Stochastic Representations of the Solution of Hamilton-Jacobi-Bellman Equation for Nonlinear Control Systems

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Abstract—In this paper, stochastic representations are investigated for the solution of Hamilton-Jacobi-Bellman equation which is necessary to be solved for realizing the optimal control of stochastic nonlinear systems.

Key Words: Hamilton-Jacobi-Bellman equation, stochastic nonlinear system, Feynman-Kac formula, Schrödinger equation

1 Introduction

In order to realize the optimal control for dynamical systems, it is required to solve the Hamilton-Jacobi-Bellman (HJB) equation whether the system is linear or nonlinear. The HJB equation is a nonlinear partial differential equation of second-order or first-order according to the stochastic or deterministic control system. It is well-known that in the LQG (linear-quadratic-Gaussian) case for stochastic systems or in the LQ case for deterministic ones the solution of the HJB equation is quadratic in the system state. On the contrary, for the nonlinear stochastic/deterministic systems its solution is not given, in general, in the analytical form. For the way to solve the HJB equation for nonlinear systems, there seems mainly two approaches: One is to solve it by some iterative method, and the other is to transform it to a linear partial differential equation using the Cole-Hopf transformation, and then the resultant linear equation is solved via the Feynman-Kac formula.

As for the first approach mentioned above, a successive approximation method has been proposed in Ref. 1) by the author for both stochastic and deterministic nonlinear systems. The crucial key is to introduce the quasilinearization to the HJB equation. As for the second approach, several papers have appeared up to the present time (e.g., see Ref. 2), 3)). In this paper, the author investigates further the second approach from the stochastic representation point of view for the solution to the HJB equation.

2 Hamilton-Jacobi-Bellman Equation

Consider the nonlinear stochastic system with affinely connected with a control term,

\[ \begin{align*}
    dx(t) &= f(t, x(t)) dt + Cu(t) dt + G dw(t), \\
    x(0) &= x_0, \quad 0 \leq t \leq T
\end{align*} \]

where \( x(t) \in \mathbb{R}^n; u(t) \in \mathbb{R}^l (l \leq n) \) is a control to be determined; and \( w(t) \in \mathbb{R}^d \) is a Wiener process with zero-mean and covariance matrix \( Q(>0) \) (\( \mathcal{E}\{[dw(t)][dw(t)^T]\} = Q dt \)). The drift term \( f(t, x) \) is an \( n \)-vector-valued nonlinear function in \( x \) satisfying the Itô conditions, viz. Lipschitz and growth conditions \(^3\)). The matrices \( C, G, Q \) are constant.

Let \( S(t, x) \) be the minimal cost functional (value function),

\[ S(t, x) = \min_{u(t)} \mathcal{E}\left\{ \phi[x(T)] + \int_t^T \{L[\tau, x(\tau)] + u(\tau)Nu(\tau)\} d\tau \middle| x(t) = x\right\}, \quad (2) \]

where \( \phi(\cdot) \) and \( L(\cdot, \cdot) \) are positive-definite functions; and \( N \) is an \( \ell \times \ell \)-positive-definite and symmetric matrix. Then, via the Bellman’s principle of optimality, the optimal control \( u^*(t) \) is given by

\[ u^*(t) = -\frac{1}{2} N^{-1} C^T \frac{\partial S(t, x)}{\partial x}, \quad (3) \]

where \( S(t, x) \) is given as the unique positive-definite solution to the nonlinear partial differential equation of second-order called the Hamilton-Jacobi-Bellman (HJB) equation \(^4\):

\[ \begin{align*}
    -\frac{\partial S(t, x)}{\partial t} &= L(t, x) + f^T(t, x) \frac{\partial S(t, x)}{\partial x} \nonumber \\
    &+ \frac{1}{2} \text{tr} \left\{ GQG^T \frac{\partial}{\partial x} \left( \frac{\partial S(t, x)}{\partial x} \right)^T \right\} \\
    &- \frac{1}{4} \left( \frac{\partial S(t, x)}{\partial x} \right)^T C N^{-1} C^T \frac{\partial S(t, x)}{\partial x} \\
    S(T, x) &= \phi(x), \quad x \in \mathbb{R}^n. \quad (4)
\end{align*} \]

