Crack Arrest by Strong Short Fibers in a Brittle Composite

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Using the mechanics of inclusions, the effect of strong fibers on the arrest of a mode I crack in a brittle material has been studied. Two effects of fibers on the crack arrest are considered; bridging action and sliding. The fibers bridge the crack and the bridging action reduces the energy release rate during the crack extension. The sliding of the fibers results in an effective increase in the fracture toughness, since the sliding under non-vanishing friction on matrix-fiber interfaces dissipates energy. The reduction in the energy release rate and the increase in the fracture toughness are given in analytical forms which involve the size of a crack, the geometry and distribution of fibers and external and frictional stresses. Diagrams to indicate the stability and growth of a crack reinforced by fibers are presented.

KEY WORDS: composite; fiber; toughening; inclusion; sliding.

1. Introduction

A brittle material such as ceramics can be toughened by mixing strong fibers to form a composite. The presence of fibers, which are stronger or tougher than a brittle matrix, can suppress the extension of a crack in a composite. This effect comes from two causes. First, fibers bridge the two surfaces of a crack and lowers the opening of the crack. The bridging action is operative as long as resistance to pulling-out of fibers exists. The resistance is caused by debonding of and friction at the interface between matrix and fibers. Second, the pulling-out of fibers with non-vanishing friction along interfaces dissipates energy. The energy dissipation results in an effective increase in the resistance to the extension of a crack.

This subject was studied on the basis of a standard fracture mechanics approach by Marshall et al.,1) Budiansky et al.,2) also analyzed the effect of fibers on the extension of a crack, using the shear lag model by Avenston and Kelly3) for deformation around fiber-matrix interfaces. Rose adopted a simpler model in which fibers act just as springs existing inside a crack and offers a bridging action.4) In a more sophisticated manner, Hori and Nemat-Nasser analyzed the bridging action of fibers under rather complicated circumstances.5)

As shown in the celebrated paper of Eshelby,6) the mechanics of a crack can be examined by the inclusion method which involves far simpler calculations. Using this advantage, Mori and Mura analyzed the effect of fibers on the arrest of a crack in a composite with a brittle matrix.7) The analysis was limited to complete bridging. Mori et al. extended this analysis to the case where fibers could be pulled out.8) In the present paper, we will extend the analysis and perform more precise analyses. These analyses will be conducted, similar to the previous study,9) under the assumption that bridging fibers can slide uniformly along the length of the fibers when a critical interfacial shear stress is reached. We will present some diagrams, with the stability and extension of a bridged crack can be readily discussed. The main point of the present study is that every relevant expression is given in an analytical form.

2. Model and Analysis

2.1. Model and Stress Analysis

Let us consider a crack, perpendicular to the X₃ direction, containing many aligned and randomly distributed short fibers which have radius r and length 2l with average spacing λ as schematically shown in Fig. 1. We approximate the shape of a crack Ω₀ by a thin oblate spheroid as

\[ \frac{(X_1^2 + X_2^2)}{a^2} + \frac{X_3^2}{c^2} \leq 1, \quad \frac{c}{a} \ll 1. \]  

(1)

a corresponds to the radius of the penny-shaped crack. 2t is the fictitious thickness of the crack. As will be seen later, the absolute value of c does not appear in the relevant equations to be derived later. The cross section Ω of a fiber in the crack is also approximated as

\[ \frac{(X_1^2 + X_2^2)}{r^2} + \frac{X_3^2}{c^2} \leq 1, \quad \frac{c}{r} \ll 1. \]  

(2)

A remote uniform tensile stress, σ₀₄=σ₄', is applied. To simulate a change in the stress state caused by the bridged crack, the eigenstrains ε₀₅ are introduced into Ω₀ and Ω's. Specifically,

\[ \varepsilon_5 = \varepsilon_p \text{ in } \Omega_0 - \Omega's, \]  

(3)

\[ \varepsilon_5 = \sigma_5 \text{ in } \Omega's. \]  

(4)

Here, σ₅ is the parameter which characterizes the slid-
International, occurs effect), 1989, following 10. The fibers have radius \( r \) and length \( 2l \) with the average spacing of \( \lambda \).

