Analytical Procedure on Nozzle Flow of a Two-phase Mixture

1. Introduction

Nozzle flows of a gas carrying suspended condensed particles are of great technical importance for various engineering applications. A system of equations, which had previously been described by Zucrow and Hoffman, was rearranged so that the system gives a fit to the case where a continuous distribution function of particle size is present. The system of equations governs steady and quasi-one-dimensional nozzle flows of gas–particle mixtures.

The theoretical model was applied to the case where a gas-particle mixture is composed of air and water-particles in relation to the mist nozzle flow utilized to the secondary cooling zone of continuously cast slabs (see Refs. 2 and 3).

The incipience of the present paper is on the basis of the comments of a reviewer of our paper, which was previously submitted. He pointed out that it seems not to be easy for the general reader to follow the theoretical procedure, because the governing equations to be solved have been expressed on the dimensionless space, and that the perturbation procedure should be addressed in full length.

Then we wish to note the general concept of the system of equations governing the nozzle flow of a gas-particle mixture. Also, we consider the problem concerning the perturbation procedure from the equilibrium flow to the nonequilibrium one from a numerical point of view.

2. Equations for Nozzle Flow

The analysis is based on the several usual assumptions (see Refs. 2 and 3). It should previously be remembered that the governing equations are followed on the dimensional space, and the dimensional quantities are denoted by the overbar in relation to Refs. 2 and 3).

The constant mass flow rate of the particles is given by

\[
\overline{M}_p = \int \overline{n}_p(\xi)p_0 d\xi_p = \int \overline{\rho}_p(\xi_p)A\overline{V}_p(\xi_p)d\xi_p = \int \overline{\rho}_p(\xi_p)(4\pi/3)p_0^{2/3}A\overline{V}_p(\xi_p)d\xi_p = \text{constant} \tag{1}
\]

in which the above indicates the definite integration taken over all sizes in \([\xi_p, \xi_p, \xi_p]\). It is noted that for the indication of the integration with respect to \(\xi_p\) the lower and upper limits will be omitted in the later also.

Next, the particle momentum equation for a particle of \(\xi_p\) yields

\[
\frac{4}{3} \pi^3 \rho_0 \beta_{np} \overline{P}_p(\xi_p) \frac{d}{d\xi} \overline{P}_p(\xi_p) = \frac{1}{2} C_p \pi^2 \beta_0 |\overline{V} - \overline{P}_p(\xi_p)| |\overline{V} - \overline{P}_p(\xi_p)| \tag{2}
\]

In the Stokes flow regime, the drag coefficient is expressed as

\[
C_{p,\text{stokes}} = \frac{24}{Re_p} = \frac{24\beta_0}{(2\xi_p) |\overline{V} - \overline{P}_p(\xi_p)|} \tag{3}
\]

Introducing the drag coefficient \(C_{p,\text{stokes}}\) by definition, we put

\[
f_p = C_{p,\text{stokes}} = C_{p,\xi_p} |\overline{V} - \overline{P}_p(\xi_p)| / 12 \beta_0 \tag{4}
\]

Then, combining Eqs. (2) and (4), one has

\[
\overline{P}_p(\xi_p) \frac{d}{d\xi} \overline{V}_p(\xi_p) = \frac{(9/2)\beta_0}{\rho_0 \xi_p^2} \int \overline{V} - \overline{P}_p(\xi_p) \tag{5}
\]

Again, the particle energy equation for a particle of \(\xi_p\) is

\[
\frac{4}{3} \pi^3 \rho_0 \beta_{np} \overline{P}_p(\xi_p) \frac{d}{d\xi} \overline{h}_p(\xi_p) = \overline{a} \{ \overline{\rho}_p(\xi_p) [\overline{T} - \overline{T}_p(\xi_p)] \} \tag{6}
\]

The Nusselt number and the Prandtl number are defined by

\[
\eta = \overline{a}(2\xi_p)|\overline{V}| \quad \text{and} \quad \text{Pr} = \overline{C}_{pp} \beta_0 |\overline{V}| \quad \tag{7}
\]

In the Stokes flow regime, \(\eta_{\text{stokes}} = 2\). By definition, let

\[
g_p = \eta / \eta_{\text{stokes}} = \eta / 2 \quad \tag{8}
\]

Combining Eqs. (7) and (8), we have

\[
\overline{a} = g_p \overline{\rho}_p (\xi_p |\overline{V}| \eta) \quad \tag{9}
\]

Substituting Eq. (9) into Eq. (6) gives

\[
\overline{P}_p(\xi_p) \frac{d}{d\xi} \overline{h}_p(\xi_p) = \frac{3g_p \overline{C}_{pp} \beta_0}{Pr \overline{\rho}_p \xi_p^2} \{ \overline{T} - \overline{T}_p(\xi_p) \} \quad \tag{10}
\]

Here, incorporating the particle equation of state,

\[
\overline{h}_p(\xi_p) = \overline{C}_{pp} \overline{T}_p(\xi_p) \quad \tag{11}
\]

into Eq. (10), we obtain
\[
\frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} = \frac{3\beta \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right)}{\rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right)} \left[ T - T_0 \right] \]
\]

