Liquid Flow Accompanying Liquid Solid Transition*

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Synopsis

The two-dimensional velocity and pressure fields within the liquid during a solidification process are solved analytically from equations of momentum and continuity. The results show that the pressure field is dominated by the gravitational effect, whereas the velocity field is governed by the freezing rate and the ratio of the liquid and solid densities.

I. Introduction

When the density of a substance in the liquid state is different from that in the solid state, the volume change associated with the liquid-solid phase transformation can induce a flow in the liquid, and hence an associated pressure field. This flow of liquid can transport energy through the bulk flow, and hence affect the freezing or melting rate. The induced pressure field can exert additional force at the liquid-solid interface which, when sufficiently high, might assist irregular growth of the solid shell. For the case of solidification of alloys that have a concentration gradient accompanying the temperature gradient near the liquid-solid interface, the liquid flow will cause redistribution of solute and hence result in macrosegregation.

Despite the potential importance of this density-change-induced flow, very few studies have been made. When the velocity field has been computed, in most theoretical studies in the field of solidification, it has been estimated by considering conservation of mass only.¹⁾ The complete analysis of the flow field should include both the continuity and momentum equations. In addition they should be coupled with considerations of heat and mass (solute) transfer and of other mechanical behavior such as the deformation of the solid shell, which can affect the character of heat transfer and hence the flow field. A complete mathematical model of the solidification process consisting of a number of simultaneous differential equations, some nonlinear, is not likely to be analytically soluble, even in an approximate manner. Hence in the present work it is assumed that the nature of the heat transfer involved in solidification is independent of the induced flow field.

For the simple one-dimensional freezing process the liquid flow field can also be considered as one-dimensional for static casting. However, for continuous casting the liquid flow becomes two-dimensional in a Eulerian coordinate system although the freezing process under steady state can still be approximated by a one-dimensional solution. Further-

more, the liquid-solid region (or mushy region) is not included in the present work. Hence the solution can be applied only to the early stage of freezing, or to low-solute alloy systems where the liquid-solid region is relatively small. The flow of the liquid is considered as a laminar incompressible flow with small Reynolds number. Hence inertia effects are neglected. The Navier-Stokes equations and continuity are used to solve for the pressure and the velocity of the induced flow field. An integral method equivalent to Karman's momentum integral²⁾ and Goodman's heat-balance integral³⁾ is used in this work.

II. Mathematical Formulation

A schematic representation of the problem to be analyzed and the coordinate system used in the present work is shown in Fig. 1, where the vertical coordinate y and the time variable t are related by $y = Vt$, where V is the casting velocity. The Navier-Stokes equation for incompressible flow can be written as

Du/Dt = X - (1/rho) * grad P + nu * [grad grad u - grad x (grad x u)](1)

where, u, X: the velocity and body force vectors, respectively,
P: the pressure.

Continuity for incompressible flow is given by

grad . u = 0(2)

Equations (1) and (2) form the system for solving the

Fig. 1. Schematic representation of the problem.

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flow field. The boundary conditions used for the solutions are (1) the pressure at the top is constant; (2) the center plane ($x=D$) is a plane of symmetry; and (3) the velocity at the liquid-solid interface ($x=E$) satisfies the conservation law from the heat transfer analysis.

To simplify the equations, it is now assumed that the Reynolds number is small so that the inertia terms in Eq. (1) can be neglected. Under steady state and for Cartesian coordinates Eq. (1) and (2) are reduced to the following three equations in dimensionless form:

$$\frac{\partial \bar{P}}{\partial \bar{x}} = \frac{\partial^2 \bar{U}_x}{\partial \bar{x}^2} + \frac{1}{R^2} \cdot \frac{\partial^2 \bar{U}_x}{\partial \bar{y}^2} \dots\dots\dots(3)$$

$$\frac{\partial \bar{P}}{\partial \bar{y}} = \bar{B} + \bar{R}^2 \cdot \frac{\partial^2 \bar{U}_y}{\partial \bar{x}^2} + \frac{\partial^2 \bar{U}_y}{\partial \bar{y}^2} \dots\dots\dots(4)$$

