulic radius *R*, e.g., in the particle boundary in structured packed bed.

I. Introduction

D’Arcy’s law for the flow in porous media has in general the following form.⁽⁶⁾

$$-\nabla p = \mathbf{A} \cdot \mathbf{v} \dots\dots\dots(1)$$

This means if the medium is nonisotropic, then the pressure gradient and the velocity vector are no longer in the same direction. But neither experimental nor theoretical expression of tensor **A** has ever appeared.

On the other hand, many authors have based their numerical analyses not on Eq. (1) but on Eq. (2).^(10,11,15–17,20)

$$-\nabla p = a \mathbf{v} \dots\dots\dots(2)$$

They all use Ergun’s correlation^(6,8) as their starting point:

$$\Delta p/L = f_1(\text{Re}_{\text{local}}) \cdot (\rho v^2/R) \dots\dots\dots(3)$$

where

$$f_1 = 7/4 + 150/\text{Re}_{\text{local}}, \text{Re}_{\text{local}} = \rho v R/\mu \dots\dots\dots(4)$$

The definition of *a* is^(10,11,15–17,20)

$$a = f_1 \rho v/R \dots\dots\dots(5)$$

Equation (2) is considered to be applicable for small nonisotropy. But it seems that, even in structured packed beds, experimentally observed isobar lines are not too different from the computational ones. This fact comes from the dependency of *a* on local *R* and *v*, inherently included in Ergun’s correlation. Nevertheless, it must be pointed out that the use of a scalar in place of a tensor is only an approximation and it makes isobar lines always perpendicular to streamlines.⁽⁴⁾

Some authors utilized a quasi-tensorial variation of Eq. (2) because of mere simplicity.^(13,18,19,21–23) Their equation of motion in two-dimensional cartesian form is

This diagonalized tensorial form means that the principal axes of **A** are the coordinates themselves. Though their results*** show that isobars are not perpendicular to streamlines, this arbitrary form is controversial, because this coincidence of axes is not justified.

The objective of this research note is to find a proper form of tensor **A** and to show why Eq. (2) has produced rather satisfactory results.

II. Derivation

Since Ergun’s correlation is best explained by the capillary model,⁽⁶⁾ it seems reasonable to base our discussion on that model.

Suppose a volume which is large enough to be approximated by a bundle of capillaries but small enough compared with the whole packed bed. The point is that this hypothetical capillary exists along the direction of **v**. If there is a change of *R* in *x*-direction as shown in Fig. 1, gradient of *R* is definable in this volume. Since each capillary is interconnected to its neighbors, it is possible to define another hypothetical bundle of divergent or convergent capillaries in *x*-direction as well. Two types of shaded areas in Fig. 1 represent this idea. In this case we can not obtain a straight streamline in *y*-direction in this volume even the total flow is only in *y*-direction. As the fluid particles are always striking the bed particles and acquiring some momentum orthogonal to the mean local velocity **v**, their trajectory will bend to the less resistant direction. This effect is called ‘cross-flow’.^(1–3,5,14) In other words, in the presence of positive $\partial R/\partial x$ there may be a positive pressure gradient in *x*-direction to keep **v** straight in *y*-direction.

This consideration leads us to some basic conditions for tensor **A**.

- 1°: **A** has two directions, i.e., ∇R and **v**. Direction of ∇p is excluded because we need an explicit form in Eq. (1).
- 2°: If $\nabla R = 0$, for an isotropic case, **A** must reduce to *a*.
- 3°: If ∇R is parallel to **v**, then ∇p must be parallel to **v**.
- 4°: A cross-flow may arise if ∇R is not parallel to

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*** It is easily shown that rotation of coordinates generates non-diagonal terms in the tensorial factor of Eq. (6),⁽⁴⁾ so it behaves like Eq. (1).

Fig. 1. A small volume in which hydraulic radius gradient exists (Shaded areas are 'capillaries'.)

v.

5°: Dimension of **A** is equal to $[\rho v/R]$.

