On symmetric nonlinear systems

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Abstract—This paper is concerned with balanced realization for symmetric nonlinear systems which will provide the basis of model order reduction. A new notion of symmetry for nonlinear systems was characterized recently. It plays an important role in linear systems theory and is expected to provide new insights on nonlinear systems. In this paper, we provide a novel framework of balanced realization for this class of systems.

Key Words: Symmetric systems, balanced realization

1 Introduction

Symmetric systems attract the attention of researchers since it can describe important class of systems such as gradient systems and passive ones. In fact, many physical systems intrinsically have such properties and there are many model order reduction techniques to preserve special properties of the original system. For example, passivity and dissipativity are important properties to be preserved. Symmetry of dynamical systems are also an attractive research topic so far and some results are reported such as analysis with cross Gramians, on symmetric linear systems, characterization via Sylvester equations, nonlinear gradient systems, etc. The present paper provides a new definition of a nonlinear symmetric system which can be used for model order reduction.

In this paper, we pay attention to a special nonlinear map called cross map which is a nonlinear generalization of the cross Gramian originally given in and plays an important role in model order reduction in the linear case. By introducing a balanced representation of the cross map, we provide a balanced realization for symmetric systems which is used for model order reduction preserving the symmetry.

2 Preliminaries

This section refers to preliminary results on nonlinear balanced realization in. Consider an input-affine, time invariant, asymptotically stable nonlinear system

\[ \Sigma: \begin{cases} \dot{x} &= f(x) + g(x) u \\ y &= h(x) \end{cases} \]  

with \( f(0) = 0 \) where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, \) and \( y(t) \in \mathbb{R}^p \). Its controllability function \( L_c(x) \) and observability function \( L_o(x) \) are defined by

\[ L_c(x^0) = \inf_{\delta x \in L_2} \frac{1}{2} \|u\|_{L_2}^2 \]  

\[ L_o(x^0) = \frac{1}{2} \| x \|_{L_2}, \quad x(0) = x^0, \quad u(t) \equiv 0. \]

In the linear case,

\[ L_c(x) = \frac{1}{2} x^T W^{-1} x, \quad L_o(x) = \frac{1}{2} x^T M x \]

hold with the controllability and observability Gramians \( W \) and \( M \), respectively. The functions \( L_c(x) \) and \( L_o(x) \) fulfill the following Hamilton-Jacobi equations.

\[ \frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial^2 L_c(x)}{\partial x^2} g(x) g^T(x) \frac{\partial L_c^T}{\partial x} = 0 \]

\[ \frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial^2 L_o(x)}{\partial x^2} h(x) h^T(x) = 0 \]

Here \( \dot{x} = -f - g g^T(\partial L_c(x)/\partial x)^T \) is asymptotically stable in a neighborhood of the origin.

Here we review the nonlinear balanced realization.

Theorem 1 Suppose that \( L_c(x) \) and \( L_o(x) \) exist and that Hankel singular values of the Jacobian linearization of \( \Sigma \) are nonzero and distinct.

1. Then there exist a neighborhood \( X \subset \mathbb{R}^n \) of the origin and a coordinate transformation \( x \mapsto \tilde{x} \) on \( X \) converting the system into the following form

\[ \tilde{L}_c(\tilde{x}) = \frac{1}{2} \sum_{i=1}^n \frac{\tilde{x}_i^2}{\sigma_i(\tilde{x}_i)} \]  

\[ \tilde{L}_o(\tilde{x}) = \frac{1}{2} \sum_{i=1}^n \tilde{x}_i^2 \sigma_i(\tilde{x}_i) \]  

where \( \tilde{L}_c(\tilde{x}) \) and \( \tilde{L}_o(\tilde{x}) \) are the functions \( L_c(x) \) and \( L_o(x) \) in the new coordinate \( \tilde{x} \), respectively.

