Mesoscopic Analysis of Mortar under High-Stress Creep and Low-Cycle Fatigue Loading

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Abstract

Mesoscopic analyses of mortar failure under high-stress creep and low-cycle fatigue loading are presented using a newly developed time-dependent constitutive model for Rigid Body Spring Model, which is a discrete analysis method. The failure process over time was successfully expressed by adopting a four-component combined mechanical model as the time-dependent model of connected springs, and by developing a new method for determining the failure state for load-controlled analysis. The numerical model provides reasonable results not only for the stress-strain characteristics under cyclic loading but also for the inapplicability of Miner’s law under varying stress levels. The mechanism of the time-dependent failure of mortar was clarified by investigating the local stress-strain behaviors.

1. Introduction

Concrete failure due to fatigue and creep is one of the primary problems for infrastructure that must resist traffic or wave loads. The failure process for concrete, as well as metals, can be explained by initial micro cracking that increases over time, which reduces the material’s strength to below that of the applied stress, as reported by Bennett et al. (1969) in their experiment. That is, fatigue failure occurs in concrete due to the buildup of damage inside the material over time. This experimental fact shows that fatigue and creep are forms of material deterioration. Even if the number of loading cycles is less than fatigue life cycles, the structure may need repair or reconstruction because the accumulated damage diminishes the structure’s mechanical properties and durability. Therefore, we must know not only when failure occurs but also how much damage has accumulated during the structure’s service life.

In the current design code for concrete structures, fatigue is represented by the S-N curve and the Miner’s law (cumulative damage law). Using these methods, we can determine the relationship between the applied load and the fatigue life. However, we cannot determine the structure’s serviceability and durability because the other properties, such as deformation and cracking are unknown. Additionally, we cannot determine the structure’s restorability because the progression of damage during

the service life of the structure is also unknown. In other words, the current design for fatigue is not applicable to performance-based design. To solve this problem, we need a new design method that can account for not only the fatigue life but also the changes in other properties over time due to fatigue.

Past studies of fatigue subjected to properties other than fatigue life generally have been experimental studies of the stress-strain characteristics (For example, Hatano et al. 1962, Muguruma et al. 1970 and Matsushita et al. 1979.) These studies clarified the relationship between concrete deformation and fatigue loading cycle, the transition of unloading-reloading curve, and changes in the internal stiffness. In addition, the common point curve (Kosaka et al. 1978), storage energy concept (Fujimoto et al., 2000), and stress-strain model (Matsumoto et al. 2004) have been used to quantitatively estimate the deformational behavior. Analytical studies are now starting to be applied. Tanabe et al. (1998) analytically expressed the effect of the strain rate and unloading/reloading on concrete behavior using a rheological model. El-Kashif et al. (2004) proposed a time-dependent concrete constitutive model that incorporates a time effect into the stress-strain relationship. Maekawa et al. (2006) carried out fatigue FE analysis of concrete members by extending El-Kashif’s model to the fatigue problem. Substance transfers and chemical reactions inside the material, which depend on microstructures such as voids and micro cracks, significantly affect material durability. Therefore, it is difficult to clarify deterioration mechanisms in real environments if concrete is treated as a homogeneous material, as it has been in previous studies.

In this study, the behavior of mortar, which is the main constituent of concrete, is targeted as a fundamental study of concrete behavior by mesoscopically analyzing mortar failure under high stress creep and low cycle fatigue. Mortar was modeled as a heterogeneous material, and the fracture mechanism, which is explained as a

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process of micro crack generation, stress release-redistribution, and crack growth over time, was investigated.

2. Method of the analysis

The RBSM (Rigid Body Spring Model) developed by Kawai and Takeuchi employs a discrete numerical analysis method (Kawai 1977, Kawai and Takeuchi 1990). Compared with common discrete analysis methods, such as the Distinct Element Method (Cundall and Strack 1979), RBSM is suitable for small deformation problems. Bolander and Saito (1998), Ueda et al. (1988) and Nagai et al. (2004) have used RBSM for analysis of cement-based materials and structures.