![Diagram of fibers and crack](image)

Fig. 1. A crack of radius \( a \) is bridged by aligned short fibers.

When the fibers along fiber–matrix interfaces when a critical interfacial shear stress is reached. For the complete bridging (no sliding of fibers), \( \alpha = 0 \) and for the complete sliding of the fibers (no bridging effect), \( \alpha = 1 \). \( \varepsilon_p \) is determined by the condition that the sum of the external stress and internal stress due to Eqs. (3) and (4) is zero in \( \Omega_0 - \Omega \)'s unbridged domain of the crack. The introduction of \( \varepsilon^*_p \) in \( \Omega_0 - \Omega \)'s and \( \Omega \)'s is equivalent to the following hypothetical sequence. First, \( \varepsilon^*_p = \varepsilon_p \) is given to the whole domain of \( \Omega_0 \)

\[ \varepsilon^*_p = \varepsilon_p \text{ in } \Omega_0. \] ..........................(5)

Next, the additional eigenstrain of \( \varepsilon^*_p = (\alpha - 1)\varepsilon_p \) is given to \( \Omega \)'s

\[ \varepsilon^*_p = (\alpha - 1)\varepsilon_p \text{ in } \Omega. \] ..........................(6)

This hypothetical sequence combined with the results of Eshelby makes the calculation of stresses simple. The stress due to Eq. (5) is uniform in \( \Omega_0 \) and that due to Eq. (6) uniform in \( \Omega \)'s. Further, conducting the averaging process employed by Mori et al.,\( ^\text{80} \) we have the average stresses in \( \Omega_0 - \Omega \)'s and \( \Omega \)'s (unbridged domains) given as

\[ \sigma_{\alpha, \gamma} = \frac{-\beta \pi a \varepsilon_p}{2(1-\varepsilon)} - \frac{\beta \pi (1 - \alpha) a \sigma_p}{2(1-\varepsilon)} \] ..........................(7)

\[ \sigma_{\alpha, \gamma} = \frac{-\beta \pi a \varepsilon_p}{2(1-\varepsilon)} + \frac{\beta \pi (1 - \alpha) a \sigma_p}{2(1-\varepsilon)} \] ..........................(8)

Here, \( \langle \rangle \) denotes the average defined in the respective domain. \( \beta \) is the shear modulus and \( \pi \) the Poisson ratio of the composite under consideration. \( \beta \) and \( \pi \) are approximated by the law of mixture. \( f \) is the volume fraction of fibers and is equal to \( \pi^2/\lambda^2 \). See Ref. 8) for details of the calculations.

The condition for \( \varepsilon_p \) to simulate the crack bridged by the fibers is written as

\[ \sigma_{\alpha} + \langle \sigma_{\beta} \rangle _{\alpha, \gamma} = 0. \] ..........................(9)

Eqs. (7) and (9) lead to

\[ \varepsilon_p = \frac{2(1-\varepsilon) \sigma_{\alpha}}{\beta \pi c [1 + (1 - \alpha)(a/r) f]} \] ..........................(10)

We will determine \( \alpha \) in Eq. (10) in the next section.