$$\frac{\partial \bar{U}_x}{\partial \bar{x}} + \frac{\partial \bar{U}_y}{\partial \bar{y}} = 0 \dots\dots\dots(5)$$

where, $\bar{P} = (P - P_0)/P_0$

$$\bar{x} = x/D$$

$$\bar{y} = y/L$$

$$\bar{R} = L/D$$

$$\bar{B} = \rho g L / P_0$$

$$\bar{U}_x = \mu U_x / P_0 D$$

$$\bar{U}_y = \mu (U_y - V) / P_0 L$$

Notations, unless defined as they appear are given in the Table of *Nomenclature*. The boundary conditions can be written as

$$\bar{P} = 0 \quad \text{at} \quad \bar{y} = 0 \dots\dots\dots(9)$$

$$\frac{\partial \bar{U}_y}{\partial \bar{x}} = 0, \bar{U}_x = 0 \quad \text{at} \quad \bar{x} = 1 \dots\dots\dots(7)$$

$$\bar{U}_x = -F\bar{\varepsilon}', \bar{U}_y = 0 \quad \text{at} \quad \bar{x} = \bar{\varepsilon} \dots\dots\dots(8)$$

$$\text{where, } F = \frac{V \left(\frac{\rho_s}{\rho_l} - 1 \right) \mu}{P_0 L}$$

$$\bar{\varepsilon}' = \frac{d\bar{\varepsilon}}{d\bar{y}}$$

The first boundary condition states that pressure at the top of the ingot is maintained at a constant value. The second one indicates symmetry at the center location of the ingot, and the last boundary condition is just the continuity at the liquid-solid interface for one-dimensional solidification problem. Since the volume changes from solidification, $F\bar{\varepsilon}'$, will be compensated by horizontal component of flow of liquid, $\bar{U}_x$.

Integrating Eq. (5) with respect to $\bar{x}$ between $\bar{x} = \bar{\varepsilon}$ and $\bar{x} = 1$ gives

$$\bar{U}_x(\bar{x} = 1) - \bar{U}_x(\bar{x} = \bar{\varepsilon}) + \int_{\bar{\varepsilon}}^1 \frac{\partial \bar{U}_y}{\partial \bar{y}} d\bar{x} = 0$$

$$\text{Let } \phi = \int_{\bar{\varepsilon}}^1 \bar{U}_y \cdot d\bar{x}$$

$$\frac{d\phi}{d\bar{y}} = -\bar{U}_y(\bar{x} = \bar{\varepsilon})\bar{\varepsilon}' + \int_{\bar{\varepsilon}}^1 \frac{\partial \bar{U}_y}{\partial \bar{y}} d\bar{x} \dots\dots\dots(9)$$

Substituting this and Eqs. (7) and (8) into Eq. (9) provides

$$\frac{d\phi}{d\bar{y}} + F\bar{\varepsilon}' = 0$$

which yields the solution $\phi = F(1 - \bar{\varepsilon})$ with the aid of boundary condition $\phi(\bar{y} = 1) = 0$ from the definition of ϕ .

Now assume that $\bar{U}_y = C(\bar{x} - \bar{\varepsilon})[(\bar{x} - \bar{\varepsilon}) - 2(1 - \bar{\varepsilon})]$ which satisfies boundary condition (7) and (8). The integration constant C can be evaluated by substituting $\bar{U}_y$ into the solution of ϕ . Hence the solution for $\bar{U}_y$ is obtained as

$$\bar{U}_y = \frac{3}{2} F \frac{1}{(1 - \bar{\varepsilon})^2} (\bar{x} - \bar{\varepsilon}) [2(1 - \bar{\varepsilon}) - (\bar{x} - \bar{\varepsilon})] \dots\dots\dots(10)$$

Substituting $\bar{U}_y$ into Eq. (5) together with the boundary condition in Eq. (8) gives the solution for $\bar{U}_x$

$$\bar{U}_x = -F\bar{\varepsilon}' \frac{(1 - \bar{x})^3}{(1 - \bar{\varepsilon})^3} \dots\dots\dots(11)$$