In RBSM, the analytical model is divided into polyhedron elements whose faces are interconnected by springs. Each element has two transitional and one rotational degree of freedom at the center of gravity. Normal and shear springs are placed at the boundary of the elements (see Fig. 1). Since cracks initiate and propagate along the boundary face, the mesh arrangement may affect fracture direction. To avoid the formation of cracks in a certain direction, random geometry is introduced using a Voronoi diagram (see Fig. 2). A Voronoi diagram is a collection of Voronoi cells. Each cell represents a mortar element in the analysis. For Voronoi meshing, geometric computational software developed by Sugiura (1998) is applied. Size of each element is controlled to be 2-3 [mm²].

In the nonlinear analysis, a stiffness matrix is constructed on the principle of virtual work (Kawai and Takeuchi 1990), and the Modified Newton-Raphson method is employed for the convergence algorithm. In the convergence process, displacements that cancel the unbalanced force of elements are added to the elements. The displacements are calculated using the stiffness matrix. Convergence in the model occurs when the ratio of the sum of the squares of the unbalanced forces of the elements in the model to the sum of the squares of applied force becomes less than $10^{-5}$. When the model does not converge at the given maximum iterative calculation number, analysis proceeds to the next step. The maximum iterative number is set to 100 in this study. These values were determined based on sensitivity analysis. The remaining unbalanced forces of elements after the iteration process are added at the next step.

3. Constitutive model

3.1 Mechanical model

Macroscopic compression failure of concrete is caused by the accumulation of mesoscopic tensile and shear fractures (Nagai et al. 2004). The deformational concept that is employed in this study is shown in Fig. 3. That is, deformation in the normal and shear directions are a combination of deformation of uncracked part and deformation at crack. Furthermore, dividing each deformation into time-dependent and time-independent components produces the four-component combined mechanical model shown in Fig. 4. In this study, RBSM is extended to the time-dependent problem by replacing the spring in Fig. 1 with the four-component combined model. The total strain in the four-component combined
model is the summation of the elastic, visco-elastic, plastic, and visco-plastic strains.

3.2 Model in normal direction
3.2.1 Elastic component
The elastic component is an elastic spring element. The stress-strain relationship follows Hooke’s law.

\[
\sigma = k \varepsilon_e
\]  

(1)

where, \( \sigma \) is stress of the elastic spring, \( k \) is elastic modulus of the elastic spring, and \( \varepsilon_e \) is elastic strain.

3.2.2 Visco-elastic component
The visco-elastic component is an elastic spring element and a dashpot that are connected in parallel (Voigt model). It corresponds to a component that does not cause damage. The stress-strain relationship of the dashpot follows Newtonian viscosity law.

\[
\sigma = k \varepsilon_{ve}
\]  

(2)

\[
\sigma = c \frac{d\varepsilon_{ve}}{dt}
\]  

(3)

where, \( \sigma \) is stress of the visco-elastic spring, \( k \) is elastic modulus of the visco-elastic spring, \( \varepsilon_{ve} \) is visco-elastic strain, \( c \) is stress of the visco-elastic dashpot, \( c \) is viscosity coefficient of the visco-elastic dashpot, and \( t \) is time.

3.2.3 Plastic component
Deformation due to cracking is the sum of the time-independent component (plastic strain) and the time-dependent component (visco-plastic strain). Plastic strain, which is described by a slider element, represents the static tensile softening characteristic in the mesoscale. Therefore, mechanical behavior of the plastic component should be described by crack opening displacement (COD), which is related to plastic strain by following equation.

\[
\omega_p = (h_1 + h_2) \varepsilon_p
\]  

(4)

where, \( \omega_p \) is the time-independent COD, \( \varepsilon_p \) is the plastic strain, \( h_1 \) and \( h_2 \) are length of perpendicular lines from the element computational point to the face as shown in Fig. 1.