2. There also exist weaker versions of balanced realizations satisfying the following properties.

\[ \tilde{x}_i = 0 \iff \frac{\partial L_c(\tilde{x})}{\partial \tilde{x}_i} = 0 \iff \frac{\partial L_o(\tilde{x})}{\partial \tilde{x}_i} = 0 \]

\[ \sigma_i(\tilde{x}_i) = \frac{L_o(0, \ldots, 0, \tilde{x}_i, 0, \ldots, 0)}{L_c(0, \ldots, 0, \tilde{x}_i, 0, \ldots, 0)} \]

Here the singular value functions \( \sigma_i(x) \) are in order such as

\[ \min_{\tilde{x} = \tilde{x}_n} \sigma_i(\tilde{x}) > \max_{\tilde{x} = \tilde{x}_n} \sigma_{i-1}(\tilde{x}), \quad i = 1, 2, \ldots, n-1 \]

for any sufficiently small positive number \( c \).

The Hankel singular value functions \( \sigma_i(x) \)'s represent the importance of the state variables \( \tilde{x}_i \)'s with respect to
the input-output behavior of the system. Therefore we can get a reduced order model by removing the less important states. This technique is called balanced truncation. Let us now suppose that the system $\Sigma$ as in Eq. (1) is balanced in the sense of Theorem 1 and that
\[
\min_{x^{\omega} \in \mathcal{X}} \sigma_i(x) \gg \max_{x^{\omega} \in \mathcal{X}} \sigma_{i+1}(x)
\]
holds. Divide the state space accordingly as
\[
x = \begin{pmatrix} x^{\omega} \\ x^b \end{pmatrix},\quad x^{\omega} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}, \quad x^b \equiv \begin{pmatrix} x_{l+1} \\ \vdots \\ x_n \end{pmatrix}
\]
and
\[
\begin{pmatrix} x^{\omega} \\ x^b \end{pmatrix} = \begin{pmatrix} f^{\omega}(x^{\omega},x^b) \\ f^b(x^{\omega},x^b) \end{pmatrix} u + g^{\omega}(x^{\omega},x^b) y = h(x^{\omega},x^b).
\]
Then the balanced truncation with respect to the important state $x^{\omega}$ is given by
\[
\Sigma^{\omega} : \begin{cases} x^{\omega} = f^{\omega}(x^{\omega},0) + g^{\omega}(x^{\omega},0) u \\ y = h(x^{\omega},0) \end{cases} \quad (4)
\]
The reduced order system thus obtained satisfies the following properties.

**Theorem 2** 6) The controllability and observability functions $L^c_\omega(x^{\omega})$ and $L^o_\omega(x^{\omega})$ of the system $\Sigma^{\omega}$ satisfy
\[
\begin{align*}
L^c_\omega(x^{\omega}) & = L^c(x^{\omega},0) \\
L^o_\omega(x^{\omega}) & = L^o(x^{\omega},0)
\end{align*}
\]
Furthermore, Hankel singular values $\sigma_i^{\omega}(x^{\omega})$ of $\Sigma^{\omega}$ satisfy
\[
\sigma_i^{\omega}(x^{\omega}) = \sigma_i(x), \quad i = 1, \ldots, l.
\]

This theorem implies that several important properties of $\Sigma$ are preserved such as controllability, observability and Lyapunov stability. Local asymptotic stability is also preserved.

3 Main results

3.1 Symmetric nonlinear systems

This section gives novel characterization of symmetric nonlinear systems which is a nonlinear generalization of the linear case result 3.2.9. Symmetric systems form an important class of systems in the linear case. They correspond to passive and/or gradient systems.

First of all, the difference between linear and nonlinear symmetric systems are summarized in TABLE 1.

Consider the nonlinear system $\Sigma$ as in Eq. (1) with $r = m$, that is, the dimensions of input and output are the same. For this class of systems we can define a cross map as follows.

**Definition 1** The cross map $\Phi$ of $\Sigma$ is defined by
\[
\Phi(x) \equiv \mathcal{C} \circ \mathcal{O}(x).
\]

<table>
<thead>
<tr>
<th>Table 1: Linear v.s. nonlinear symmetric systems</th>
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<tbody>
<tr>
<td>Name</td>
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<td>system parameter</td>
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<td>system parameter</td>
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<td>Controllability func.</td>
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<td>Observability func.</td>
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<td>Cross map</td>
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<td>Pseudo-metric map</td>
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<td>state of $\mathcal{C}$</td>
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<td>state of $\partial \Phi$</td>
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Here $\mathcal{C}$ and $\mathcal{O}$ are controllability and observability operators of the system $\Sigma$, respectively, defined by
\[
\mathcal{C} : u \in L_{2-} \mapsto x(0) : \dot{x} = f(x) + g(x) u, \quad x(-\infty) = 0
\]
\[
\mathcal{O} : x(0) \mapsto y \in L_{2+} : \begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}
\]
The cross map in the linear case becomes
\[
\Phi(x) = X x
\]
with the cross Gramian $X$. It satisfies the following property.