Using these solutions for $\bar{U}_x$ and $\bar{U}_y$, $\bar{P}$ can be solved from Eq. (3), thus,

$$\begin{aligned} \bar{P} = & 3F\bar{\varepsilon}' \frac{(1 - \bar{x})^2}{(1 - \bar{\varepsilon})^3} + \frac{F}{4\bar{R}^2} \frac{(1 - \bar{x})^4}{(1 - \bar{\varepsilon})^3} \left[12 \frac{\bar{\varepsilon}'^3}{(1 - \bar{\varepsilon})^2} \right. \\ & \left. + 9 \frac{\bar{\varepsilon}'\bar{\varepsilon}''}{(1 - \bar{\varepsilon})} + \bar{\varepsilon}''' \right] + \bar{b}(\bar{y}) \dots\dots\dots(12) \end{aligned}$$

where $\bar{b}$ is the integration constant, and can be determined in the following manner. Integrating Eq. (4) with respect to $\bar{x}$ between $\bar{x} = \bar{\varepsilon}$ & $\bar{x} = 1$ provides

$$\begin{aligned} \frac{d}{d\bar{y}} \int_{\bar{\varepsilon}}^1 \bar{P} \cdot d\bar{x} + \bar{P}(\bar{x} = \bar{\varepsilon})\bar{\varepsilon}' = & \bar{B}(1 - \bar{\varepsilon}) \\ & + \bar{R}^2 \left[\frac{\partial \bar{U}_y}{\partial \bar{x}}(\bar{x} = 1) - \frac{\partial \bar{U}_y}{\partial \bar{x}}(\bar{x} = \bar{\varepsilon}) \right] + \int_{\bar{\varepsilon}}^1 \frac{\partial^2 \bar{U}_y}{\partial \bar{y}^2} d\bar{x} \dots\dots\dots(13) \end{aligned}$$

The value of $\bar{b}$ is therefore obtained from Eq. (13) by substituting Eqs. (10) to (12).

$$\begin{aligned} \bar{b} = & \bar{b}_0 + \int_0^{\bar{y}} \left\{ \bar{B} - 3F\bar{R}^2 \frac{1}{(1 - \bar{\varepsilon})^2} - \frac{2F}{(1 - \bar{\varepsilon})} \left[\bar{\varepsilon}'' + \frac{3(\bar{\varepsilon}')^2}{(1 - \bar{\varepsilon})} \right] \right. \\ & - \frac{F}{\bar{R}^2} \left[\frac{1}{20} (1 - \bar{\varepsilon}) \bar{\varepsilon}'''' + \frac{3}{5} \bar{\varepsilon}'\bar{\varepsilon}''' + \frac{9}{20} (\bar{\varepsilon}'')^2 \right. \\ & \left. \left. + \frac{18}{5} \frac{(\bar{\varepsilon}')^2 \bar{\varepsilon}''}{(1 - \bar{\varepsilon})} + \frac{3(\bar{\varepsilon}')^4}{(1 - \bar{\varepsilon})^2} \right] \right\} d\bar{y} \dots\dots\dots(14) \end{aligned}$$

$$\begin{aligned} \text{where, } \bar{b}_0 = & -F\bar{\varepsilon}'_0 - \frac{F}{\bar{R}^2} \left[\frac{3}{5} (\bar{\varepsilon}'_0)^3 + \frac{9}{20} \bar{\varepsilon}'_0 \bar{\varepsilon}''_0 + \frac{1}{20} \bar{\varepsilon}''_0{}^2 \right] \\ & \bar{\varepsilon}'_0 = \bar{\varepsilon}'(\bar{y} = 0) \text{ etc.} \end{aligned}$$