3 Stochastic Representation of Solution of the HJB Equation

The logarithmic Cole-Hopf transformation for the nonlinear partial differential equation (4),

\[ S(t, x) = -\ln p(t, x), \quad (5) \]
The terminal condition is given from
produced to a linear one by setting the design-parameter
of the scalar function
equation,
term in the R.H.S. of (6) disappears to give a linear
process (1) with
in which
is the differential generator of the control-
process described by
Hence, the HJB equation (4) is converted to the linear
Hence, the HJB equation (4) is converted to the linear
under the postulation (7),
for given $C$, $G$ and $Q$. In such a case, the fourth
term in the R.H.S. of (6) disappears to give a linear equation,

\[
\frac{\partial p(t,x)}{\partial t} = \mathcal{L} p(t,x) - L(t,x)p(t,x) \tag{8}
\]

The terminal condition is given from $S(T,x) = \lim_{t\to T} S(t,x) = \mathcal{E}\{\phi[x(T)] | x(T) = x\} = \phi(x), \quad p(T,x) = \exp\{-\phi(x)\}. \tag{9}

Hence, the HJB equation (4) is converted to the linear equation under the postulation (7),

\[
\begin{aligned}
- \frac{\partial p(t,x)}{\partial t} &= \mathcal{L} p(t,x) - L(t,x)p(t,x) \\
p(T,x) &= \exp\{-\phi(x)\}, \quad x \in \mathbb{R}^n
\end{aligned} \tag{10}
\]

in which $\mathcal{L}$ is the differential generator of the control-free process (1) with $u(t) \equiv 0$,

\[
\mathcal{L}(\tau) = \frac{1}{2} \text{tr}\left\{GQG^T \frac{\partial}{\partial x} \left( \frac{\partial (\cdot)}{\partial x} \right)^T \right\} + f^T(t,x) \frac{\partial (\cdot)}{\partial x}. \tag{11}
\]

Here, note that Eq. (10) is the Kolmogorov backward equation with killing rate $L(t,x)$, and the problem of solving (10) for given $L$ and $\phi$ is the Cauchy problem.

Associated with (10) there is an $n$-vector stochastic process described by

\[
d\xi(t) = f[\tau, \xi(t)] \, d\tau + G \, dv(\tau), \quad t \leq \tau
\]

where the drift term $f(\cdot, \cdot)$ is the same nonlinear function as in (1) and $\tau(\cdot)$ is a Wiener process having the same covariance matrix $Q$ as $w(t)$-process does, and the solution of the Cauchy problem (10) can be represented in terms of the process $\xi(t)$. To make its starting point $\xi(t) = x$ clear, denote (12) as

\[
\xi_{t,x}(\tau) = x + \int_t^\tau f[\sigma, \xi_{t,x}(\sigma)] \, d\sigma + \int_t^\tau G \, dv(\sigma) \tag{13}
\]

on the interval $[t,T]$. Then the solution of (10) is given by

\[
p(t,x) = \mathcal{E}\left\{ \exp\left[ - \int_t^T L[\tau, \xi_{t,x}(\tau)] \, d\tau \right] \cdot \exp\left\{ -\phi(\xi_{t,x}(T)) \right\} \right\}. \tag{14}
\]

Such a stochastic representation as (14) is given in the literature on the stochastic processes (e.g. Gihman and Skorohod5, Chap.3), Friedman6, Thm 5.3), Karlin and Taylor7, Chap. 15), and is now known as the Feynman-Kac formula. A plainer proof of the Feynman-Kac formula than that made in Ref. 7, 8) is given in Appendix.

Comment on (7): The above augment holds under the restriction (7), so that the weight matrix $N$ should be selected to satisfy (7) as a user-defined parameter if $C$, $G$ and $Q$ are given a priori.

