Numerical Analysis of Thermo-electrically Conducting Fluids in a Cubic Cavity Using Vector Finite Element Method for Induction Equations

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(Received on November 7, 2002; accepted in final form on January 13, 2003)

The purpose of this study is to apply vector finite element method to magnetohydrodynamics. Vector finite element method is one of the popular methods in the field of electromagnetism. Two types interpolation functions are defined. One is facet element and another is edge element. In applying vector finite element method to the Inductions equations solenoidal condition is satisfied automatically without iterative corrections. But classical finite element method like B method by Oki et al. needs to solve Poisson equations to satisfy the solenoidal condition. In the present study Induction equations are numerically analyzed with vector finite element method. Flow field and temperature field are analyzed using GSMAC finite element method. We call this analysis technique GSMAC-VFEM (generalized simplified marker and cell vector finite element method). Three-dimensional natural convection in a cavity under a constant magnetic field is analyzed to determine the accuracy and the efficiency of the method. Computational results are compared with B method to verify this numerical scheme. Since the numerical results obtained here agreed well with other numerical results, the new numerical method for solving Induction equations using vector finite element method was verified. Calculation time of new numerical scheme was faster than the other numerical method. The reason is that using vector finite element method for solving Induction equations solenoidal condition for magnetic flux density satisfies automatically.

KEY WORDS: numerical simulation; vector finite element method; magnetohydrodynamics; Induction equations; computational fluid dynamics; cubic cavity flow.

1. Background and Objective

One of the most important problems in solving Induction equations numerically is to satisfy the solenoidal condition of magnetic flux density. It is very difficult to achieve this condition in numerical analysis. Sato et al. numerically analyzed thermal electromagnetic flows in a square cavity treating Induction equations as only advection diffusion equations without satisfying solenoidal conditions. Time marching algorithm was forward Euler method and no iterative operations were used. In this problem temperature distributions were quite similar to the experimental results. The magnetic flux density profile was obtained and evaluation was not good enough. Another numerical results was the B method proposed by Oki et al. The B method is one of the iterative solvers to satisfy the solenoidal conditions for the magnetic flux density. This idea is based on the HSMAC algorithm. In calculating incompressible fluid flows, Navier-Stokes equations are numerically analyzed under the condition of the mass conservation law. Velocity and pressure are iteratively corrected in one time step. So the solenoidal condition for velocity is satisfied in every time step. B method is the expansion of this idea to the Induction equations. To apply the iterative algorithm, deformation of the Induction equations is needed. Just like the pressure term in the Navier-Stokes equations the gradient of the scalar variable of the Induction equations is explicitly formulated as follows:

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = \nabla \mathbf{R} + \nu_m \nabla^2 \mathbf{B} \quad \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

where \( \mathbf{R} \) is a numerical residual. The same algorithm is applied to these equations and the solenoidal condition is satisfied however this algorithm is time consuming. Oki analyzed the same problem with Sato’s and the profiles of the magnetic flux density, temperature and velocity were in good agreement with each other.

On the other hand in the field of electromagnetism, other numerical approach is used, which is based on the Yee’s algorithm. Recently it has been used as the FDTD (Finite Difference Time Domain) method. This is one of the finite difference methods but the formation of variables are very different from the prescribed method. Usually the variables are positioned at the vertex or the center of the element, however in case of the FDTD method variables are defined at the edge on the element or the surface on it. That is suitable for analyzing the Maxwell equations. One of the examples of numerical approach of this algorithm to the
Induction equations is MOCCT (the Method of Characteristic Constrained Transport)\(^\text{13}\) method which is one of the numerical technique for astrophysical MHD. This method usually treats only ideal MHD which is advection equation of the Induction equations so the diffusion term of the Induction equations is not considered. The generalization of the FDTD method to arbitral elements is the vector finite element method.\(^8-12\) The vector finite element has two types of element just like variables of FDTD method are defined at the edge and the surface center on the element. One is facet element and the other is edge element. Using vector finite element method, solenoidal condition is automatically satisfied in numerical evolution without iterative correction. We can find a lot of numerical examples using vector finite element method in electromagnetism, but we have never seen the numerical examples which concern the induced magnetic field i.e. Induction equations. In this present paper Induction equations are numerically analyzed with vector finite element method. On the other hand flow field and temperature field are analyzed using GSMAC finite element method. We call this analysis technique GSMAC-VFEM (generalized simplified marker and cell vector finite element method). Three-dimensional natural convection in a cavity under a constant magnetic field is analyzed to determine the accuracy and the efficiency of the method. Computational results are compared with other numerical method to verify this numerical scheme.