The solution for solidification of a slab with a constant heat-transfer coefficient at the surface is derived in *Appendix*. Expressions for $\bar{\varepsilon}$, $\bar{\varepsilon}'$, etc. needed for the present solutions are

$$\begin{aligned} \left(\frac{1}{2} + G \right) [(1 + H\bar{\varepsilon})^2 - 1] - \ln(1 + H\bar{\varepsilon}) = & 2H^2 \bar{R}^2 \bar{\alpha} \bar{y} \\ \dots\dots\dots(15) \end{aligned}$$

$$\bar{\varepsilon}' = \frac{H\bar{\alpha}\bar{R}}{(1/2+G)(1+H\bar{\varepsilon})-1/2\frac{1}{1+H\bar{\varepsilon}}} \dots\dots\dots(16)$$

Higher order derivatives are easily obtained by differentiating Eq. (16).

For the case similar to the process of continuous casting of steel, Eq. (14) can be approximated by an empirical correlation:

$$\bar{b} = B\bar{y} \dots\dots\dots(17)$$

III. Numerical Calculations

For a case similar to typical continuous casting of steel, the following values are assigned to the dimensionless parameters needed for the numerical evaluation of the present analytical solutions

$$\begin{aligned} H &= 10, & G &= 1, \\ \bar{\alpha} &= 0.00888, & \bar{R} &= 100, \\ \bar{B} &= 37.73. \end{aligned}$$

Other required data are

$$\begin{aligned} D &= 50 \text{ (cm)} \\ \mu &= 0.065 \text{ (poise)}. \end{aligned}$$

The horizontal components of velocity $\bar{U}_x$ are calcu-

Fig. 2. Distribution of Horizontal component of velocity.

lated by using Eq. (11) for various locations and are shown in Fig. 2. The magnitude is negative, being maximum at the freezing front and approaching zero at the center of the ingot. However, it only changes at the top of ingot (or the initial stages of solidification), then remains at constant value. The vertical components of velocity are evaluated from Eq. (10) and are shown in Figs. 3 and 4. It can be seen that the vertical component changes its value in both directions. The pressure field is shown in Fig. 5. The value of the pressure varies significantly along the vertical direction. For a gravitational constant of 980 (cm/cm²) it is practically unchanging with respect to $\bar{x}$ and to ratios of solid and liquid densities up to 12.

However, if the gravitational effect is neglected, the pressure will reverse its sign (from positive to negative, or from compressive to tensile) as shown in Fig. 6. For this case the pressure is directly proportional to the ratio of solid and liquid densities, but remains practically independent of horizontal location.

IV. Conclusions and Recommended Future Work

Although a liquid flow is induced by the density change accompanying liquid–solid transition in low-alloy steels, the pressure field within the liquid is dominated by the gravitational effect. Because mechanical behavior of steel at elevated temperature varies with composition as demonstrated by some previous experimental work, and the liquid velocity in the horizontal direction has its maximum value near the liquid–solid interface, it is reasonable to explain that the irregular solid shell growth is due to the redistribution of solute from liquid motion rather than to the pressure exerted at the interface. The gravitational effect can be regulated by means of magnetic field, whereas the velocity is governed by freezing rate. These should be the direction for future analysis of the solidification problem.

Nomenclature

- C*: Specific heat (cal/g°C)
- D*: Thickness of cast (cm)
- h*: Heat transfer coefficient (cal/cm²-s°C)
- K*: Thermal conductivity (cal/cm-s°C)
- L*: Height of cast at complete solidification (cm)
- L'*: Heat of fusion (cal/g)
- P*: Pressure (dyn/cm²)
- P*₀: Pressure at the top of cast (dyn/cm²)
- t*: Time (s)
- T*: Temperature (°C)
- T*_c: Temperature of coolant (°C)
- T*_m: Melting temperature (°C)
- V*: Casting speed (cm/s)
- x*: Horizontal distance (cm)
- y*: Vertical distance (cm)
- α*: Thermal diffusivity (cm²/s)
- ρ*: Density (g/cm³)
- ρ*_l: Density of liquid (g/cm³)
- ρ*_s: Density of solid (g/cm³)
- μ*: Viscosity (g/cm-s)
- ν*: Kinematic viscosity (cm²/s)

Fig. 3. Distribution of vertical component of velocity.

