Flow Dynamics of Granular Materials in a Hopper*

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Synopsis
A mathematical model, capable of describing the macroscopic movement of an assembly of particles in a hopper, was developed based on the constitutive equations described by Voigt-Kelvin rheological model with a slider and on the equations of motion for translation and rotation of each particle. The validity of the present model was confirmed by comparing the results, such as the order of particles discharged from the hopper, arrangement of particles after charging in the hopper and stress distribution at the wall, obtained from calculations with the corresponding ones obtained by experiments as well as Janssen's equation.

The present model was successfully applied to the macroscopic flow of the assembly of granular materials consisting of different particle size under gravitational force in a hopper.

The installation of a repulsion box in the upper part of the hopper was also simulated and resulted in the decrease in the variation of particle size during discharge through the suppression of small particle segregation during charge, while it had insignificant effect on the deposit profile and the order of discharge.

The present model was found to precisely describe frictional wall effect in solid flow and abnormal flow behavior, such as bridge formation, in comparison with the conventional continuous potential flow model.

Key words: ironmaking process; solid flow; hopper; granular materials; Voigt-Kelvin rheological model; blast furnace simulation; flow dynamics.

I. Introduction

In the ironmaking process, theoretical analysis on the flow dynamics of granular materials, such as coke and ore is essential for the improvement of segregation in the process of material transportation as well as the realization of stable burden descent in a blast furnace. However, theoretical research on the flow dynamics of granular materials in this field has recently been initiated and the fruitful method of analysis has not yet been established.

There are several approaches previously reported for the flow behavior of granular materials under the assumption of continuum.

In the first place, the potential flow approach was initially applied by Kuwabara et al.11 for the analysis of burden flow in a blast furnace. The inner state of the furnace was studied by Sugiyama et al.21 through the use of an integrated mathematical model, in which the potential flow of burden was considered as well as gas flow, chemical reactions, and heat transfer. Although the potential flow approach is successful in the estimation of burden trajectory in the furnace, it is difficult to estimate the stress field and the effect of gas flow on burden flow due to the absence of equations based on mechanical laws. Furthermore, the abnormal phenomena of burden flow, such as bridge formation at the exit of a hopper, hanging or slipping in the furnace can not be analyzed because of the assumption of continuum. Besides, the potential flow approach requires the boundary conditions on the extent of stagnant region which must be determined prior to calculation on the basis of either stress analysis without plastic deformation or experiments. Therefore, the limiting exists in the application of potential flow approach to the actual processes.

Secondly, the probability approach6,40 in which the movement of each distinct granular material is approximated by random motion has also been attempted to describe the motion of granular materials. In this approach, the converging flow in a hopper can be described without introducing the specified boundary conditions by the use of a parameter which represents the extent of random motion. However, in the probability approach the adoption of the assumption of continuum, which is similar to the potential flow approach, results only in the estimation of the trajectory of burden flow. Therefore, the probability approach has also restrictions on the application.

The third approach is the application of viscoelastic theory with the consideration of plastic deformation.50 The characteristics of the approach is that the velocity field can be determined from the stress field. This approach, however, also has restrictions on the application, since the assumption of continuum is used.

In contrast to the approach based on the assumption of continuous media for granular materials, distinct particle approach which is based on the equations of motion on each particle with the consideration of interaction between particles was proposed for the analysis of macroscopic behavior of an assembly of particles. Campbell and Brennen60 proposed "hard" particle model and reported the effectiveness of the model for the analysis of the movement of particles accompanied with instantaneous collision, such as the motion of particles in a chute. On the other hand, Cundall and Strack73 proposed "soft" particle model to analyze the movement of particles in a wide range from the static stress field to the rapid change of the flow of particles in a short period of time.

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The model uses Voigt–Kelvin rheological model for the interaction between particles and has no restrictions on the contact time of particles. However, the viscous force was not properly considered in the estimation of sliding condition to result in the low analytical accuracy.