2. Numerical Method

2.1. Assumptions

In the following numerical analyses, the assumptions are made to simplify the governing equations.
(1) the fluid is Newtonian,
(2) the flow is laminar and incompressible,
(3) the displacement current is neglected,
(4) material values are constant.

2.2. Governing Equations

Dimensionless governing equations in the thermo-electromagnetic field are conservation laws, Navier–Stokes equations, energy equation and Maxwell equations as follows:

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{(2)}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} = \nabla \mathbf{u} + \frac{1}{\text{Gr}} (\nabla^2 \mathbf{u} - 2 \mathbf{e}) + \frac{\text{Ha}^2}{\text{Gr}} \mathbf{J} \times \mathbf{B} \quad \text{(3)}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\text{Rem}} \nabla \times (\nabla \times \mathbf{B}) \quad \text{(4)}
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{(5)}
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \frac{1}{\text{Ra}} \nabla^2 T + \frac{\text{Ha}^2 \text{E}^2}{\text{Gr}} J^2 \quad \text{(6)}
\]

\[
\mathbf{J} = \frac{1}{\text{Rem}} \nabla \times \mathbf{B} \quad \text{(7)}
\]

where \( \mathbf{B}, \mathbf{J}, p, T, t, \mathbf{u} \) are magnetic flux density, electric current, pressure, temperature, time and velocity, respectively. Dimensionless parameters \( \text{Ec}, \text{Gr}, \text{Ha}, \text{Ra}, \text{Rem} \) are Eckert number, Grashof number, Hartmann number, Rayleigh number and magnetic Reynolds number, respectively.

2.3. Time Marching Algorithm

The Navier–Stokes equations are discretized using GSMAC finite element method. Considering that solenoidal condition for the velocity field, the following finite element formulations are obtained.

\[
\mathbf{M}_{ab} \frac{\mathbf{u}_{\beta}}{\Delta t} - \mathbf{u}_{\beta} = - \mathbf{u}_e^k \cdot A_{ab} \mathbf{u}_{\beta}^n + C_{ab} P^n - \frac{1}{\text{Gr}} D_{ab} \mathbf{u}_{\beta}^n + \frac{1}{\text{Gr}} \mathbf{M}_{ab} T_{\beta}^n + \frac{\text{Ha}^2}{\text{Gr}} \mathbf{M}_{ab} \mathbf{J} \times \mathbf{B}_{\beta} \quad \text{(8)}
\]

\[
\mathbf{M}_{ab} \frac{\mathbf{T}_{\beta}^n}{\Delta t} = - \mathbf{u}_e^k \cdot A_{ab} T_{\beta}^n - \frac{1}{\text{Ra}} D_{ab} \mathbf{u}_{\beta}^n + \frac{\text{Ha}^2 \text{E}^2}{\text{Gr}} \mathbf{M}_{ab} \mathbf{J} \times \mathbf{B}_{\beta} \quad \text{(17)}
\]

On the other hand Induction equations are discretized using vector finite element method. Induction equations are divided into two phases and following time marching algorithms are obtained.

\[
\mathbf{B}_{\beta}^{n+1} - \mathbf{B}_e^n = - \nabla \times \mathbf{E}_e^n \quad \text{(18)}
\]

\[
\mathbf{E}_e^n = - \mathbf{u}_e^{n+1} \times \mathbf{B}_e^n + \frac{1}{\text{Rem}} \nabla \times \mathbf{B}_e^n \quad \text{(19)}
\]