Fig. 4. Distribution of vertical component of velocity.

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Fig. 5. Pressure distribution along vertical direction.

Fig. 6. Pressure distribution. (Neglect gravitational effect).

Appendix

Solidification of a Slab with Constant Heat-transfer Coefficient

By using the rectangular coordinate system and placing the x -axis along the solidification direction and $x=0$ at the cooling surface, temperature distribution and the location of the solidification front are calculated from the following mathematical formulation for the case of no super heat in liquid phase.

$$\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} = \frac{\partial \bar{T}}{\partial \bar{t}} \dots\dots\dots(A-1)$$

Boundary conditions:

- 1) $\bar{T}(x = \bar{\varepsilon}) = 1 \dots\dots\dots(A-2)$
- 2) $\frac{\partial \bar{T}}{\partial \bar{x}}(\bar{x} = \bar{\varepsilon}) = G \frac{d\bar{\varepsilon}}{d\bar{t}} \dots\dots\dots(A-3)$
- 3) $\frac{\partial \bar{T}}{\partial \bar{x}}(\bar{x} = 0) = H \bar{T}(\bar{x} = 0) \dots\dots\dots(A-4)$

Initial condition:

$$\bar{\varepsilon}(\bar{t} = 0) = 0 \dots\dots\dots(A-5)$$

where the dimensionless parameters are defined.

$$\begin{aligned} \bar{T} &= (T - T_c)/(T - T_m) \\ \bar{x} &= x/D; \\ \bar{t} &= \alpha t/D^2 \\ H &= hD/K \\ G &= L'/\{C(T_m - T_c)\} \end{aligned}$$

Integrating Eq. (A-1) with respect to $\bar{x}$ between $\bar{x}=0$ and $\bar{x}=\bar{\varepsilon}$, and using boundary conditions (A-3) and (A-4) gives

$$G \frac{d\bar{\varepsilon}}{d\bar{t}} - H \bar{T}(\bar{x} = 0) = \frac{d\theta}{d\bar{t}} - \frac{d\bar{\varepsilon}}{d\bar{t}} \dots\dots\dots(A-6)$$

where, $\theta = \int_0^{\bar{\varepsilon}} \bar{T} \cdot d\bar{x}$

Assume $\bar{T} = 1 + C_1(\bar{\varepsilon} - \bar{x})$ which satisfies boundary condition (A-2), then C_1 can be calculated from boundary condition (A-4)

Hence,
$$\bar{T} = \frac{1 + H\bar{x}}{1 + H\bar{\varepsilon}} \dots\dots\dots(A-7)$$

Substituting this into Eq. (A-6) provides

$$\frac{d\bar{\varepsilon}}{d\bar{t}} = \frac{H(1+H\bar{\varepsilon})}{G+(1+2G)H\bar{\varepsilon}+(1/2+G)H^2\bar{\varepsilon}^2} \dots\dots(A-8)$$

$\bar{\varepsilon}$ is therefore solved from Eq. (A-8) together with initial condition (A-5):

$$(1/2+G)[(1+H\bar{\varepsilon})^2-1]-\ln|1+H\bar{\varepsilon}| = 2H^2\bar{t} \dots\dots\dots(A-9)$$

As $y = V \cdot t; \quad \bar{y} = y/L; \quad \bar{t} = \alpha t/D^2;$
 $\bar{R} = L/D; \quad \bar{\alpha} = \alpha/VD,$

Eqs. (A-9) and (A-8) become

$$(1/2+G)[(1+H\bar{\varepsilon})^2-1]-\ln(1+H\bar{\varepsilon}) = 2H^2\bar{R}^2\bar{\alpha}\bar{y}$$

and

$$\bar{\varepsilon}' = \frac{H\bar{\alpha}\bar{R}}{(1/2+G)(1+H\bar{\varepsilon})-1/2\frac{1}{1+H\bar{\varepsilon}}}$$

which are Eqs. (15) and (16), respectively.

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