As mentioned above, the previous theoretical researches on the flow dynamics of granular materials were defective either in the extent of application or in the analytical accuracy.

Therefore, in the present work, a mathematical model which enables to analyze the macroscopic behavior of an assembly of granular materials was developed on the basis of the equations of motion and constitutive equations described by Voigt–Kelvin rheological model with sliding condition determined by the resultant of elastic and viscous force. A better fundamental understanding was obtained from the application of the model to the segregation phenomenon of particle size during charge into a hopper as well as discharge from a hopper.

II. Outline of the Mathematical Model

1. Interaction between Particles

In the present research, the interaction between particles was evaluated by Voigt–Kelvin rheological model with a slider shown in Fig. 1, in which the energy dissipation during collision between particles and the stability for the arrangement of particles in the static state were assured. Profile of a particle was assumed to be two dimensional disc. Since Voigt–Kelvin model is applied not only to the principal but also to the shear direction of strain, the present model enables to analyze the motion both in translation and rotation.

The instability of the arrangement of particles caused by slipping was described by application of Coulomb's law written by Eq. (1) to the resultant of spring and dashpot force in the present study, while the law was applied only to spring force in the previous study.\(^7\)

\[ |f_{ij}^{s}/f_{ij}^{D}| \leq \mu \]  

where positive values of \(\mu\) correspond to the counterclockwise rotation of particle and the displacement of particle in the approaching direction, respectively. The infinitesimal displacement between particles in the tangential direction \(\Delta\) is represented by Eq. (2) with an addition of angular velocity of rotation.

\[ \Delta l_{ij} = (\dot{X}_i - \dot{X}_j) \cos \theta_{ij} \cdot \Delta t + (\dot{Y}_i - \dot{Y}_j) \sin \theta_{ij} \cdot \Delta t \]  

where positive values of \(\theta_{ij}\) and positive value of \(\Delta l_{ij}\) correspond to the counterclockwise rotation of particle and the displacement of particle in the approaching direction, respectively. The infinitesimal displacement between particles in the tangential direction \(\Delta l_{ij}\) is represented by Eq. (3) with an addition of angular velocity of rotation.

In Fig. 1, comparison of the rheological model with the elastic analysis on the assumption of continuum for the interaction between particles shows that \(K_n\) and \(K_s\) in the former correspond to the bulk modulus and the shear modulus in the latter, respectively, although an object in the analysis is the force in the former, while it is the stress in the latter. When the interaction of the model is compared with that in the viscous fluid dynamics, \(D_s\) and \(D_n\) in the former correspond to the first and the second viscosity\(^8\) in the latter, respectively.

Furthermore, plastic deformation can be considered in the present model, since the sliding condition is introduced through friction coefficient. Accordingly, the rheological model in Fig. 1 is regarded as the distinct model for the continuous visco-elastic media with plastic deformation.

2. Formulation of Mathematical Model

An infinitesimal displacement of particle in an infinitesimal increment of time generates the force among particles through the constitutive equations. This force in return produces the infinitesimal displacement among particles again through the equations of motion. Thus, alternative solving of constitutive equations and equations of motion on each particle with respect to variables, such as velocity, force and coordinate under the appropriate boundary condition with time marching provides the macroscopic dynamic behavior of an assembly of granular materials.

The infinitesimal displacement for the distance between particles \(i\) and \(j\) in the normal direction \(\Delta l_{ij}\) within the infinitesimal increment of time \(\Delta t\) is represented by Eq. (2) using particle velocities \(\dot{X}_i\) and \(\dot{X}_j\) and \(\theta_{ij}\) with the help of geometrical relation shown in Fig. 2.

\[ \Delta l_{ij} = (\dot{X}_i - \dot{X}_j) \cos \theta_{ij} \cdot \Delta t + (\dot{Y}_i - \dot{Y}_j) \sin \theta_{ij} \cdot \Delta t \]  

Fig. 1. Rheological model for the interaction between particles.