2.2. Energy Calculation and Determination of \( \alpha \)

The mechanical Gibbs free energy \( F \), the sum of the elastic strain and external potential energies, of the body containing the crack is calculated by the expression

\[ F = \frac{1}{2} \int_{\Omega_0} \sigma_{ij}^* \varepsilon_{ij}^* \, dV - \int_{\Omega_0} \sigma_{ij} \varepsilon_{ij}^d \, dV. \] ..........................(11)

\( \varepsilon^*_p \) is given by Eqs. (3) and (4) and \( \sigma_{ij} \) by Eqs. (7) and (8). \( F \) is, of course, the additional term due to the presence of the crack. Since \( \varepsilon_p \) is given in Eq. (10), Eq. (11) is approximately calculated as

\[ F = F_0 - \frac{2k(1/\alpha)((a/r) f)}{3\beta} \] ..........................(12)

with

\[ F_0 = \frac{4(1-\varepsilon)a^2 \sigma_p^3}{3\beta} \] ..........................(13)

The approximation is that the higher term in \( f \) can be neglected. \( F_0 \) is the Gibbs free energy for the crack without bridging fibers.

Next we will determine \( \alpha \). The total stress, the sum of the external and internal stresses, in a fiber in the crack is written as

\[ \sigma_{\alpha}(\Omega) = \sigma_{\alpha}(1 - \alpha) (a/r) 1 + (1 - \alpha) (a/r) f \] ..........................(14)

from Eqs. (8) and (10). When \( \sigma_{\alpha}(\Omega) \), the end stress of a fiber, is small, friction along the matrix-fiber interface completely suppresses the sliding (pulling-out) of the fiber, the case of \( \alpha = 0 \). This case occurs when \( \sigma_{\alpha} \) and \( a \) are small. However, when \( \sigma_{\alpha}(\Omega) \) becomes sufficiently large, the fiber starts to slide against the retarding action of the friction, \( \alpha > 0 \). We can discuss the critical condition for the sliding to occur. When the sliding occurs, the end stress and the total force due to the friction balance,

\[ \sigma_{\alpha}(\Omega) \pi^2 = 2\pi^2 k. \] ..........................(15)

Here, \( k \) is the frictional stress. Thus, Eqs. (14) and (15) give the condition for the onset of the sliding,

\[ \sigma_{\alpha} = 2k(l/\alpha)(1 + (a/r) f) \] ..........................(16)

When \( \sigma_{\alpha} > \sigma_{\alpha} \), as we have assumed that the fiber sliding occurs uniformly along the whole length and \( k \) is constant, \( \alpha \) is given by

\[ \alpha = 1 - \frac{2k(l/\alpha)}{\sigma_{\alpha} - 2k(-\alpha) f}. \] ..........................(17)

When \( \sigma_{\alpha} \leq \sigma_{\alpha} \), \( \alpha = 0 \) (complete bridging), using Eq. (17), the Gibbs free energy Eq. (12) is rewritten as

\[ F = -\frac{4(1-\varepsilon)a^2 \sigma_p^3}{3\beta(1 + (a/r) f)}, \quad (\alpha = 0), \] ..........................(18)

and

\[ F = -\frac{4(1-\varepsilon)a^2 \sigma_p^3}{3\beta[1 - (1 - \alpha)(a/r) f]} \] ..........................(19)

When \( \sigma_{\alpha} > \sigma_{\alpha} \), \( \alpha > 0 \) (pulling-out of fibers), resulting in the energy dissipation. In this case, the crack opening, \( \delta \), is defined as

\[ \delta = 2ca \varepsilon_p \] ..........................(20)
at the bridged parts. Thus, the sliding of a single fiber accompanies the energy dissipation \( w \) given by

\[
 w = 2\delta\pi k l = \frac{8\pi k l r (1-\nu) (\sigma_\alpha - 2 f k l r) - 2kl}{\mu}.
\]

Since the number of the fibers in the crack is \( \pi a^2/2 \), the total dissipated energy \( W \) is calculated as

\[
 W = \frac{8\pi k l r (1-\nu) (\sigma_\alpha - 2 f k l r) - 2kl a^2}{\mu a^2},
\]

\( (a>0 \text{ or } \sigma_\alpha > \sigma_\alpha). \)

\[\text{.........(22)}\]