The former, the equations of the magnetic field mean Faraday’s law which is discretized using facet vector finite element method and the latter, the equations of the electric
field the combination of Ampere’s law and Ohm’s law which is discretized using edge vector finite element method. Electric field contains no time concerning term. With the use of the vector finite element method, they are rewritten in element matrix form.

\[
\frac{b^{n+1}_a - b^n_a}{\Delta t} = - (\pm \varphi^a_{\text{rh}} \pm \varphi^a_{\text{rh}}) e^n_a \quad \ldots \ldots (20)
\]

where \( b \) and \( e \) mean the surface integral of the magnetic flux density \( B \) and the line integral of the electric field \( E \), respectively. Signs in the parenthesis follow the description in Appendix. The magnetic flux density and the electric field are interpolated using the vector shape functions defined as follows:

\[
B = F^a_j b^j_a \quad \ldots \ldots (22)
\]

\[
E = E^a_j e^j_a \quad \ldots \ldots (23)
\]

The element coefficient matrices in Eqs. (20) and (21) are calculated as follows:

\[
F_{ab} = \int_{\Omega} F^a_j F^b_j d\Omega \quad \ldots \ldots (24)
\]

\[
E_{ab} = \int_{\Omega} E^a_j E^b_j d\Omega \quad \ldots \ldots (25)
\]

\[
R_{ab} = \int_{\Omega} F^a_j \times E^b_j d\Omega \quad \ldots \ldots (26)
\]

Precise description of the formulation of Eqs. (20) and (21) from Eqs. (18) and (19) are written in Appendix.

3. Analytical Model

To verify the new numerical scheme for Induction equations, the natural convection of molten tin in a cavity under a constant magnetic field is analyzed numerically, using the GSMAC-VFEM. The analytical model is shown in Fig. 1. Liquid metal is filled in the unit cubic cavity under the constant gravity and the magnetic field in the \( z \) direction, respectively. The boundary condition of the velocity field is no slip condition at all walls. The boundary condition of temperature field is constant at \( T = 0.0 \) and \( T = 1.0 \) at the wall of \( x = 1.0 \) and \( x = 0.0 \), respectively. Other walls at \( y = 0.0 \), \( y = 1.0 \), \( z = 0.0 \) and \( z = 1.0 \) are adiabatic. The boundary condition of the magnetic field is a little complicated. Since Induction equations are divided into two phases, we have to consider the boundary conditions at each phase. The boundary condition of the magnetic field at \( z = 0.0 \) and \( z = 1.0 \) is that the normal component of the magnetic field equals to \( -1.0 \). The boundary condition of the magnetic flux density at \( x = 0.0 \), \( x = 1.0 \), \( y = 0.0 \) and \( y = 1.0 \) is that the normal component of the magnetic field equals to 0.0, respectively. The boundary condition of the electric filed is that the tangential component of the electric field equals to 0.0 at all boundary edges, which means that all walls are perfectly conducting. The initial conditions are fluid at rest and a uniform temperature (\( T = 0.0 \)) and magnetic flux density (\( B = (0.0, 0.0, -1.0) \)).

The computational mesh is shown in Fig. 2 and the non uniform finite element mesh consists of \( 30 \times 30 \times 30 = 27,000 \) elements and \( 29,791 \) node points. The calculation parameters are shown in Table 1 in which \( \Delta t \), \( t_{\text{end}} \) and \( \varepsilon \) are the time increment, the dimensionless calculation time and the governing criterions for Poisson equation of the velocity field, respectively. The calculation conditions of dimensionless parameters are shown in Table 2.

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4. Results and Discussion

4.1. Computational Results of Case 1

The numerical conditions of case 1 and case 2 are the same as the numerical results using B method proposed by Oki et al. The velocity distributions of case 1 at steady state at $y=0.5$ are shown in Fig. 3. It is shown that the natural convection is promoted by the twin vortices in the center of the cavity. This is caused by the effect of Lorentz force. The velocity vectors of case 1 at $x=0.8$ is shown in Fig. 4. From this flow pattern velocity distributions are almost two-di-

Fig. 3. Velocity vectors of case 1 at $y=0.5$ ($H_a=65.7, r=80,000$).