Fig. 2. Particle arrangement used for the mathematical model.
\[ \Delta l_{ij} = -([\dot{X}_i - \dot{X}_j] \sin \theta_{ij} + \Delta t + (\dot{Y}_i - \dot{Y}_j) \cos \theta_{ij} \Delta t + d(\psi_i + \psi_j) \Delta t) / 2 \] ................................. (3)

The constitutive equations for the particle \( i \) are expressed by Eqs. (4) and (5).

\[ f_{ij}^n = K_n (\Delta l_{ij} + l_{ij}) + D_n \Delta l_{ij} \] ................................. (4)

\[ f_{ij}^t = K_t (\Delta l_{ij} + l_{ij}) + D_t \Delta l_{ij} \] ................................. (5)

where, \( f_{ij}^n \) is the force acting on the particle \( i \) in the normal direction with the positive value for compression, while \( f_{ij}^t \) similarly the force acting on particle \( i \) in the tangential direction with positive value for clockwise rotation.

Equation (6), which corresponds to Eq. (1), is added on Eqs. (4) and (5) to express the particle arrangement which can be realized.

\[ K_s (\Delta l_{ij} + l_{ij}) + D_s \Delta l_{ij} \leq \mu \left| K_n (\Delta l_{ij} + l_{ij}) + D_n \Delta l_{ij} \right| \] ................................. (6)

The equations of motion on particles in two dimensional system are represented by Eqs. (7) to (9) for translation \( X \) and \( Y \) and rotation \( \psi \) with the use of \( f_{ij}^n \) and \( f_{ij}^t \).

\[ m \dot{X}_i = mg_x - \sum_j \left( f_{ij}^n \cos \theta_{ij} - f_{ij}^t \sin \theta_{ij} \right) \] ................................. (7)

\[ m \dot{Y}_i = mg_y - \sum_j \left( f_{ij}^n \sin \theta_{ij} + f_{ij}^t \cos \theta_{ij} \right) \] ................................. (8)

\[ m \dot{\psi}_i = -4 \sum_j f_{ij}^t \] ................................. (9)

Numerical integration of Eqs. (2) to (9) with respect to time provides the successive change of force, velocity and coordinate of particles with time. A time step in the calculation was chosen 1/17 as small as the critical one predicted by the damped harmonic oscillator model. Since \( \sqrt{m/K_n} \) is proportional to the critical time step, the dimensionless number \( II = mg/\sqrt{K_n d} \), which presents the ratio of gravity to elastic force, should be chosen as large as possible for the reduction of computation time under the condition that the overlapping of particles does not become noticeable.

3. Evaluation of the Properties of Particle

Friction coefficients between particle–particle and particle–wall for an acrylic bar with circular cross section were evaluated by the one dimensional shear test. The outline of the experiment and an example of transitions with time for shear force and displacement are shown in Fig. 3. The displacement is almost proportional to time and the shear force is almost constant, though slight oscillation is observed. The time-averaged value of shear force was used for the evaluation of friction coefficients. Figure 4 shows the relation between normal force and shear force for particle–particle and particle–wall. Since the normal force is proportional to the shear force, the validity for the application of Coulomb’s law represented by Eq. (1) is confirmed. The friction coefficients for particle–particle and particle–wall in acrylic system were evaluated from the gradient of straight lines in Fig. 4 as 0.36 and 0.42, respectively.

4. Verification of the Validity of the Mathematical Model

The validity of the model was examined through the comparison of the calculated results with the experimental ones for cases of charging 10 cm long cylindrical acrylic bars of 1.4 cm diameter into a rectangular or a converging hopper and the case of discharging them from a slit located at the bottom of a hopper. The size of the hopper was 23.8 cm long and 43.8 cm high and the width of a slit was 6.6 cm. A 6.7 cm × 3.4 cm repulsion box was installed in the hopper, when it was necessary. The front and the back walls of the hopper were removed to avoid the influence of friction between the walls and the cross sections of particles.

