Combined Topology and Shape Optimization of Structures using Nodal Density as Design Parameter

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Abstract
In this study, a new nodal-based distribution method of material properties is proposed for material topology optimization in linear elastostatic structures. It is based on a design domain concept of a density distribution method (Bendsøe, 1989). Nodal densities of material properties are considered as optimization design variables. Geometric boundaries are represented by fixed grids, thus an Eulerian type of formulation is used for optimal interfaces. The objective is to obtain both optimal smooth shapes and topologies based on the design domain concept avoiding an excessive number of finite elements. This approach allows us to perform a nodal-based topology and shape optimization, which can be easily implemented in existing gradient-based optimization codes. Numerical examples demonstrate the efficiency of the present method.

Keywords: nodal-based distribution; density distribution method; SIMP; Eulerian type; nodal-based topology and shape optimization; optimal topology; optimal shape; gradient-based optimization

1. Introduction
Frequently in finite element analyses, material properties are assumed to be uniform within one element. This element-based distribution of material properties is adequate when a clear boundary between different materials exists. However, it gives rise to problems in topology optimization where material properties are allowed to vary continuously from element to element. Therefore, the final designs of topology optimization may be rendered with rough layouts since they are described by raster patterns based on regular rectangular grids. Increasing the number of finite elements will improve the resolution of the topology optimization, but then computational costs may become an issue.

To overcome these difficulties, different methods were investigated in order to obtain smooth shapes. The Adaptive Topology Optimization method (ATO) of Maute and Ramm (1995) starts with a conventional material-based topology optimization. A smooth boundary is determined after several optimization steps have been completed. Additional topology optimization steps with a new discretization are performed within this domain, and the result is improved by a shape optimization that adjusts the smooth boundary.

Although this method works very well for two-dimensional problems (Schwarz et al., 2001; Kemmler et al., 2005), it is difficult to extend the procedure to general three-dimensional problems for lack of reliable mesh generators.

Apart from that, level set methods (Sethian and Wiegmann, 2000; Allaire et al., 2004), which have recently been introduced in the field of shape optimization, detect the evolution of smooth interfaces by embedding the boundary as a zero level set of the signed distance functions. However, the speed of convergence may slow substantially because of the rigorous CFL condition of a level set equation and the boundary evolution of the shape or local derivatives.

In this study, nodal densities as design variables are updated utilizing optimization methods such as OC or MMA without solving the level set equation, which results in a slow convergence. The density, which varies in different regions, is bilinearly interpolated by the shape functions for finite elements in the design domain. This density function is explicitly cut by an iso-line of constant density to represent continuous boundaries between voids and solids. The iso-line moves toward an optimal interface by means of the updated nodal density. Moreover the densities can be distributed as point densities on fine grids within elements by bilinear interpolation. In this case, the iso-line is searched with more concrete information concerning the material boundaries. Element densities which are obtained by the arithmetic average of the nodal densities are implemented into finite element formulation for structural analyses. Sensitivity
analyses and optimization method algorithms are implemented with respect to nodal densities. Through these optimization procedures the discontinuity problem of material distributions is resolved and a fast convergence is obtained.

Numerical examples demonstrate the efficiency of the present method.

2. Density Distribution Method

In continuous formulations of the topology optimization problem, the design is given by a continuous scalar function, $\Phi$, from the fixed design domain, $\Omega \subseteq \mathbb{R}^n (n=2$ or $3)$, to the allowed material density, $0 \leq \Phi \leq 1$. After the discretization process of the continuous design domain, a material density, $\Phi_i$, is assigned to each finite element and is defined by applying a penalty contour to the design variable field, i.e. as in the so-called "power law approach" or density distribution method.

According to the approach, the material density distribution has an effect on element stiffness. Thus, the element stiffness-density relationship may be expressed in terms related to Young's modulus, $E$, i.e. $E_i$ is assigned by an average element density, $\Phi_i$, of updated nodal densities, and is defined as

$$E_i(\Phi_i) = E_0 \left( \frac{\Phi_i}{\Phi_0} \right)^k ; \quad i = 1 \ldots n$$

where $E_0$ and $\Phi_0$ are the nominal values of Young's modulus and material density respectively, and $n$ is the number of elements. The penalty parameter, $k \geq 1$, penalizes intermediate material densities as shown in Fig.1, and $k=3.0$ is generally used throughout the optimization process.

![Fig.1. Penalty relationship between Young's modulus and element density with various penalty parameters](image)

An isotropic material model with a plane stress (such as a wall structure) is used here without a loss of generality, so that

$$C_i = \frac{E_i(\Phi_i)}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{bmatrix}$$

where $C_i$ is a material tensor of each finite element, $i$, and includes the updated term of Young's modulus, $E_i$, defined by an average element density, $\Phi_i$, of updated nodal densities. $\nu$ is Poisson's ratio.

3. Geometric Boundary Representation

The design domain for material distribution problems of topology optimization can be divided into solid (1) and void (0) phases through the signs of the Heaviside function and is shown in Fig.2.