Let us now derive the total drag force \( \delta \vec{D} \) exerted by all of the particles on the gas. For a particle of \( \tau_p \),
\[
\delta \rho \tau_p \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \frac{d}{dx} \begin{bmatrix} V_p \left( \tau_p \right) \end{bmatrix} = \delta \vec{D}_p \left( \tau_p \right) \quad \text{(13)}
\]
where, \( \delta \rho \tau_p \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \) is the mass of a particle of \( \tau_p \),
\( \delta \vec{D}_p \left( \tau_p \right) \); drag force on this particle.
So, \( \delta \vec{D} \) inside of the control volume is simply given by
\[
\delta \vec{D} = \int \delta \vec{D}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \quad \text{(14)}
\]
It follows from Eqs. (13) and (14) that
\[
\frac{d}{dx} \delta \vec{D} \left( \tau_p \right) = \int \delta \vec{D}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \quad \text{(15)}
\]
Here, \( \delta \vec{D} \left( \tau_p \right) \): the total drag force per unit volume.
Next, the heat transfer rate \( \delta \vec{Q}_p \left( \tau_p \right) \) between a particle of \( \tau_p \) and the gas is expressed as
\[
\delta \vec{Q}_p \left( \tau_p \right) = \delta \vec{T}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \frac{d}{dx} \begin{bmatrix} V_p \left( \tau_p \right) \end{bmatrix} \quad \text{(16)}
\]
and
\[
\delta \vec{Q} \left[ \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \right] \delta \vec{D} \left( \tau_p \right) = 0 \quad \text{(17)}
\]
in which \( \delta \vec{Q} \) indicates the total heat transfer rate for all of the particles per unit mass of the gas inside of \( \delta \vec{A} \). Incorporating Eq. (16) into Eq. (17) gives
\[
\delta \vec{Q} = \int \delta \vec{Q}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \quad \text{(18)}
\]
Here, it should be noted that \( \delta \vec{Q} \) is the energy per unit time and per unit volume.
All of the work expended on the particles by the gas leads to an increase in the kinetic energy. So that,
\[
\delta \vec{W}_p \left( \tau_p \right) = \delta \vec{T}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \frac{d}{dx} \begin{bmatrix} V_p \left( \tau_p \right) \end{bmatrix} \quad \text{(19)}
\]
and
\[
\delta \vec{W} \left[ \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \right] \delta \vec{D} \left( \tau_p \right) = 0 \quad \text{(20)}
\]
Substituting Eq. (19) into the above equation, we have
\[
\delta \vec{W} = \int \delta \vec{W}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \quad \text{(21)}
\]
We will now turn to the problem of the equations governing the gas-phase. The gas continuity equation is given by
\[
\dot{N}_g = \rho \dot{A} \dot{V} = \text{constant} \quad \text{(21)}
\]
The gas momentum equation is obtained by taking into account the total drag force per unit volume presented in Eq. (15) as
\[
\frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} = \frac{1}{\rho} \frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} + \int \frac{\delta \vec{W}_p \left( \tau_p \right)}{\rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right)} d\vec{A} \quad \text{(22)}
\]
Next, the gas energy equation is derived by the conservation law of energy. It is convenient to express the relationship on a rate basis as
\[
\frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} + \frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} = 0 \quad \text{(23)}
\]
It follows from Eqs. (1) and (21) that
\[
\frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} = \int \frac{\delta \vec{Q}_p \left( \tau_p \right)}{\rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A}} \quad \text{(24)}
\]
Integrating the aforementioned with respect to \( x \), we obtain
\[
\begin{array}{c}
\delta \vec{Q} = \int \delta \vec{Q}_p \left( \tau_p \right) \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A} \\
\int \int \frac{\delta \vec{W}_p \left( \tau_p \right)}{\rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) d\vec{A}} \\
\end{array}
\]
in which \( \delta \vec{Q} \) and \( \delta \vec{W}_p \left( \tau_p \right) \) are the stagnation enthalpy for gas and particle of \( \tau_p \), respectively.
Finally, the gas equation of state is given by
\[
\frac{d}{dx} \begin{bmatrix} \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \end{bmatrix} = \delta \vec{D} \quad \text{(25)}
\]
Up to this point we have completed the system of equations governing the nozzle flow of gas-particle mixture on the dimensional space. The system completely determines the flow if the inlet condition and the nozzle geometry are prescribed.

3. Introduction of Perturbation Method to Numerical Procedure

When \( \rho \left( \frac{P}{\rho} \right) \theta \rho \left( \frac{P}{\rho} \right) \) and \( \delta \vec{D} \) for all \( \tau_p \), the particles are said to be in velocity and thermal equilibrium with the gas. Again, such a mixture flow is called the equilibrium flow. While the flow obeying the aforementioned equations is termed the nonequilibrium flow.