1. Charging Behavior of Particles into Hoppers

The rearrangement of particles in a hopper was calculated by the model in case of charging colored particles provided uniformly in layers prior to charge at the full width of a hopper inlet. The calculation condition is shown in Table 1. The number of particles is 297 and approximately corresponds to 1/9 that of the experiment.

Figure 5 shows the calculated flow behavior of particles during charge into the converging hopper without and with a repulsion box.

Without the repulsion box, falling of particles near the inclined wall is in retard and the particles tend to gather in the central part. As a result, particles charged into the central part falls in advance in the
lower part of the hopper. On the other hand, in the upper part of a hopper, the deposit level in the central part becomes higher.

With the repulsion box, particles charged in the early time tend to fall into the aperture between the inclined wall and the repulsion box, followed by the collision of particles near the bottom of a hopper. As a result, particles tend to show higher deposit level in the central part below the repulsion box. A part of particles charged in the early time remains on the repulsion box to make a heap. However, the influence of a repulsion box on the deposit profile is small above the repulsion box.

Comparisons of the calculated results with the experimental ones on the arrangement of particles after charging into a hopper are shown in Fig. 6, including the case for a rectangular hopper with a repulsion box. Good agreement on the deposit profiles and the arrangements of particles in a hopper was obtained.

2. Stress Distribution in the Normal Direction at the Side Wall of a Hopper

Normal force on the side wall of a rectangular hopper in mechanical equilibrium state was analyzed by the model under the condition shown in Table 1 except for $d/L=1/30$. The number of particles used in the calculation was 885. Calculated result for the distribution of dimensionless stress was compared in Fig. 7 with that obtained from Janssen’s Eq. (10), which is familiar as the theoretical prediction for the stress distribution in powder.

$$\frac{fp}{\rho g L} = \frac{1}{4\mu_p} \left(1 - \exp\left(4\mu_{p-w} \frac{H}{L}\right)\right) \quad \text{......(10)}$$

With an increase in depth, the calculated stress at the side wall increases, though some local irregularity is observed. The agreement of calculated result with Janssen’s equation is reasonable except for that on the corner of a hopper. The reason for the decrease in the calculated stress on the corner is that the par-
tides existing there have fewer contact points as is obvious from Fig. 7. Similar reduction of the stress on the corner in actual hoppers may occur, though the region of reduced stress is very restricted to the small area where particles have fewer contact points.

3. Discharging Behavior of Particles from a Hopper

Figure 8 shows the calculated flow behavior of particles during discharge from the hopper obtained from the same calculation condition as shown in Table 1. Solid lines drawn from the center of each particle show the direction of movement and their lengths are proportional to the velocities. Some particles are drawn black to elucidate the successive change of the arrangement of particles. At \( t = 0.1 \) s, the arch just above the discharge slit is collapsed. At \( t = 1.0 \) s, disordered area for the arrangement of particles is enlarged from the slit to upwards together with the formation of concave surface at the top of the bed.

At \( t = 1.5 \) s, the height of a bed is decreased without the remarkable change of surface profile. Particles initially located in the central part and near the discharge slit have large descent velocity, while those initially located near the wall and on the bottom have small descent velocity.

4. Order of Discharged Particles from a Hopper

The calculated result on the order of discharged particles from the rectangular hopper with a slit at the bottom was compared with the experimental one. In the experiment, the acrylic bars with figures on their cross sections were placed in the hopper prior to discharge and the figures were recorded on discharge by the use of a high speed video camera placed in front of a slit.