The left-side of Fig.2 illustrates a boundary represented by the zero level sets of a signed distance function, which is well-known in the level set methods and the right-side of Fig.2 shows the definition of the interfaces between the voids and solids by an iso-line with a specific value of 0.5 for the density function. When compared to the zero level sets of the left-side of Fig.2, the 0.5 iso-line of the right-side of Fig.2 is utilized in this study for its smooth boundary. When densities are smaller than 0.5, their regions are assumed to be voids; otherwise, the regions are solids.

![Fig.2. Geometrical design domain and Heaviside function](image)

Fig.3. illustrates the process of constructing the 0.5 iso-line based on nodal densities as a design parameter. It is assumed that the element densities are assigned to nodal densities. A three-dimensional density function is bilinearly interpolated by the nodal densities. Then it is cut by a specific level of 0.5 and the boundaries between the voids and solids are described by 0.5 iso-lines. While the nodal densities are automatically updated during the optimization processes, the iso-line boundaries can be propagated or reduced as the topology changes so that it is possible to create voids in solid regions or solids in void regions.

Fig.4. shows the interfaces that were propagated in the discrete design domain during the optimization iterations. The inactive elements have four nodal
densities over or under 0.5; otherwise, they are labeled active elements. The inactive elements do not contribute to the construction of the 0.5 iso-line interface while the interfaces are determined from the updated density function of the active elements during the optimization iterations. The dashed and bolded lines show the interfaces for the \( n \) and \( n + 1 \) iterations, respectively.

4. Nodal-Based Material Topology Optimization Problem

(1) Problem Statements

In this paper, a linear elastostatic problem is considered for structural topology optimization. In continuous formulations of the topology optimization problem, the design is given by a continuous scalar function, \( \Phi \), from the fixed design domain, \( \Omega_x \subseteq \mathbb{R}^n \) \((n=2\ or\ 3)\) to the allowed material density value of \( 0 \leq \Phi \leq 1 \). The schematic of the topology optimization procedure for a solid structure with specified field and boundary conditions is shown in Fig.5. The general problem of structural topology optimization is specified by the objective function and constraints. By using the principle of minimum potential energy, the objective function can be written with minimal strain energy as follows:

\[
f = \min \left\{ \frac{1}{2} \int_{\Omega_x} \mathbf{e}^T \mathbf{C} \mathbf{e} \, d\Omega_x \right\}
\]

(3)

Here according to discretization, the continuous material tensor, \( \mathbf{C} \), depends on the density-stiffness relationship of the typical SIMP approach. The

Fig.3. Construction of the iso-line by bilinear interpolation function

Fig.4. Propagation of interfaces in the discrete design domain
discontinuous Heaviside function can be regularized as a smoothed and continuous form. The function can be included in the strain energy formulation since it is based on signals for a void (0) or solid (1) with respect to original Heaviside function.

\[
\Gamma = \partial \Omega_x = \Gamma_x \cup \Gamma_y \cup \Gamma_u
\]
\[
\Gamma_t : t = t_0
\]
\[
\Gamma_u : \bar{u} = u_0
\]

Fig.5. Schematic for topology optimization of structure

The inequality optimization constraint is \(0 \leq \Phi \leq 1\). An equality constraint is a linear elastostatic equilibrium, and another one is the limit on the required amount of materials in terms of the constant volume, \(V_0\), of the design domain. They can be written as follows, respectively:

\[
\int_{\partial \Omega_x} \delta e^T \sigma \, d\Omega_x = \int_{\partial \Omega_x} \delta u^T b \, d\Omega_x + \int_{\Gamma_t} \delta t \, d\Gamma_t
\]
\[
\int_{\partial \Omega_x} d\Omega_x - V_0 = 0
\]

(2) Sensitivity Analysis

In general, the sensitivity of the optimization problems such as the objective functions or constraints can be calculated by analytical or numerical method. Since the numerical sensitivity method may yield large sensitivity errors, the method is often used to verify solutions. The analytical method usually only has small errors in the solutions, so it is used for the sensitivity of optimization problem. The analytical sensitivity method is distributed as a discrete and a variational approach. In the discrete approach, optimization problems are differentiated and then the derivative is calculated. However, in the variational approach, the derivative is differentiated, and then the derivative is discretized.

The analytical sensitivity method of the variational approach is utilized here since the variational method is numerically more efficient than discrete method in certain optimization problems. Since continuous displacement fields depend on design variables, \(s\) (for instance, material densities), the total differential form of the objective function consists of parts of an explicit partial derivative and an implicit partial derivative, and it is defined by Haug et al. (1986) and Haftka and Guerdal (1992).

\[
\nabla_s f = \nabla_s^{\text{exp}} f + \nabla_s^{\text{imp}} f \nabla_s u
\]

The total partial derivative is written as

\[
\nabla_s f = -\frac{1}{2} \int_{\partial \Omega_x} t e^T \mathbf{C} \dot{e} \, d\Omega_x
\]

5. Numerical Examples

(1) Topology Optimization of 2D MBB-beam

As a test example, a MBB-beam as shown in Fig.6 is used as a source of values in the present method. For numerical efficiency, half of the MBB-beam in Fig.6 (b) is calculated.