**Lemma 1 (Nonlinear Sylvester equation)** The cross map $\Phi(x)$ satisfies
\[
\frac{\partial \Phi(x)}{\partial \xi} f(x) + f(\Phi(x)) + g(\Phi(x)) h(x) = 0. \quad (5)
\]
The Sylvester equation in the linear case is as follows.
\[
X A + A X + B C = 0
\]
The definition of the symmetry of nonlinear systems is as follows.

**Definition 2** The system $\Sigma$ is said to be symmetric if
\[
\text{Im}\mathcal{C}^\dagger = \text{Im}\mathcal{O}.
\]
A symmetric system has the following property.

**Lemma 2** For a symmetric system, the following property holds.
\[
L_o(x) = L_c(\Phi(x))
\]
We have a necessary and sufficient condition for the symmetry.

**Lemma 3** A nonlinear system is symmetric if and only if the cross map $\Phi(x)$ is invertible and satisfies
\[
h(x) = g(\xi)^T \nabla L_c(\xi) \bigg|_{\xi = \Phi(x)} \quad (6)
\]
and one of the following two equations
\[
\frac{\partial \Phi(x)}{\partial \xi} f(x) = \left( -f(\xi) - g(\xi)^T \nabla L_c(\xi) \right) \bigg|_{\xi = \Phi(x)} \quad (7)
\]
\[
\frac{\partial \chi(x)}{\partial \xi} f(x) = \left( f(\xi) + g(\xi)^T \chi(x) \right)^T \bigg|_{\xi = \Phi(x)} \chi(x). \quad (8)
\]
Eq. (6) corresponds to
\[ Cx = B^T W^{-1} X x = B^T T x \]
in the linear case. If the system is symmetric, the three dynamics \( \dot{\xi}, \dot{\xi}^T \), and \( \dot{\xi}^T (\dot{\xi})^T (p) \) are equivalent. This equivalence is characterized by Eqs. (7) and (8) in Lemma 3 which correspond to
\[ WTAx = -AWTx - BB^T T x \]
\[ TAx = A^T T x \]
in the linear case. Further, the latter reduces to
\[ TAx = -P x = - \frac{\partial}{\partial x} \left( \frac{1}{2} x^T P x \right) \]
which implies that the system is a gradient system with a potential function \( (1/2)x^T P x \).

3.2 Balanced realization for symmetric systems

This section gives a novel characterization of balanced realization for symmetric nonlinear systems and clarifies some of their properties. Consider a nonlinear symmetric system \( \Sigma \) of the form (1) and let us define the balanced realization for them as follows.

**Definition 3**

1. The controllability function \( L_c(x) \) is said to be balanced if it is separable with respect to the states such as
\[ L_c(x) = \sum_{i=1}^{n} l_{c,i}(x_i). \]

2. The observability function \( L_o(x) \) is said to be balanced if it is separable with respect to the states such as
\[ L_o(x) = \sum_{i=1}^{n} l_{o,i}(x_i). \]

3. The cross map \( \Phi(x) \) is said to be balanced if it satisfies
\[ \Phi(x) = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_2) \\ \vdots \\ \phi_n(x_n) \end{pmatrix}. \]

4. A symmetric system is said to be balanced if its observability function \( L_o(x) \) and the cross map \( \Phi(x) \) are balanced.

We also have its weaker version as follows.

**Definition 4**

1. The controllability function \( L_c(x) \) is said to be weakly balanced if it satisfies
\[ x_i = 0 \iff \frac{\partial L_c(x)}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n. \]

2. The observability function \( L_o(x) \) is said to be weakly balanced if it satisfies
\[ x_i = 0 \iff \frac{\partial L_o(x)}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n. \]

3. A symmetric system is said to be weakly balanced if its observability function \( L_o(x) \) is weakly balanced and if the cross map \( \Phi(x) \) is balanced.