Fig. 4. Velocity vectors of case 1 at $x=0.8$ ($H_a=65.7, r=80,000$).

The profiles of the velocity vectors in Fig. 3 can be explained by the profiles of the Lorentz force in Fig. 6. Since the velocity vectors are restricted by the opposite force of Lorentz force, two vortices are found.

The magnetic flux density vectors of case 1 at steady state at $y=0.5$ is shown in Fig. 5. Induced magnetic flux density is influenced on by the flow field. Lorentz force vectors of case 1 at steady state at $y=0.5$ is shown in Fig. 6.

Fig. 5. Magnetic flux density vectors of case 1 at $y=0.5$ ($H_a=65.7, r=80,000$).

Fig. 6. Lorentz force vectors of case 1 at $y=0.5$ ($H_a=65.7, r=80,000$).
We compared present results with the other numerical method called B method proposed by Oki et al. Using B method, Induction equations are numerically solved iteratively. Although Fig. 4 shows that the velocity profiles are quasi-two dimensional profiles, we compared the results of present method at \( y=0.5 \) with the results of three-dimensional results of B method. Figure 7 shows the comparison of the velocity profiles in the x direction at \( x=0.5 \) and \( y=0.5 \) at steady state of case 1 between present results and the results of B method. Figure 8 also shows the comparison of the velocity profiles in the z direction at \( y=0.5 \) and \( z=0.5 \) of case 1 between present method and the results of B method. Both results are in good agreement with each other, so the new numerical scheme is verified for the velocity field. The comparison of the magnetic flux density profiles in the x direction at \( x=0.5 \) and \( y=0.5 \) between present method and the results of B method at the steady state is shown in Fig. 9. Since both results are in good agreement with each other, the computational results of Induction equations analyzed using the vector finite element method agree well with the results using B method. Temperature distributions of present method and B method in the x direction at \( y=0.5 \) at steady state are shown in Fig. 10. The comparison of the temperature distributions at \( z=0.0 \), \( z=0.5 \) and \( z=1.0 \), respectively is in good agreement with each other.
Figure 11 shows the transient profiles of the velocity vectors and the magnetic flux density vectors of case 1 at $t=5000$, $t=10000$ and $t=15000$, respectively. The center of vortex transfers from left to right. We can understand that the induced magnetic flux density vectors are induced with the growth of the velocity distributions.

4.2. Computational Results of Case 2

The computational condition of case 2 has larger Hartmann number than that of case 1. It means that the Lorentz force in case 2 is larger than that in case 1. Figure 12 shows the velocity vector of case 2 at steady state at $y=0.8$. And the velocity vectors of case 2 at steady state at $x=0.8$ is shown in Fig. 13. From Figs. 12 and 13 flow field is not disturbed and the distribution is almost two-dimensional at all field without boundary layer. In Fig. 13 the flow field near the walls forms boundary layer which has no vortices just like formed in Fig. 4. The magnetic flux density vectors in case 2 at steady state at $y=0.5$ is shown in Fig. 14. Induced magnetic field is very small and the magnetic field is not affected by the flow field. Figure 15 shows us the Lorentz force vectors of case 2 at steady state at $y=0.5$. The direction of Lorentz force vectors is almost $x$ direction. The spatial variation in the $z$ direction of the Lorentz force of case 2 is smaller than that of case 1. This is because the positions of the center of vortices in case 2 are different from that in case 1.