For example, in case of $C = I_n$ (unit matrix of dimension $n$), then $N$ is given such as $N = \frac{1}{2}(GQG^T)$. Or, in case of $\ell \neq n$ and $N$ is set to be diagonal with the same entry $\alpha$, i.e. $N = \text{diag}\{\alpha_0, \cdots, \alpha_0\} = \alpha_0 I_t$, then $N^{-1} = (1/\alpha_0) I_t$, so that from (7) we can get

\[
\alpha_0 = \frac{\text{tr}(CC^T)}{2\text{tr}(GQG^T)} \quad (GQG^T > 0).
\]

Otherwise, assuming that rank $C = \ell$ to use the pseudoinverse $C^+ = (C^TC)^{-1}C^T \in R^{\ell \times n}$ of $C$, we can set $N$ as

\[
N = \frac{1}{2}(C^T C G Q G^T C)\, (C^T C G Q G^T C)^{-1}.
\]

4 Computation of Optimal Control

Based on the stochastic representation (14), $p(t,x)$ can be computed using the sample path generated by the process (12). So, we compute the expectation in (14) via ensemble average of $M$ sample paths under the condition $x(t) = x = \xi_{t,x}(t)$. Then, $p(t,x)$ is approximated by

\[
p_M(t,x) = \frac{1}{M} \sum_{k=1}^M \exp\left\{ - \int_t^T L[\tau, \xi_{t,x}^{(k)}(\tau)] \, d\tau \right\} \cdot \exp\left\{ -\phi(\xi_{t,x}^{(k)}(T)) \right\}, \tag{15}
\]

where $\xi_{t,x}^{(k)}(\tau) (\tau \in [t,T])$ is the $k$th sample path generated by

\[
d\xi_{t,x}^{(k)}(\tau) = f[\tau, \xi_{t,x}^{(k)}(\tau)] \, d\tau + G \, dv^{(k)}(\tau), \quad \xi_{t,x}^{(k)}(t) = x \quad \text{for all } k = 1, 2, \cdots, M. \tag{16}
\]

So, $S(t,x)$ may be approximated by

\[
S_M(t,x) := -\ln p_M(t,x). \tag{17}
\]
However, in order to compute the optimal control, the gradient ∂S(t, x)/∂x is required. This can be obtained by using S_M(t, x + Δx) and S_M(t, x - Δx) which are computed, respectively, using (17) at two spatial points x ± Δx (Δx > 0) nearby the realization x = x(t) at the present time t, i.e.,

\[
\frac{∂S(t, x)}{∂x} \approx \frac{S_M(t, x + Δx) - S_M(t, x - Δx)}{2Δx}
\]

= \frac{1}{2Δx} [ln p_M(t, x - Δx) - ln p_M(t, x + Δx)] \quad (18)

if we employ the familiar central difference scheme. Here, p_M(t, x ± Δx) are computed by (15) starting at ξ_{t,x±Δx}(t) = x ± Δx, respectively.

5 Real Schrödinger Equation

In this section, assume that \( f(t, x) = f(x) \) and \( L(t, x) = L(x) \). Write as \( Q_0 = GQG^T \).

**Theorem.** Suppose there exists a scalar function \( F(x) \) with continuous first partial derivative such that

\[
\frac{∂F(x)}{∂x} = Q_0^{-1} f(x).
\]

Then, the Cauchy problem (10) is converted by the gauge transformation,

\[
p(t, x) = ψ(t, x) e^{-F(x)},
\]

(20)
to

\[
\begin{align*}
\frac{∂ψ(t, x)}{∂t} &= \mathcal{H}ψ(t, x) \\
ψ(T, x) &= \exp\{F(x) - φ(x)\}
\end{align*}
\]

in which

\[
\mathcal{H}(·) = -\frac{1}{2} \text{tr} \left\{ Q_0 \frac{∂}{∂x} \left( \frac{∂(·)}{∂x} \right)^T \right\} + V(x)(·)
\]

(22)
is a Hamiltonian operator with potential function

\[
V(x) = \frac{1}{2} \text{tr} \left\{ Q_0 \frac{∂}{∂x} \left( \frac{∂F(x)}{∂x} \right)^T \right\} + \frac{1}{2} \left( \frac{∂F(x)}{∂x} \right)^T Q_0 \frac{∂F(x)}{∂x} + L(x).
\]

(23)

Equation (21) is referred to as the *imaginary-time analogue of the Schrödinger equation* with potential \( V(x) \).