The calculated order of discharged particles is shown in Fig. 9 together with the experimental one. Particles initially located in the region with low figure in Fig. 9 are discharged in early time. In the larger half time from the beginning of discharge, particles initially located in the area surrounded by inclined lines with an angle of 60° originated from the ends of discharge slit are concentrically discharged in the order of those at the bottom toward those at the top. Formation of 60° sliding angle results from the geometrical condition that the system is composed of particles in equal size, which is expected from the particle movement shown in Fig. 8. Similarity between the calculated order and experimental one for the discharged particle is found to be reasonably good.

5. Transition of Number of Particles Discharged from the Hopper

Transitions for the number of particles discharged from the hopper, which are obtained from the same experiment and calculation condition as those in the preceding chapter are shown in Fig. 10. The number of discharged particles increases monotonously with time both in the calculation and in the experiment, except for the last period of discharge. Independence of the discharging rate from the height of...
packed bed, which is one of the inherent characteristics of granular materials, is clearly described by the calculation. It is found that the transition number of discharged particles obtained from the calculation agrees well with that obtained by the experiment.

Accordingly, the validity of the mathematical model has been confirmed through the comparisons of the calculated results with the experimental ones on the deposit profile, stress at the side wall, the order of discharged particles and the transition number of discharged particles.

III. Application of the Mathematical Model to Size Segregation Phenomena

The solid flow in a hopper with particles consisting of different sizes is important in the ironmaking process. For example, transition of particle size during discharge from the top bunker of a bell-less blast furnace has significant influence on the radial and circumferential distribution of gas permeability in the furnace. An installation of a repulsion box in the upper part of a hopper was experimentally investigated to improve size segregation in a hopper.9 In the present study, the installation of a repulsion box in a hopper was investigated by the mathematical model. The calculation condition is the same as shown in Table 1 except for \( d/L = 1/29 \) and \( 1/58 \) and \( p = 1 \). In addition, the experiment using an acrylic hopper of 29 cm diameter and aluminum balls of 3 and 7 mm diameter was performed.

1. Flow Behavior of Particles during Charge

Figure 11 shows the calculated intermediate state of flow behavior of particles during charge without a repulsion box (case (a)) and with a repulsion box (case (b)). Since the position of peaks in case (b) is related with the falling trajectory of particles, the peak position during charge generally depends on the height as well as the size of a repulsion box.

Distribution of small particles after charge is shown in Fig. 12. The installation of a repulsion box changes the trajectory of particles to affect the distribution of small particles in a hopper.

Without a repulsion box in case (a), small particles segregate in the central part of the hopper. On the contrary, with a repulsion box in case (b), small particles deposit slightly both in the central part and in the wall part, while they deposit more in the middle part where peaks are formed during charge.

For the benefit of quantitative evaluation of the distribution of small particles, the variation of small particles was defined by Eq. (11).

\[
\sigma = \sqrt{\sum_{i=1}^{n} (s_i - \bar{s})^2 / n} \tag{11}
\]
where, \( s \): horizontal distance of small particle from the vertical side wall of the hopper (m)
\( \bar{s} \): average of \( s \) (m)
\( n \): number of small particles; 450 (–).

\( \sigma \)'s calculated are 0.36 and 0.42 for cases (a) and (b), respectively. The installation of the repulsion box suppresses the variation of small particles in the horizontal direction during charge by 17%.

2. Flow Behavior of Particles during Discharge

The effect of the installation of a repulsion box on the discharging order of particles is shown in Fig. 13. Without a repulsion box in case (a), particles initially located near the discharge slit are discharged first. Particles initially located above the slit are subsequently discharged from the bottom to the top. Finally, particles initially located near the wall of a hopper are discharged. Therefore, the solid flow, so called “funnel shaped flow”, is simulated.

With a repulsion box in case (b), the discharge behavior of particles is similar to that of case (a) except for particles initially located around the repulsion box.

Therefore, it is concluded from the calculation with respect to the flow behavior of particles during charge and discharge in a hopper that the installation of a repulsion box at the upper part of a hopper improves the uneven distribution of small particles in the radial direction, while it insignificantly affects the deposit profile and the discharging order of particles.