The comparison of the velocity vectors in the $x$ direction at steady state at $x=0.5$ and $y=0.5$ between present results and the results of B method is shown in Fig. 16. The comparison of the velocity vectors in the $z$ direction at steady state at $y=0.5$ and $z=0.5$ between present results and the results of B method is shown in Fig. 17. Since both computational results are in good agreement with each other, the
Lorentz force evaluated by the electric current and the magnetic flux density which is interpolated using the vector finite element method in the present method is validated. The velocity magnitude shown in Figs. 16 and 17 is smaller than that in Figs. 7 and 8, due to the difference of the Lorentz force. Figure 18 shows the comparison of the magnetic flux density profiles in the \(x\) direction at \(x=0.5\) and \(y=0.5\) between present method and the results of B method. These comparisons assure the evaluation of the results of the vector finite element method for Induction equations. The computational results of the temperature profiles between present results and the results of B method are in good agreement with each other. The spatial difference of temperature in case 2 is smaller than the results of case 1. This is because the strong induced Lorentz force with the large Hartmann number restricted the velocity field and the convection term had weak influence on the temperature profiles.

CPU time for the present method and B method is shown.
in Table 3. Programming language is Fortran 95 and CPU time was measured using the intrinsic subroutine ‘call cpu_time’. Calculation was carried out under the Pentium IV 2.53 GHz processor with 1.0 Gbytes total memory. This means that the present method is superior to the B method in the calculation time.

As a whole computational results using vector finite element method for electromagnetic field and the finite element method for both the velocity field and the temperature field is in good agreement with the results using the convective finite element method for whole field. Computation time is also superior to the B method.

5. Conclusions

The conclusions of the present paper are summarized as follows.

(1) Since the numerical results obtained here agree well with other numerical results, the new numerical method for solving Induction equations using vector finite element method is verified.

(2) Calculation time of new numerical scheme is faster than the other numerical method. The reason is that using vector finite element method for solving Induction equations solenoidal condition for magnetic flux density satisfies automatically.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{ab}$</td>
<td>Advection matrix</td>
</tr>
<tr>
<td>$B$</td>
<td>Dimensionless magnetic flux density vector</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Applied magnetic flux density</td>
</tr>
<tr>
<td>$b_a$</td>
<td>Surface integral of dimensionless magnetic flux density at node $a$</td>
</tr>
<tr>
<td>$C_a$</td>
<td>Gradient vector</td>
</tr>
<tr>
<td>$D_{ab}$</td>
<td>Diffusion matrix</td>
</tr>
<tr>
<td>$E$</td>
<td>Dimensionless electric field</td>
</tr>
<tr>
<td>$E_a$</td>
<td>Edge vector shape function</td>
</tr>
<tr>
<td>$E_{ab}$</td>
<td>Coefficient matrix of edge vector shape function</td>
</tr>
<tr>
<td>$e_a$</td>
<td>Line integral of dimensionless electric field at node $a$</td>
</tr>
<tr>
<td>$E_c$</td>
<td>Eckert number ($= \frac{U_r^2}{\rho C_p D T}$)</td>
</tr>
<tr>
<td>$e$</td>
<td>Gravity direction unit vector</td>
</tr>
<tr>
<td>$F_a$</td>
<td>Facet vector shape function</td>
</tr>
<tr>
<td>$F_{ab}$</td>
<td>Coefficient matrix of facet vector shape function</td>
</tr>
<tr>
<td>$G_r$</td>
<td>Grashof number ($= \frac{\rho V B_0 L_r}{\eta}$)</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational acceleration</td>
</tr>
<tr>
<td>$H_a$</td>
<td>Hartmann number ($= \frac{\sqrt{\sigma e \rho V B_0 L_r}}{m U_r L_r}$)</td>
</tr>
<tr>
<td>$J$</td>
<td>Dimensionless current density vector</td>
</tr>
<tr>
<td>$L_r$</td>
<td>Representative length</td>
</tr>
<tr>
<td>$M_a$</td>
<td>Mass coefficient vector</td>
</tr>
<tr>
<td>$M_{ab}$</td>
<td>Mass matrix</td>
</tr>
<tr>
<td>$N_a$</td>
<td>Shape function</td>
</tr>
<tr>
<td>$P_r$</td>
<td>Prandtl number ($= \nu/\eta$)</td>
</tr>
<tr>
<td>$p$</td>
<td>Dimensionless pressure</td>
</tr>
<tr>
<td>$R_a$</td>
<td>Rayleigh number ($= P_r \cdot G_r$)</td>
</tr>
<tr>
<td>$R_{ab}$</td>
<td>Coefficient matrix of facet and edge vector shape function</td>
</tr>
<tr>
<td>$R_{em}$</td>
<td>Magnetic Reynolds number ($= \frac{\sigma e \mu U_r L_r}{m U_r \eta}$)</td>
</tr>
<tr>
<td>$T$</td>
<td>Dimensionless temperature</td>
</tr>
<tr>
<td>$t$</td>
<td>Dimensionless time</td>
</tr>
<tr>
<td>$U_i$</td>
<td>Representative velocity governed by the balance between buoyancy and viscosity</td>
</tr>
<tr>
<td>$u$</td>
<td>Dimensionless velocity vector</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Relaxation parameter</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Magnetic permeability</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Dynamic viscosity</td>
</tr>
<tr>
<td>$v_m$</td>
<td>Magnetic dynamic viscosity</td>
</tr>
</tbody>
</table>
\[ \sigma_e: \text{Electric conductivity} \]
\[ \rho: \text{Density} \]
\[ \phi_0: \text{Modified velocity potential} \]
\[ V: \text{Element volume} \]