**Proof of Theorem:** From (20) we know that the L.H.S. of the first equation in (10) becomes as

\[
\frac{∂p}{∂t} = -e^{-F} \frac{∂ψ}{∂t},
\]

while the R.H.S. becomes

\[
\mathcal{L}p - L(x)p = - \left\{ \frac{1}{2} \text{tr} \left\{ Q_0 \frac{∂}{∂x} \left( \frac{∂ψ}{∂x} \right)^T \right\} + \frac{1}{2} \left( \frac{∂ψ}{∂x} \right)^T Q_0 \frac{∂ψ}{∂x} + f^T(x) \frac{∂ψ}{∂x} \right\} - e^{-F} \left\{ \frac{1}{2} \text{tr} \left\{ Q_0 \frac{∂}{∂x} \left( \frac{∂F}{∂x} \right)^T \right\} + \frac{1}{2} \left( \frac{∂F}{∂x} \right)^T Q_0 \frac{∂F}{∂x} + f^T(x) \frac{∂F}{∂x} + L(x) \right\} ψ,
\]

where the relation \( \text{tr}\{Q_0ab^T\} = b^TQ_0a \) (a, b ∈ ℝ^n) has been used. So, multiplying both sides by \( e^F \), we obtain

\[
- \frac{∂ψ}{∂t} = \frac{1}{2} \text{tr} \left\{ Q_0 \frac{∂}{∂x} \left( \frac{∂ψ}{∂x} \right)^T \right\} - \left( \frac{∂F}{∂x} \right)^T Q_0 \frac{∂ψ}{∂x} + f^T(x) \frac{∂F}{∂x} + L(x) \]

(24)

The terminal condition is given from (9) and (20). (Q.E.D.)

For (21), there corresponds a stochastic process \( ξ_{t,x}(τ) \in ℝ^n \) governed by

\[
\dot{ξ}_{t,x}(τ) = G \dot{v}(τ), \quad t ≤ τ ≤ T
\]

(25)
or

\[
ξ_{t,x}(τ) = x + Gv(τ) \quad (v(t) = 0),
\]

(26)
where \( v(τ) \) is a Wiener process having the same covariance matrix \( Q \) as \( ω(·) \) and \( v(·) \) do. So, the stochastic representation of the solution to (21) is given by

\[
ψ(t, x) = \mathcal{E} \left\{ \exp \left[ -\int_t^T V(ξ_{t,x}(τ)) \, dτ \right] \cdot \exp \left\{ F[ξ_{t,x}(T)] - φ[ξ_{t,x}(T)] \right\} \right\}
\]

(27)
in which the expectation is taken under the measure of the Wiener process \( v(τ) \).

Noting that the solution process \( ξ_{t,x}(τ) \) given by (13) is generated by the Wiener process \( v(τ) \) whose covariance is the same as that of \( v(τ) \). By the Girsanov’s theorem, \( v(τ) \) and \( v(τ) \) are equivalent. Using the Girsanov’s theorem, \( ξ_{t,x}(τ) \) coincides with \( ξ_{t,x}(τ) \), i.e., \( ξ_{t,x}(τ) = ξ_{t,x}(τ) \) if

\[
Gv(τ) = Gv(τ) + \int_t^T f[ξ(σ)] \, dσ,
\]

(28)
Then, \( f \) satisfies the same equation as (21) is obtained as far as potential or \( df \) which gives just the above equation for \( \pi \).

\[
V(x) = \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^T f(x) + \frac{1}{2} \left\| f(x) \right\|^2_{Q_0^{-1}} + L(x)
\]

or

\[
V(x) = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} f_i(x)[Q_0^{-1}]_{ij} f_j(x) + L(x).
\]

**Remark:** If we use the transformation \( p(t,x) = \pi(x)\psi(t,x) \) (\( \pi(x) \): scalar function) instead of (20), the same equation as (21) is obtained as far as \( \pi(x) \) satisfies

\[
\frac{\partial \pi(x)}{\partial x} + Q_0^{-1} f(x) \pi(x) = 0.
\]

In view of (20) we see that \( \pi(x) = e^{-F(x)} \); so (19) leads us to

\[
Q_0^{-1} f(x) = \frac{\partial F(x)}{\partial x} = -\frac{1}{\pi(x)} \frac{\partial \pi(x)}{\partial x}
\]

which gives just the above equation for \( \pi(x) \).