The simplest form of **A** which satisfies all these conditions is

$$\mathbf{A} = (\rho/R)[f_1 \mathbf{v} \mathbf{E} - f_2 \{\nabla R \mathbf{v}^T + \mathbf{v}(\nabla R)^T\}] \dots (7)$$

or in three-dimensional kernel-index notation this is read as

$$A_{ij} = (\rho/R)[f_1 v_{ij} - f_2 \{(\partial R/\partial x_i)v_j + v_i(\partial R/\partial x_j)\}], \\ (i, j = 1, 2, 3) \dots (7')$$

The constant f_2 is a function of Re_{local} . It is noted that **A** is symmetric. Equation (7) is written more simply as

$$\mathbf{A} = a \mathbf{E} - b[\nabla R \mathbf{v}^T + \mathbf{v}(\nabla R)^T] \dots (8)$$

where $b = f_2 \rho/R$. The reciprocal of Eq. (1) is written as

$$-\mathbf{v} = \mathbf{A}^{-1} \cdot \nabla p \dots (9)$$

Now, let us see how the above-mentioned conditions are met by Eq. (8). First, 1° is already materialized in the square bracketed term. Second, if $\nabla R = 0$ then $\mathbf{A} = a \mathbf{E}$. Insertion of the last equation into Eq. (1) leads to

$$-\nabla p = a \mathbf{E} \cdot \mathbf{v} = a \mathbf{v} \dots (10)$$

So, the condition 2° is fulfilled. Third, if ∇R is parallel to **v**, then we may write $\nabla R = \lambda \mathbf{v}$. Insert this into Eq. (8) and then into Eq. (1) gives

$$\mathbf{A} = a \mathbf{E} - 2b\lambda(\mathbf{v} \mathbf{v}^T) \\ -\nabla p = a \mathbf{v} - 2b\lambda(\mathbf{v} \mathbf{v}^T) \cdot \mathbf{v} = (a - 2b\lambda v^2) \mathbf{v} \dots (11)$$

Thus 3° is satisfied. Fourth, if ∇R is not parallel to **v**, we take an arbitrary vector **w** which is orthogonal to **v**. Then we decompose ∇R into two mutually orthogonal directions as

$$\nabla R = \lambda \mathbf{v} + \kappa \mathbf{w}$$

After a few rearrangements, it can be shown that

$$(a - b\lambda v^2) \mathbf{v} = -\nabla p + b v^2 \nabla R \dots (12)$$

From this result it is obvious that **v** has not only a component parallel to the pressure gradient but also one parallel to the hydraulic radius gradient. This is the cross-flow effect of non-vanishing ∇R .

Finally, let us see a simple example. Take Fig. 1 as an example in two-dimensional cartesian coordinate, where $\partial R/\partial x > 0$ and $\partial R/\partial y = 0$. Equation (1) becomes in this case

$$-\partial p/\partial x = a v_x - 2b v_x^2 (\partial R/\partial x) \\ - b v_y [v_x (\partial R/\partial y) + v_y (\partial R/\partial x)] \dots (13-x) \\ -\partial p/\partial y = a v_y - 2b v_y^2 (\partial R/\partial y) \\ - b v_x [v_x (\partial R/\partial y) + v_y (\partial R/\partial x)] \dots (13-y)$$

If $v_x > 0$ and $v_y = 0$, we find

$$-\partial p/\partial x = [a - 2b v_x (\partial R/\partial x)] v_x < a v_x, \quad -\partial p/\partial y = 0 \\ \dots (14)$$

This suggests that the pressure drop is lessened by a divergent capillary. Next, if $v_x = 0$ and $v_y > 0$, then it becomes

$$-\partial p/\partial x = -b v_y^2 (\partial R/\partial x) < 0, \quad -\partial p/\partial y = a v_y \dots (15)$$

This shows that a positive pressure gradient in x -direction is necessary to keep $\mathbf{v} = v_y \mathbf{j}$.