3. Transition of Particle Size during Discharge

Calculated result on the transition of discharged particle size with time is shown in Fig. 14 together with a three dimensional experimental result using aluminum balls.

In the model calculation, without the repulsion box in case (a), small particles are preferentially discharged in the early period of time, while large ones are preferentially discharged in the final period of time. With the repulsion box in case (b), the size of discharged particle varies less during discharge in contrast with case (a), since the distribution of small particles in the radial direction is improved. Similar tendency was also observed in the experiment as shown in Fig. 14.

It is predicted by the model that the variation of small particles without a repulsion box is \( 3.3 \times 10^{-2} \) and it reduces approximately by 40% due to the installation of a repulsion box. In the calculation of variation \( \sigma \), \( s = n(t) - n_0(t) \) was substituted for \( s \) in Eq. (11), where \( n(t) \) is the calculated total number of discharged small particles at time \( t \) and \( n_0(t) \) is the ideal one at time \( t \) without size segregation.

In the experiment, variation of small particles without a repulsion box was \( 2.9 \times 10^{-2} \) and it reduced approximately by 60%. Although the suppression effect of size variation by the installation of a repulsion box is rather large in the three dimensional experiment, the tendency to the suppression of size variation can be quantitatively described by the mathematical model.

### IV. Discussion

1. Solid Flow

Solid flow in a packed bed with particles was analyzed by the present model (model (a)) and the result was compared with that obtained by the potential flow model (model (b)), which was a typical one in the previous analysis for the solid flow. The shape of a vessel is rectangular and the particles are discharged from the region near the side wall on the bottom.

In model (a), the calculation condition is the same as shown in Table 1 except for \( d/L=1/10 \). Calculation proceeded until the number of particles discharged became twice the number of those initially charged. In model (b), Laplace's equations repre-
sent in the curvilinear coordinate were solved.

\[ g_{22} \frac{\partial^2 \psi}{\partial x^2} - 2g_{12} \frac{\partial^2 \psi}{\partial x \partial y} + g_{11} \frac{\partial^2 \psi}{\partial y^2} = 0 \] ..........................(12)

\[ g_{22} \frac{\partial^2 \phi}{\partial x^2} - 2g_{12} \frac{\partial^2 \phi}{\partial x \partial y} + g_{11} \frac{\partial^2 \phi}{\partial y^2} = 0 \] ..........................(13)

where, \( \psi \): stream function

\( \phi \): velocity potential.

\( g_{ij} \) are metric tensors written by Eqs. (14) to (16).

\[ g_{22} = \psi^2 + \phi^2 \] ..........................(14)

\[ g_{12} = \psi \psi_y + \phi \phi_y \] ..........................(15)

\[ g_{11} = \psi^2 + \phi^2 \] ..........................(16)

The stagnant region of solid flow with a slide angle of 45° was assumed as the boundary condition based on the result derived from model (a).

Figure 15 shows the arrangement of particles whose charging order is expressed by the differently blackened tracers and the equal time line of solid flow obtained by models (a) and (b), respectively. In model (a) tracer particles deposit on the stagnant slope formed by the initially charged tracers denoted as \( \bigcirc \). Therefore, this slope can be regarded as the profile of a deadman. Although the calculated results obtained from the both models are generally in agreement, a slight contradiction is observed in the solid flow near the side wall. As a characteristic of model (b), the descent velocity is the largest on the side wall AB. On the other hand, in model (a), the descent velocity is not necessarily the largest on the side wall AB due to the presence of friction between particles and wall, which leads to the existence of the largest descent velocity at a little inner region from the side wall. This phenomenon was also experimentally observed in the previous study.10

Consequently, the present model can precisely describe the effect of friction between particles and wall on solid flow, while the previous potential flow model can not describe the phenomenon so precisely.