**Example 1.** For \( n = 1 \), let \( f(x) = \tanh x \) and \( Q_0 = 1 \). Then, \( df(x)/dx = \text{sech}^2 x \), so that we have \( df(x)/dx + f^2(x) = \text{sech}^2 x + \tanh^2 x = 1 \). Then, the potential \( V(x) \) is given by

\[
V(x) = \frac{1}{2} + L(x).
\]

This example is suggestive. If \( f(x) \) satisfies the Riccati equation such as \( df(x)/dx + f^2(x) = ax^2 \), then \( V(x) = \frac{1}{2}ax^2 + L(x) \), so the potential can become quadratic.

**Example 2.** Let \( x \in \mathbb{R}^n \) and \( Q_0 = I_n \), and consider

\[
F(x) = \ln \frac{\sinh r}{r} \quad (r = \|x\| = (\Sigma x_i^2)^{1/2}).
\]

Then, \( f(x) \) is given by

\[
f(x) = \frac{\partial F(x)}{\partial x} = \left( \coth r - \frac{1}{r} \right) \frac{1}{r} x.
\]

In this case, somewhat tiresome calculations yield that

\[
\left( \frac{\partial}{\partial x} \right)^T f(x) + \left\| f(x) \right\|^2 = \frac{1}{r^2} \left\{ r^2 + (n-3)r \coth r + (3-n) \right\},
\]

so if \( n = 3 \), this reduces to 1. Hence, the potential becomes also \( V(x) = \frac{1}{2} + L(x) \).

The quadratic potential seems familiar and for such a case the eigenvalue problem of the Hamiltonian operator is fundamental in quantum mechanics for the time-independent Schrödinger (e.g., see Ref. 11).

### 6 LQG Case

It will be worthwhile to investigate the LQG case. Let \( f(x) = Ax + b \), where \( x \in \mathbb{R}^n \); and \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \), \( b \in \mathbb{R}^n \) are constant. And let \( Q_0 = I_n \). Since

\[
\left( \frac{\partial}{\partial x} \right)^T (Ax + b) = \sum_i^n \sum_j^n a_{ij} x_j
\]

the potential becomes as

\[
V(x) = \frac{1}{2} \text{tr} A + \frac{1}{2} x^T A x + b^T x + \frac{1}{2} b^T b + L(x).
\]

Thus, the potential \( V(x) \) is expressed in terms of \( f(x) = Ax + b \) without using \( F(x) \).

However, it is required to compute \( \psi(t,x) \) by (21). In this case, if \( A \) is symmetric, \( F(x) \) is given by \( F(x) = \frac{1}{2} x^T A x + b^T x \) by intuition. Indeed, from (19) we have \( \partial F(x)/\partial x = Ax + b \), or \( F(x) = f(Ax + b)^T dx \). The symmetric matrix \( A \) is expressed as \( A = O^T A O \) with an orthogonal matrix \( O \) and the diagonal \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) whose entries are the eigenvalues of \( A \). Hence,

\[
F(x) = \int x^T O^T A O dx + \int b^T dx.
\]

Here, let \( y = O x \in \mathbb{R}^n \). Then,

\[
\int x^T O^T A O dx = \int y^T A dy \quad (dy = O dx)
\]

\[
= \int (\lambda_1 y_1 dy_1 + \cdots + \lambda_n y_n dy_n)
\]

\[
= \frac{1}{2} (\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2) = \frac{1}{2} y^T A y
\]

\[
= \frac{1}{2} x^T (O^T A O)x = \frac{1}{2} x^T A x,
\]

so that we get \( F(x) = \frac{1}{2} x^T A x + b^T x \).