**Subscript**

- \( k \): simultaneous relaxation step
- \( n \): Time step
- \( e \): Element averaged value

**REFERENCE**


**Appendix**

We show how to discretize the Induction equations using vector finite element method.

First of all we define the vector shape function. The vector shape function has two types of shape functions where one is face shape function and another is edge shape function shown in Fig. 20. The node of the facet element is defined at the center of the surface element. On the other hand, the node of the edge element is defined at the center of the line element. The facet shape functions for any three-dimensional hexahedra are written as follows:

\[
F_1 = -\frac{1}{8} (1 - \eta) \nabla \eta \times \nabla \xi, \quad F_2 = \frac{1}{8} (1 + \eta) \nabla \eta \times \nabla \xi, \\
F_3 = -\frac{1}{8} (1 - \zeta) \nabla \zeta \times \nabla \xi, \quad F_4 = \frac{1}{8} (1 + \eta) \nabla \zeta \times \nabla \xi, \\
F_5 = -\frac{1}{8} (1 - \zeta) \nabla \zeta \times \nabla \eta, \quad F_6 = \frac{1}{8} (1 + \zeta) \nabla \zeta \times \nabla \eta.
\]

\[\text{Example of facet and edge shape functions.}\]

\[\text{Fig. 20.} \]

where the values in the parenthesis means the size of the vectors and rotation of two gradients of \( \xi, \eta \) or \( \zeta \) means the direction of the vector. The edge shape functions for any three-dimensional hexahedra are given as follows:

\[
\begin{align*}
E_1 &= \frac{1}{8} (1 - \eta)(1 - \zeta) \nabla \xi, & E_2 &= \frac{1}{8} (1 + \eta)(1 - \zeta) \nabla \xi, \\
E_3 &= \frac{1}{8} (1 + \eta)(1 + \zeta) \nabla \xi, & E_4 &= \frac{1}{8} (1 - \eta)(1 + \zeta) \nabla \xi, \\
E_5 &= \frac{1}{8} (1 - \zeta)(1 - \xi) \nabla \eta, & E_6 &= \frac{1}{8} (1 + \xi)(1 - \zeta) \nabla \eta, \\
E_7 &= \frac{1}{8} (1 + \zeta)(1 + \xi) \nabla \eta, & E_8 &= \frac{1}{8} (1 - \zeta)(1 + \xi) \nabla \eta, \\
E_9 &= \frac{1}{8} (1 - \xi)(1 - \eta) \nabla \zeta, & E_{10} &= \frac{1}{8} (1 + \eta)(1 - \xi) \nabla \zeta, \\
E_{11} &= \frac{1}{8} (1 + \xi)(1 + \eta) \nabla \zeta, & E_{12} &= \frac{1}{8} (1 - \zeta)(1 + \eta) \nabla \zeta.
\end{align*}
\]

\[\text{Example of facet and edge shape functions.}\]

\[\text{Fig. 20.} \]

where the product of the values of two parenthesis means the size of the vector and the gradients of \( \xi, \eta \) or \( \zeta \) means the directions of the vector. It is interesting to note that relationship between the facet shape function and the edge shape function satisfies following conditions.