2.3. Energy Release Rate and Friction-induced Resistance to the Extension of a Crack

The energy release rate \( G \) for the extension of a crack is defined as \( -\partial W/\partial(\pi a^2) \). Thus, from Eqs. (18) and (19), we have

\[ G = G_0 \frac{1+(2/3)(a/r) f}{(1+a/r) f^2}, \quad a = 0, \quad \text{.........(23)} \]

\[ G_0 = \frac{2(1-\nu) \sigma_\alpha}{\pi b}, \quad \text{.........(24)} \]

\[ G = \frac{2(1-\nu) r [(3a^2 - 12 f/2(k_1^2 (l/r)^2 f^2) - 2k_2 (l/r)^2 f)]}{3\pi b^2}, \quad \alpha > 0. \quad \text{.........(25)} \]

The resistance, \( \Delta G_f \), caused by the friction-induced energy dissipation, is similarly defined by \( \partial W/\partial(\pi a^2) \). It is calculated as

\[ \Delta G_f = \frac{4(1-\nu) kl a [(3a^2 - 12 f/2(k_1^2 (l/r)^2 f^2) - 2k_2 (l/r)^2 f)]}{\mu a^2}, \quad \alpha > 0. \quad \text{.........(26)} \]

Of course, when fiber sliding does not occur,

\[ \Delta G_f = 0, \quad a = 0. \quad \text{.........(27)} \]

3. Discussion

We will discuss the growth and stability of a crack in a fiber reinforced composite, where matrix has fracture toughness \( G_c \). A crack is stable when \( G_c + \Delta G_f > G \) and extends when \( G_c + \Delta G_f < G \). A convenient method to discuss this subject is to draw diagrams where \( G_c + \Delta G_f \) and \( G \) are plotted against the crack size \( a \). Using the results in the previous section, we have drawn such diagrams for various combinations of external and frictional stresses and the geometry and distribution of fibers. Fig. 2, the case for \( 4f_2^2 < G_c < 8/\pi^2 \), covers all the possible cases with which the stability and extension of a bridged crack can be examined. Here, \( s \) is the aspect ratio \( l/r \) of fibers and \( G_c \) the normalized fracture toughness of the matrix defined by

\[ G_c = \frac{G_c}{N}, \quad \text{.........(28)} \]

with

\[ N = \frac{2(1-\nu) l r^2}{3\bar{f} \pi}, \quad \text{.........(29)} \]

Fig. 2. Normalized energy release rate \( \tilde{G} \) and the sum of the normalized fracture toughness \( \tilde{G}_c \) and friction-induced resistance \( \tilde{\Delta G}_f \) are plotted against the normalized crack size \( \tilde{a}(=a/r) \).
In the following discussion and diagrams, \( \Delta G_f \) and \( G \) are similarly normalized by \( N \). \( a \) is also normalized by \( \pi r \) to \( \tilde{a} (= a/\pi r) \).

Fig. 2(a) applies for the case when the friction at the matrix–fiber interface completely suppresses the sliding of fibers, the case of \( \Delta G_f = 0 \). When \( \sigma_d/k > \sqrt{\Delta G_f/2} \), \( G \) of any crack is less than \( G_c \) and the crack is stable. If \( \sigma_d/k \) exceeds \( \sqrt{\Delta G_f/2} \) but is less than \( 2\beta s \), the crack larger than \( \tilde{a}_c \), indicated in Fig. 2(a), extends to an infinite size. From Eqs. (23) and (28), \( \tilde{a}_c \) is obtained as

\[
\tilde{a}_c = \frac{2G_c f - 3(\sigma_d/k)^2 + \sqrt{9(\sigma_d/k)^4 - 4G_c f(\sigma_d/k)^3}}{4f(\sigma_d/k)^2 - 2G_c f^3}.
\]

However, the composite does not fracture, as the crack is supported by fibers which cannot slide because of the friction.