To show an occurrence of the cross-flow, we consider the case of $\partial p/\partial x = 0$ and $\partial p/\partial y < 0$. It is easily shown using Eq. (9) that

$$-v_x = [b v_y / \det(\mathbf{A})] (\partial R/\partial x) (\partial p/\partial y) \dots (16-x) \\ -v_y = [(a - 2b v_x \cdot \partial R/\partial x) / \det(\mathbf{A})] (\partial p/\partial y) \dots (16-y)$$

and hence

$$v_x [(f_1 v / f_2) (\partial R/\partial x)^{-1} - v_x] = v^2 \dots (17)$$

The last equation is an evidence for existence of a cross-flow.

III. Discussion and Conclusion

At this stage, nothing is known about magnitude and functional form of f_2 . Since f_1 and f_2 are friction factors, their orders of magnitude may not be very different. Comparing the order of magnitude of the f_1 and the f_2 terms in Eq. (7),

$$(\text{order of the } f_1 \text{ term}) / (\text{order of the } f_2 \text{ term}) \simeq |\nabla R| \\ \dots (18)$$

It is easily seen that the truly tensorial f_2 term plays a minor role except in the neighborhood of sharp boundary between particles of different sizes.

Thus, it is concluded that the isobars will be nearly orthogonal to the streamlines in the most part of a bed excluding particle boundaries. This conclusion leads us to the validity of computations already done by numerous authors based on Eq. (2). It must be stressed again that this by no means justifies the usage of Eq. (6) as criticized by Araki and Moriyama,⁴⁾ though Eq. (6) is reasonably accepted for nearly parallel flow fields.²⁰⁾

Another important point is as follows. Some authors added to Eq. (2) an inertial^{9,12)} or viscous⁷⁾ term. But these 'hybrid' equations of motion are erroneous, because the Ergun's correlation is already an experimental integration of the original rigorous differential equation of motion under very compli-

cated boundary conditions of a packed bed, and it is valid only for a small, but never infinitesimally small, volume in the bed. Without a theory similar to that of turbulence, no equation of motion other than Eq. (2) is justified. We are forced to solve a set of first-order differential equations, namely the equation of continuity and Eq. (2), with necessary and sufficient boundary conditions.

The author is doing computational works on Eq. (7) and expecting to estimate the magnitude of the second friction factor f_2 by comparing with some measurements.

Nomenclature

- a : $f_1 \rho v / R$, $[M/L^3 T]$
 b : $f_2 \rho / R$, $[M/L^4]$
 c_1, c_2 : arbitrary constants
 D_p : particle diameter, $[L]$
 f_1, f_2 : friction factor, $[-]$
 L : length of the bed, $[L]$
 p : pressure, $[M/LT^2]$
 Δp : pressure drop across the bed, $[M/LT^2]$
 R : hydraulic radius $= D_p \cdot \varepsilon / (1 - \varepsilon)$, $[L]$
 Re_{local} : local Reynolds number $= R \rho v / \mu$, $[-]$
 v : $|\mathbf{v}|$, $[L/T]$
 $\mathbf{i}, \mathbf{j}, \mathbf{k}$: unit vector in x -, y -, z -direction, respectively, $[-]$
 $\mathbf{v}$: $\mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z$; mean local velocity in pore, $[L/T]$
 $\mathbf{x}$: $\mathbf{i}x + \mathbf{j}y + \mathbf{k}z$; position, $[L]$
 $\mathbf{w}$: arbitrary vector orthogonal to $\mathbf{v}$, $[L/T]$
 ∇ : $\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$; space derivative operator, $[1/L]$
 $\mathbf{A}$: flow resistance tensor, $[M/L^3 T]$
 $\mathbf{E}$: unit tensor $= \delta_{ij}$, $[-]$
 ε : porosity, $[-]$
 κ, λ : arbitrary constants, $[T/L]$
 μ : viscosity, $[M/LT]$
 ρ : density, $[M/L^3]$

Suffix

x, y : direction

Prefix

- T: transpose of a vector or a tensor
 -1: reciprocal of a scalar or a tensor

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