\[
\begin{align*}
\nabla \times E_1 &= -F_3 + F_5, & \nabla \times E_2 &= +F_5 - F_4, \\
\nabla \times E_3 &= -F_6 + F_4, & \nabla \times E_4 &= +F_6 - F_3, \\
\nabla \times E_5 &= -F_2 + F_1, & \nabla \times E_6 &= +F_2 - F_1, \\
\nabla \times E_7 &= -F_3 + F_5, & \nabla \times E_8 &= +F_3 - F_5, \\
\nabla \times E_9 &= -F_2 + F_1, & \nabla \times E_{10} &= +F_2 - F_1, \\
\nabla \times E_{11} &= -F_4 + F_2, & \nabla \times E_{12} &= +F_4 - F_2.
\end{align*}
\]

where this relationship is very important for Galerkin formulation of the Inductions equations.

The finite element formulation of Eq. (18) is given as the following equations:

\[
\int_{\Omega_b} F_0 \cdot \frac{\mathbf{B}^{n+1} - \mathbf{B}^n}{\Delta t} \ d\Omega = -\int_{\Omega_b} F_0 \cdot \nabla \times F_0 \cdot \mathbf{E}^n \ d\Omega \quad \text{...(A-4)}
\]

Using the Eq. (A-3), we can show

\[
\int_{\Omega_b} F_0 \cdot F_0 d\Omega \int_{\Omega_b} \frac{b_{0}^{n+1} - b_{0}^{n}}{\Delta t} d\Omega = \int_{\Omega_b} F_0 \cdot (\pm F_{0b} - F_{b2}) d\Omega e_{0}^{n}
\]

\[\text{...(A-5)}\]

The sign of the first and second term of the right hand side satisfies the relationship shown in Eq. (A-3). The subscript \( b1 \) and \( b2 \) also follow the Eq. (A-3). Applying the mass lumping algorithm on each side of the Galerkin formulation, the final form of the magnetic field is given by
\[
\sum_{b} \int_{\Omega_{b}} \mathbf{F}_{b} \cdot \mathbf{g}_{b} \, d\Omega = \frac{b_{n}^{+1} - b_{n}^{-}}{\Delta t}
\]

where this equation means the time variation of the surface integral of the magnetic flux density equals to the negative value of the line integral of the electric field.

On the other hand, Galerkin formulation of the electric field is given by

\[
\sum_{b} \int_{\Omega_{b}} \mathbf{F}_{b} \cdot \mathbf{g}_{b} \, d\Omega = \sum_{b} \int_{\Omega_{b}} \mathbf{F}_{b} \cdot (\pm \mathbf{F}_{b} \pm \mathbf{F}_{b} \pm \mathbf{F}_{b}) \, d\Omega e_{b} \quad \text{.........(A-6)}
\]

Applying the mass lumping algorithm on the left hand and using the vector identity \( \nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \times (\nabla \times \mathbf{a}) - \mathbf{a} \times (\nabla \times \mathbf{b}) \) where \( \mathbf{a} \) and \( \mathbf{b} \) are arbitrary vectors, the Galerkin formulation of the electric field is given as follows:

\[
\sum_{b} \int_{\Omega_{b}} \mathbf{F}_{b} \cdot \mathbf{g}_{b} \, d\Omega = - \mathbf{u}_{e}^{+1}, \int_{\Omega_{b}} \mathbf{F}_{b} \times \mathbf{E}_{b} \, d\Omega e_{b}^{+1} + \frac{1}{\text{Rem}} \sum_{b} \int_{\Omega_{b}} \mathbf{F}_{b} \cdot (\pm \mathbf{F}_{b} \pm \mathbf{F}_{b}) \, d\Omega e_{b}^{+1} \quad \text{.........(A-8)}
\]