Fig. 2(b) is for the range of \( 2\beta s < \sigma_d/k < \sigma_c \) (\( \sigma_c \) will be defined later). In this case, a crack larger than \( \tilde{a}_c \) extends to \( \tilde{a}_c \). However, if the crack further extends slightly, fibers start to slide and \( G_c + \Delta G_f \) exceeds \( G \). Thus, the crack remains stable at \( \tilde{a}_c \). \( \tilde{a}_c \) is determined as

\[
\tilde{a}_c = \frac{2s}{\sigma_d/k - 2\beta s}, \quad \text{(31)}
\]

from the condition of \( \sigma_d = \sigma_c \). \( \tilde{a}_c \) is the crack size at which the pulling-out of fibers starts. \( G \) jumps at \( \tilde{a} = \tilde{a}_c \). This is firstly because \( F \) depends on \( a \), Eqs. (18) and (19), and secondly because \( \alpha \) does not depend on \( a \) and remains zero when \( a < \alpha_c \), while \( \alpha \) depends on \( a \) when \( a > \alpha_c \). Thus, \( G \) defined by \(-\dot{\Phi}/\dot{a}(\pi a^2) \) changes discontinuously at \( \dot{a} = \tilde{a}_c \).

When an external stress is in the range of \( \tilde{a}_c < \sigma_d/k < 4\beta s \), the \( G \) and \( G_c + \Delta G_f \) vs. \( a \) curves take the forms given in Fig. 2(c). In this case, \( G_c + \Delta G_f \) is less than \( G \) for \( a > \tilde{a}_c \). Thus, when the size of an existing crack is in this range, the crack extends to \( \tilde{a}_c \). Of course this process accompanies the sliding of fibers (\( \alpha > 0 \)) when or after the crack size becomes larger than \( \tilde{a}_c \). \( \tilde{a}_c \) is determined as

\[
\tilde{a}_c = \frac{-16\beta s^2 + \tilde{G}_c}{3(\sigma_d/k - 2\beta s)(\sigma_d/k - 4\beta s)}, \quad \text{(32)}
\]

from the solution of \( \tilde{G}_c + \Delta G_f = \tilde{G}_c \). \( \tilde{a}_c \) is the value of \( \sigma_d/k \) with which \( \tilde{a}_c = \tilde{a}_c \) is satisfied.

Fig. 2(d) is for the case of \( 4\beta s < \sigma_d/k \). A crack larger than \( \tilde{a}_c \) extends to an infinite size. Different from the case of Fig. 2(a), the resistance by the friction at the matrix–fiber interfaces cannot support the external force and the composite material fractures.