The nonequilibrium flow can be treated as a perturbation from an equilibrium reference flow. Therefore, we briefly describe the problem of the constant
lag approximation. Let us now define the following two lag factors,
\[ K_p(t_p) = \frac{\bar{V}_p(t_p)}{\bar{V}} \quad \text{and} \quad L_p(t_p) = \frac{T_p - T}{T - T_0} \]
(27)
Again, the relation between \( K_p(t_p) \) and \( L_p(t_p) \),
\[ L_p(t_p) = \left[ 1 + 3 \beta p \frac{C_{pp}^p}{C_{pp}^g} \left( 1 - K_p(t_p) \right) \right]^{-1} \]
(28)
is valid on the specified assumptions, and can easily be derived.

Now, consider the constant lag flow \( (K_p(t_p) = \text{constant}) \). Substituting Eq. (27) into Eq. (5) yields
\[ \frac{dV}{dx} = \frac{(9/2)\bar{h} f_p}{\beta_{pp} p} f_p - \frac{1 - K_p(t_p)}{K_p(t_p)} \]
(29)
Here, one should bear in mind that \( dV/dx \) is not a function of \( t_p \), and that \( f_p = 1 \) in the Stokes flow regime. Thus,
\[ K_p(t_p) = \left[ (1 + 4 \theta_p)^{1/2} - 1 / 2 \theta_p \right] (\theta > 0) \]
(30)
and
\[ K_p(t_p) = 1 \quad (\theta = 0) \]
(31)
in which
\[ \theta = \frac{1}{G} \frac{dV}{dx} = \text{constant} \quad (G = \frac{9}{2} \beta_{pp}) \]
(32)
Again, we should like to note that \( L_p(t_p) = 1 \) if \( K_p(t_p) = 1 \) for all \( t_p \), and the mixture directly flows through a nozzle. It is of course to consider that the velocity of mixtures is zero at the reservoir. The initial situation where \( V = 0 \) and \( A \rightarrow \infty \) does not enable us to promote the numerical calculations at the initial computational step. So, in the numerical sense, it is necessary to assume that all of the particles are in velocity and thermal equilibrium with the gas only in the short distance from reservoir. For the case of equilibrium flows, the governing equations can be reduced to the identical form as that for the equations describing the steady and quasi-one-dimensional flow of a perfect gas, except that only the ratio of the specific heats \( \gamma \) in the isentropic flow equations is replaced by the modified parameter \( \bar{\gamma} \) (see Ref. 2).

In short, all of the flow properties in the nonequilibrium flow may be obtained by the perturbation between the equilibrium and nonequilibrium flows.

4. Concluding Remarks

We have derived the system of equations governing the nozzle flow of a gas-particle mixture on the dimensional space, and discussed the general concept of equilibrium flow in the mixture and the problem of the perturbation procedure from the equilibrium flow to the nonequilibrium one.

We believe that this paper is useful for understanding the physical meaning of equations governing the nozzle flows of two-phase mixtures. In this sense, this is a medium for the presentation of interpretations in connection with the papers previously published in the Institute journal.5,6

In conclusion, we hope that our theoretical model is widely applied to the establishment of the conditions for optimum nozzle performance.

Nomenclature

\[ \begin{align*}
A: & \quad \text{sectional area of nozzle} \\
C_p: & \quad \text{drag coefficient} \\
C_{pp}^p: & \quad \text{gas specific heat at constant pressure} \\
C_{pp}^g: & \quad \text{specific heat of particle material} \\
\bar{h}, \bar{h}_p: & \quad \text{enthalpy for gas and particle, respectively} \\
(f_p, f_p): & \quad \text{velocity and thermal lag factors defined in Eq. (27), respectively} \\
\bar{m}_p: & \quad \text{mass flow rate function of particles} (\bar{m}_p \equiv \bar{m}_p(t_p)) \\
\bar{M}_p, \bar{M}_p: & \quad \text{total mass flow rate of gas and particle, respectively} \\
\bar{n}_p: & \quad \text{number density function of particle} (\bar{n}_p \equiv \bar{n}_p(t_p)) \\
\bar{p}: & \quad \text{pressure} \\
\bar{R}_p: & \quad \text{particle Reynolds number} \\
\bar{r}_p: & \quad \text{particle radius} \\
\bar{T}, \bar{T}_p: & \quad \text{gas- and particle-phase temperatures} (\bar{T} \equiv T_p(t_p), \text{respectively}) \\
\bar{V}, \bar{V}_p: & \quad \text{gas- and particle-phase velocities} (\bar{V} \equiv \bar{V}_p(t_p), \text{respectively}) \\
\bar{\xi}: & \quad \text{coordinate along nozzle axis} \\
\bar{\mu}: & \quad \text{gas viscosity} \\
\bar{\rho}: & \quad \text{gas density} \\
\bar{\rho}_{pp}: & \quad \text{particle material density} \\
\bar{\rho}_p: & \quad \text{particle density function} (\bar{\rho}_p \equiv \bar{\rho}_p(t_p))
\end{align*} \]

Subscripts

0: reservoir condition

g: gas

\[ \begin{align*}
\bar{p}: & \quad \text{particles}
\end{align*} \]

REFERENCES


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