In the sequel, consider the quadratic potential. To do this, let \( L(x) = x^T M x \) and \( \phi(x) = x^T M_T x \) for symmetric, positive-definite matrices \( M \) and \( M_T \). Then,
it is a simple exercise to show that the solution to (21) is given by
\[
\psi(t, x) = \exp \left\{ -[x^T P(t) x + \alpha^T (t) x + \beta(t)] \right\},
\]
where \( P(t) \in \mathbb{R}^{n \times n} \), \( \alpha(t) \in \mathbb{R}^n \) and \( \beta(t) \) (scalar function) are solutions to
\[
\begin{align*}
\dot{P}(t) + \frac{1}{2} A^T P + P A - 2 P^2 A &= 0, \quad P(0) = P_0,
\end{align*}
\]
\[
\begin{align*}
\dot{\alpha}(t) + A^T \alpha(t) + \beta(t) &= 0, \quad \alpha(0) = -\beta.
\end{align*}
\]
\[
\begin{align*}
\dot{\beta}(t) + \frac{1}{2} \text{tr} P(t) + \frac{1}{2} b^T b - \frac{1}{2} \alpha^T (t) \alpha(t) &= 0, \\
\beta(T) &= 0.
\end{align*}
\]
Whenever \( M_T \) is selected such that \( M_T \geq \frac{1}{2} A \), the above matrix Riccati equation has a unique positive-definite solution \( P(t) \). Note that if \( b = 0 \), then \( \alpha(t) \equiv 0 \) for all \( t \).

7 Conclusion

In this paper, we have investigated how the solution of the HJB equation can be represented by introducing new stochastic processes. Furthermore, a real Schrödinger equation has been derived using the gauge transformation (20) whose scalar function \( F(x) \) is obtained by (19) from the nonlinear system function \( f(x) \). However, to solve (19) for an arbitrary vector-valued nonlinear process may be, in general, hard.

References


APPENDIX: Derivation of Feynman-Kac Formula

Feynman-Kac Formula: Consider a Cauchy problem for a scalar function \( u(t, x) \),
\[
\begin{align*}
- \frac{\partial u(t, x)}{\partial t} &= \frac{1}{2} \text{tr} \left\{ Q(t, x) \frac{\partial^2 u(t, x)}{\partial x^2} \right\} \\
+ f(t, x) \frac{\partial u(t, x)}{\partial x} - V(t, x) u(t, x)
\end{align*}
\]
\[
u(T, x) = \rho(x),
\]
where \( f(t, x), \ Q_0(t, x) \) are \( n \)-vector-valued and \( n \times n \)-matrix-valued functions, and \( V(t, x) \) is a bounded scalar function. Then, its solution can be expressed as a conditional expectation
\[
u(t, x) = \mathcal{E} \left\{ \exp \left[ -\int_t^T V(\tau, \xi(\tau)) d\tau \right] \rho(\xi(T)) \bigg| \xi(t) = x \right\}
\]
in which \( \{\xi(\tau), t \leq \tau \leq T\} \) is an \( n \)-vector Itô stochastic process driven by the Wiener process with covariance matrix \( Q(t) (> 0) \),
\[
d\xi(\tau) = f(\tau, \xi(\tau)) d\tau + G(\tau, \xi(\tau)) d\omega(\tau), \quad \xi(t) = x \tag{A.3}
\]
and \( Q_0(t, x) \) in (A.1) is identified with \( Q_0(t, x) = G(t, x) Q(t) G^T(t, x) \).

Proof: Let
\[
\eta(\tau) = e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} u(\tau, \xi_\tau), \quad t \leq \tau \leq T \tag{A.4}
\]
where \( \xi_\tau = \xi(\tau) \) is the solution process of (A.3).