Fig. 3 summarizes the above discussion and shows the path of the crack extension and the stability of a crack. It also shows whether fibers are pulled out (\( \alpha > 0 \)) or not (\( \alpha = 0 \)). Suppose that a crack, indicated by line 2 exists and we apply an external stress \( \sigma_d \) gradually. When the stress is in region (1), the crack remains in the original size. When the stress reaches the value indicated by \( \tilde{a}_c \), the crack suddenly extends to \( \tilde{a}_c \) and stays there. As the stress further increases, the fibers continue to slide but the crack keeps its size, \( \tilde{a}_c \). When the stress becomes equal to that given by line \( \tilde{a}_s \), the crack again starts to extend along line \( \tilde{a}_s \) as the stress increases. The material eventually fractures when the stress becomes \( 4\beta s \). The path of the response of a crack to an external stress is depicted by the arrows on the dashed lines and line \( \tilde{a}_d \) in Fig. 3. Whether the crack extends or remains stable can be discerned by examining the domain occupied by the combination of the stress and crack size, domains (1)—(5) in Fig. 3. These domains are bordered by the \( \tilde{a}_1 \), \( \tilde{a}_2 \) and \( \tilde{a}_3 \) lines. The positions and crossing points of these lines depend on \( C_0, \ k, f \) and \( s \). However, only Figs. 4 to 6 in addition to Fig. 3 describe the characteristics of the crack size—stress diagrams with which the extension and stability of a crack bridged by fibers can be discussed. Fig. 4 applies for \( 0 < \tilde{G}_c < 4\beta s^2 \), Fig. 5, \( 8\beta s^2 < \tilde{G}_c < 16\beta s^2 \) and Fig. 6, \( \tilde{G}_c > 16\beta s^2 \). The dashed-dotted lines in Figs. 3 to 6 divide the stable from the unstable region of a crack without fibers, the case of a Griffith crack. The reinforcement effect of fibers is exhibited in such a manner that all the solid lines, \( \tilde{a}_1 \), \( \tilde{a}_2 \) and \( \tilde{a}_3 \), lie in the upper-right region from the dash-dotted line. The shift from the dash-dotted line to line \( \tilde{a}_2 \) is caused directly by the bridging effect of fibers. The presence of lines \( \tilde{a}_1 \) and \( \tilde{a}_3 \) is due to the fiber sliding which dissipates energy needed to extend a crack bridged by fibers. A crack keeps its size in the region bounded by lines \( \tilde{a}_1 \) and \( \tilde{a}_3 \) during an increase in an external stress, while the amount of fiber sliding increases. The further extension of the crack occurs along line \( \tilde{a}_3 \) after an external stress reaches line \( \tilde{a}_2 \). This indicates that a crack smaller than a critical size always grows in a stable manner until an external stress becomes equal to \( 4\beta s \). The critical size is obtained by putting \( \sigma_d/k = 2\beta s \) in Eq. (30). There is another critical size, given by \( \tilde{a}_1 \) with \( \sigma_d/k = 4\beta s \). A crack smaller than this size extends to an infinite size and complete fracture occurs when an external stress becomes equal to or larger than \( 4\beta s \).

Figs. 3 to 6 also tell an effective method to toughen a brittle material with strong fibers. It is clear that for a given volume fraction the mixing of smaller
radius fibers result in larger toughening. Physically, this is due to larger interfacial areas possessed by finer fibers. Similarly, fibers with a larger aspect ratio are more effective to toughening. This is demonstrated in Figs. 3 to 6 where the abscissas are represented by $f_s$. The physical reason for this effect is identical to that for the effect of the fiber radius.

The above discussion is valid for fibers which are extremely strong and do not fracture. We can account for the fracture of fibers with finite strength. The maximum tensile stress, $\sigma_u(L)$, of a fiber in a crack is obtained by Eq. (15), when sliding does not occur. When the sliding occurs, the maximum stress is of course

$$\sigma_u(L) = 2ks, \quad a>0.$$  \hspace{1cm} (33)

Thus, we can divide the effect of a finite strength $\sigma_f$ of fibers on the extension of a crack into two cases; $\sigma_f < 2ks$ and $\sigma_f > 2ks$. Two examples of the former case are shown by dotted lines 1 and 2 in Fig. 7, where lines $\sigma_1$, $\sigma_2$, and $\sigma_3$ in Fig. 3 are reproduced. These lines, 1 and 2, are given by equating Eq. (14) ($a=0$) to $\sigma_f$. The upper-right region of these lines loses its meaning discussed before. When $\sigma_f > 2ks$, the solid lines in Fig. 7 are always valid, since the maximum stress of the fibers in a crack is less than the strength of the fibers.

4. Conclusion

Using an inclusion method, the toughening by mixing strong short fibers into a brittle material has been examined. The fibers suppress the extension of a crack through two effects. One is the direct effect of fiber bridging which prevents the opening of the crack. The other is caused by the fiber sliding which accompanies energy dissipation due to friction when the crack extends. These effects are separately evaluated and formulated as the energy release rate of a fiber bridged crack and the resistance to the crack extension by the energy dissipation. The crack size-external stress diagrams are presented with which the stability and extension path of the bridged crack can be readily discerned. The effect of the strength of fibers is also discussed.

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