Note that the balancing and weak balancing defined above correspond to the balanced realizations 1) and 2) in Theorem 1, respectively. We will prove some properties using those balanced realizations.

In the above definitions, observability requirement for symmetry can be replaced by controllability as follows.

**Proposition 1**

1. A symmetric system is balanced if and only if the controllability function \( L_c(\xi) \) and the cross map \( \Phi(x) \) are balanced.

2. A symmetric system is weakly balanced if and only if the controllability function \( L_c(\xi) \) is weakly balanced and the cross map \( \Phi(x) \) is balanced.

**Proof.**

1. Sufficiency: Since the system is symmetric, it follows from Lemma 2 and Definition 3 that
\[ L_o(x) = L_o(\Phi(x)) = \sum_{i=1}^{n} l_{o,i}(\xi_i)|_{\xi = \Phi(x)} = \sum_{i=1}^{n} l_{o,i}(\phi_i(x)) \]
which implies that \( L_o(x) \) is balanced.

   Necessity:
\[ L_c(\xi) = L_o(\Phi^{-1}(\xi)) = \sum_{i=1}^{n} l_{o,i}(x_i)|_{\xi = \Phi^{-1}(\xi)} \]
\[ = \sum_{i=1}^{n} l_{o,i}(\phi_i^{-1}(\xi)) \]
which implies that \( L_c(\xi) \) is balanced.

2. This part can be proved in a similar way to part 1).

Proof is completed. \( \square \)

The pseudo-metric map \( p = \chi(x) \) satisfies the following property.

**Lemma 4**

If a symmetric system is balanced, then its pseudo-metric map \( \chi(x) \) is also balanced such as
\[ \chi(x) = \begin{pmatrix} \chi_1(x_1) \\ \chi_2(x_2) \\ \vdots \\ \chi_n(x_n) \end{pmatrix}. \]
Proof. The statement can be readily obtained by
\[
\chi(x) = \nabla L_c(\xi)_{\xi=\Phi(x)} = (f_{\xi,1}(\xi_1), \ldots, f_{\xi,n}(\xi_n))^T|_{\xi=\Phi(x)} \\
= (f_{\xi,1}(\Phi_1(x_1)), \ldots, f_{\xi,n}(\Phi_n(x_n)))^T \\
= (\chi_1(x_1), \ldots, \chi_n(x_n))^T
\]
which completes the proof. □

If the system is symmetric and balanced, we can obtain the following input-normal realization which shows the intrinsic nature of the symmetric system.

**Theorem 3** Consider a symmetric system \( \Sigma \) of the form \( (1) \). Suppose that the assumptions of Theorem 1 hold and that \( \Sigma \) is balanced in the sense of Definition 3. Then there exists another balanced realization satisfying the following properties.

\[
\Phi'(x) = \Phi(x) = \begin{pmatrix} \pm x_1 \sigma_1(x_1) \\ \vdots \\ \pm x_n \sigma_n(x_n) \end{pmatrix} \tag{9}
\]

\[
L_c(\xi) = \frac{1}{2} \xi^T \xi \tag{10}
\]

\[
L_0(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \sigma_i(x_i)^2 \tag{11}
\]

Proof. First of all, once we obtain a realization balanced in the sense of Definition 3, its form is invariant for any coordinate transformation of the form

\[
x_i \mapsto \tilde{x}_i(x_i), \quad i = 1, 2, \ldots, n \tag{12}
\]

that is, expansion with respect to the coordinate axis. Applying the proof of Theorem 1-1 to this realization, then the input-normal balanced realization satisfying Eqs. (10) and (11) can be obtained by a coordinate transformation of the form (12). Then the result follows immediately (e.g., using Lemma 4).

In this way, the cross map \( \xi = \Phi(x) \) and the pseudometric map \( p = \chi(x) \) play important roles in characterizing balanced realizations of nonlinear symmetric systems as in the linear case. The following subsection discusses on model order reduction preserving symmetric property of the original system.

4 Conclusion

This paper proposed a new definition of a nonlinear symmetric system and discussed some of its properties with its balanced realization. This has been done by extending the well-known notions for linear system such as gross Gramians and Sylvester equations. It is expected to provide a new insight to nonlinear systems theory. The future work would take care of the existence of the solutions, the computational issue, and physical application and so on.