2. Bridge Formation

The bridge formation phenomenon can not be described by the potential flow model, since the outflowing particle from an exit should be supplied from an entrance in consequence of the equation of continuity. On the contrary, the present model may describe the bridge formation phenomenon when the friction coefficient is large, since the model is based on the distinct particles. Therefore, the model calculation was tried for the solid flow in a rectangular hopper shown in Fig. 8 filled with the rough surface particles of \( \rho = 0.72 \). The calculated result is shown in Fig. 16.

At 0.2 s after the beginning of discharge, an arc shaped bridge was formed just above the discharge slit, followed by the stopping of discharge. The solid lines connecting particles represent the normal force acting between particles. The length of the solid line is proportional to the strength of the normal force. Just above the discharge slit, the interaction forces are distributed in arc shape. The interaction force is rather strong in the region between the oblique lines with an angle of 60° originated from both ends of discharge slit and the hopper walls.

Similar phenomenon was also observed in the experiment and the bridge profile is also shown in Fig. 16 by the solid line in bold stroke. The agreement of the calculated result with the experimental one is good.

Accordingly, the present model can precisely describe the bridge formation phenomenon, while the previous models based on the assumption of continuum can not describe it.

V. Conclusion

A mathematical model, capable of describing the macroscopic movement of an assembly of particles, was developed based on the equations of motion on each particle and constitutive equations described by appropriate Voigt-Kelvin rheological model.

The validity of the model was confirmed through experiments with respect to the order of particles discharged from a hopper, the transition of the number
of discharged particles, the arrangement of particles after filling a hopper, and the calculation with respect to stress distribution in the vertical direction of a hopper obtained from Janssens' equation.

The model led to a better fundamental understanding for the flow behavior of particles during charge and discharge in a hopper and the following results were obtained.

1. The installation of a repulsion box in the upper part of a hopper significantly affects the trajectory of particles, while it affects slightly the deposit profile and the discharge order.

2. The trajectory of particles during charge affects the small particle distribution in a hopper. Without a repulsion box, small particles segregate in the central part of a hopper, while with a repulsion box, they segregate in the middle part.

3. The installation of a repulsion box in a hopper using particles consisting of different size decreases the particle size variation during discharge through the suppression of small particle segregation during charge.

4. The present model can precisely describe the frictional wall effect in solid flow and bridge formation.

**Nomenclature**

- $a$: length of a cylindrical particle (m)
- $d$: diameter of a cylindrical particle (m)
- $D$: dashpot factor (kg/s)
- $f$: force (N)
- $g$: gravitational acceleration (m/s$^2$)
- $g_{ij}$: metric tensor (—)
- $H$: height of hopper (m)
- $K$: elastic constant of spring (kg/s$^2$)
- $L$: width of hopper (m)
- $l$: displacement of particle (m)
- $m$: mass of particle (kg)
- $n$: number of particle (—)
- $p$: stress in the vertical direction of packed bed (N/m$^2$)
- $t$: time (s)

$X$: coordinate in horizontal direction (m)

$Y$: coordinate in vertical direction (m)

$\theta_{ij}$: angle between particle $i$ and particle $j$ (rad)

$k$: soil pressure coefficient $= \frac{1 - \sin \mu_{p-p}}{1 + \sin \mu_{p-p}}$ (—)

$\mu$: friction coefficient (—)

$H$: $mg/Kd$ (—)

$\rho$: density of particle (kg/m$^3$)

$\sigma$: variation (—)

$\Phi$: velocity potential (m$^2$/s)

$\Psi$: stream function (m$^2$/s)

$\phi$: rotation angle of particle (rad)

**Subscript and Superscript**

- $i, j$: identification for particle
- $n$: normal direction
- $p-p$: particle–particle
- $p-w$: particle–wall
- $s$: shearing direction
- $\cdot$: time derivative

**REFERENCES**