![Fig.6. Analysis model (a) MBB-beam (b) half MBB-beam](image)

Three square finite elements, 30\times10, 60\times20, and 90\times30, with four nodes are utilized for discretization of the design domain (L/2(6m) \times H(2m)). For numerical simplicity, the material parameters are Young’s modulus, \(E = 1.0 \, \text{kN/m}^2\), and Poisson’s ratio, \(\nu = 0.2\). A load of \(P = 1.0 \, \text{kN}\) is concentrated at the center of upper surface of the structure.

The penalty parameter is \(k = 5.0\) for the SIMP approach. The penalty formulation includes pure element densities in an element-based topology optimization method, and element densities are averaged by nodal densities in nodal-based topology

![Fig.7. Histories of convergence in EBTO and NBTO](image)
optimization. Minimal strain energy is defined as an objective function. Half of the total volume is fixed during the entire optimization procedure.

Table 1. shows the problem types and characteristics of a typical element-based topology optimization (EBTO), typical nodal-based topology optimization (NBTO), and nodal-based topology and shape optimization (NBTSO). Here, Matlab codes of Sigmund (2001) execute topology optimization. When different discretizations of the design domain are considered, convergent histories of the objective function in element and nodal-based topology optimization methods are kept and are shown in Fig.7.

The changing objective function values in NBTO have slower streams than those in EBTO since the design variable in NBTO is not the element densities but the averaged element densities and their values change very slowly. In addition, NBTO uses many design parameters, i.e. nodal densities, to analyze sensitivity analyses and complete the optimization method. Therefore, convergence becomes very slow in NBTO.

Fig.8. shows convergent objective function values using different discretizations in EBTO and NBTO. As can be seen, the finite elements are more refined in EBTO, and the maximal stiffness of optimal structure decreases gradually. This outcome is related to the numerical instability associated with checkerboard patterns and stress concentrations. The high stiffness that finite elements with checkerboard patterns have results from element locking (Bendsøe, 1989; Youn and Park, 1997). However, NBTO prevents element locking by redistributing the averaged element densities in the troubled regions.

![Fig.8. Convergent values of objective function](image)

The optimal contours of density distribution and continuous 0.5 iso-lines in NBTSO (nodal-based topology and shape optimization) are shown in Fig.9. Both continuous optimal shapes and optimal topology can be obtained using NBTSO. Fig.10. shows the changes to the continuous boundaries with each iteration. The iso-lines of the nodal density function are designed to move the boundaries as the topology changes and yield continuous shapes. The result is similar to that of the level set method. However, this approach does not calculate PDE for boundary variation, and therefore, it does not execute excessive computations. Note that the continuous boundary variations cut by the specific 0.5 level of the nodal density function is one kind of shape optimization. The boundary variations of an intermediate optimization stage in NBTSO are shown in Fig.11. For visual clearness Fig.11. uses a vertically great scale. From Fig.11., it can be seen that there is little variation except near the boundaries in the design domain.

(2) Topology Optimization of 2D Michell-Structure

The second example is a 2D Michell beam structure as shown in Fig.12. The design conditions are the same as those of example (1). The roller is supported at the bottom left corner and the hinge, the bottom right. At the bottom center, a concentrated load is applied. The 6m×3m design domain was discretized using 120×60 square finite elements with four nodes.
Fig. 13. shows the optimal contours of the material density distribution by volume constraints of 30 to 60% for EBTO and NBTO. With EBTO, unclear and jagged contours such as checkerboards are distributed in the design domain, but NBTO yields obvious boundaries between the material and non-material regions.

According to NBTSO, the improved optimal topologies of NBTO are transferred as continuous optimal shapes through a 0.5 level of the three-dimensional nodal density function and the results in the initial and final stages are shown in Fig. 14. The three-dimensional density function that was interpolated by nodal density is illustrated in the right sides of Fig. 14. The function cut by 0.5 on the z-coordinate, i.e. density and the iso-lines describe continuous interfaces between solids and voids. The interfaces are changed topologically while they are merged or split with other partial material regions during iterations.

The optimal topology in NBTSO is determined and therefore it is not the same as the shape typically produced by optimization based on boundary variation of shape derivatives. In addition, NBTSO produces continuous optimal shapes such as conventional shape optimization, not discontinuous optimal shapes of typical topology optimization based on element or nodal density.

6. Conclusions

This paper proposes a combined approach to material topology optimization and a special kind of shape optimization using nodal density values as design parameters. Most often a constant density within a finite element is used. In contrast to this procedure, density values are interpolated by element shape functions and are used as design parameters.
in material topology optimization. This increase of design parameters results in a layout which is much more detailed than using constant element densities as design parameters. It also avoids checkerboard patterns for low order finite elements. These density values are used in order to determine a smooth iso-line to describe the boundary of the optimization layout. The nodal densities are now utilized to move this boundary on a fixed grid. This approach allows us to perform a nodal-based topology and shape optimization, which can be easily implemented in existing gradient-based optimization codes.

Numerical examples demonstrate the efficiency of the proposed method with respect to optimal solutions and convergence compared with the typical resolutions of element-based topology optimization and nodal-based topology optimization.

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**References**