Applying Itô formula \( 3) \) to \( \eta(\tau) \), we have
\[
d\eta(\tau) = d\left( e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} u(\tau, \xi_\tau) \right) = d\left( e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} u(\tau, \xi_\tau) \right) + d\left( e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} d\omega(\tau, \xi_\tau) \right) + d\left( e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} du(\tau, \xi_\tau) \right). \tag{A.5}
\]
Here,
\[
d(e^{-\int_t^\tau V(\sigma, \xi_\sigma) d\sigma}) = e^{-\int_t^\tau \frac{\partial u(\tau, \xi_\sigma)}{\partial \tau} + L u(\tau, \xi_\sigma) d\tau} \\
+ e^{-\int_t^\tau V(\sigma, \xi_\sigma) d\sigma} \left( \frac{\partial u(\tau, \xi_\sigma)}{\partial \tau} \right)^T G(\tau, \xi_\sigma) dw(\tau) + o(d\tau)
\]
and applying Itô formula to \( u(\tau, \xi_\tau) \) yields
\[
du(\tau, \xi_\tau) = \left[ \frac{\partial u(\tau, \xi_\tau)}{\partial \tau} + L u(\tau, \xi_\tau) \right] d\tau \\
+ \left( \frac{\partial u(\tau, \xi_\tau)}{\partial \tau} \right)^T G(\tau, \xi_\tau) dw(\tau),
\]
where \( L \) is the differential generator defined by (11) in which \( GQG^T \) is replaced by \( Q_0(t, x) = G(t, x)Q(t)G^T(t, x) \).

From these two we know that the last term in the most R.H.S. of (A.5) is of order \( o(d\tau) \). Therefore, we get
\[
d\eta(\tau) = e^{-\int_t^\tau V(\sigma, \xi_\sigma) d\sigma} \left[ \frac{\partial u(\tau, \xi_\tau)}{\partial \tau} + L u(\tau, \xi_\tau) \right] d\tau \\
+ e^{-\int_t^\tau V(\sigma, \xi_\sigma) d\sigma} \left( \frac{\partial u(\tau, \xi_\sigma)}{\partial \tau} \right)^T G(\tau, \xi_\sigma) dw(\tau) + o(d\tau)
\]
in which (A.1) has been used.

Integrating both sides of the resultant equation from \( t \) to \( T \) and taking expectations conditioned on \( \xi(t) = x \), we get
\[
\mathcal{E}\{\eta(T) - \eta(t) \mid \xi(t) = x\} \\
= \mathcal{E}\left\{ e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} \left( \frac{\partial u(\tau, \xi_\tau)}{\partial \tau} \right)^T G(\tau, \xi_\tau) dw(\tau) \mid \xi(t) = x\right\}
\]
so it follows that
\[
\mathcal{E}\{\eta(T) \mid \xi(t) = x\} = \mathcal{E}\{\eta(t) \mid \xi(t) = x\} \\
= \mathcal{E}\left\{ e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} u(\tau, \xi_\tau) \mid \tau = t, \xi(t) = x\right\}
\]
Therefore, from (A.8) and (A.4) it follows that
\[
u(t, x) = \mathcal{E}\{\eta(T) \mid \xi(t) = x\} \\
= \mathcal{E}\left\{ e^{-\int_t^T V(\sigma, \xi_\sigma) d\sigma} u(\tau, \xi_\sigma) \mid \tau = T, \xi(t) = x\right\}
\]
or
\[
u(t, x) = \mathcal{E}\left\{ e^{-\int_t^T V(\sigma, \xi, \sigma) d\sigma} u(T, \xi_t(\xi(\tau)))\right\},
\]
where \( \xi_t(\sigma) \) denotes \( \xi(\sigma) \) at \( \sigma \geq t \) having started at \( x(t) = x \). Since \( u(T, \xi_t(T)) = p(T, \xi_t(T)) \), (A.2) is obtained. (Q.E.D.)

Replacing the Wiener process \( u(t) \) above by \( v(t) \) and identifying \( u(t, x) \), \( p(x) \) and \( V(\cdot, \cdot) \) with \( p(t, x) \), \( p(T, x) = \exp\{ -\phi(x) \} \), \( L(\cdot, \cdot) \), respectively, Eq. (14) is obtained.