The stress-COD relationship shown in Fig. 5 was determined based on the model proposed by Nagai et al.

\[
\omega_p = 0 \quad \text{(Path[A])}
\]

\[
\sigma_{pl} = f_{pl} \left( 1 - \frac{\omega_p}{\omega_{u1}} \right) \quad \text{(Path[B])}
\]  

(5)

\[
\sigma_{pl} = \frac{\omega_p}{\omega_{u1}} (\sigma_{un1} - \sigma_{r1}) + \sigma_{r1} \quad \text{(Path[C])}
\]

\[
\sigma_{pl} = 0 \quad \text{(Path[D])}
\]

where, \( \sigma_{pl} \) is stress of slider, \( f_{pl} \) is tensile strength of slider, \( \omega_{u1} \) is limit COD for transferring stress, and \( \sigma_{r1} \) is residual COD indicator.

3.2.4 Visco-plastic component
Visco-plastic strain is described by the slider element and the dashpot, which are connected in parallel, and represents the time-dependent tensile softening characteristic in the mesoscale. Relationship between the visco-plastic strain and COD is given as below.

\[
\omega_{vp} = (h_1 + h_2) \varepsilon_{vp}
\]  

(6)

where, \( \omega_{vp} \) is the time-dependent COD, and \( \varepsilon_{vp} \) is the visco-plastic strain.

The slider follows the stress-COD relationship shown in Fig. 5 and is the same as that for the plastic component.
3.2.5 Relationships among the components

Each strain and stress component has the following relationships.
\[
\varepsilon = \varepsilon_e + \varepsilon_{ve} + \varepsilon_p + \varepsilon_{vp} \\
\sigma = \sigma_1 + \sigma_{ve} + \sigma_1 = \sigma_{p1} + \sigma_2
\]
(10)
where, \(\varepsilon\) is the total strain, and \(\sigma\) is the total stress.

3.3 Model in shear direction

3.3.1 Elastic and visco-elastic component

The stress-strain relationship of the elastic spring and dashpot elements is the same as that in the normal direction.
\[
\tau_{e1} = k_3 \gamma_e \\
\tau_{e2} = k_4 \gamma_{ve} \\
\tau_{e1} = c_3 \frac{d\gamma_{ve}}{dt}
\]
(11, 12, 13)
where, \(\tau_{e1}\) is stress of the elastic spring, \(k_3\) is elastic modulus of the elastic spring, \(\gamma_e\) is the shear elastic strain, \(\tau_{e2}\) is stress of the visco-elastic spring, \(k_4\) is elastic modulus of the visco-elastic spring, \(\gamma_{ve}\) is the shear visco-elastic strain, \(\tau_{e1}\) is stress of the visco-elastic dashpot, and \(c_3\) is viscosity coefficient of the visco-elastic dashpot.

3.3.2 Plastic and visco-plastic component

Slider elements represent the shear transfer characteristic between the crack surfaces at the mesoscale. Therefore, mechanical behavior of plastic component should be described by crack slipping displacement (CSD), which is related to plastic strain by following equation.
\[
\delta_p = (h_1 + h_2) \gamma_p \\
\delta_{vp} = (h_1 + h_2) \gamma_{vp}
\]
(14)
where, \(\delta_p\) is the time-independent CSD, \(\gamma_p\) is the plastic strain, \(\delta_{vp}\) is the time-dependent CSD, and \(\gamma_{vp}\) is the visco-plastic strain.

The stress-CSD relationship follows the rigid plastic model shown in Fig. 6. The value of \(\tau_{max}\) changes according to the condition of the normal stress and strain, and is given as follows (see Fig. 7).

In case of, \(\sigma_{p1} < f_t\)
\[
\tau_{max} = \left\{ \begin{array}{ll}
\tau_{max1} & (\omega_p > 0) \\
\tau_{max2} & (\omega_p = 0) 
\end{array} \right.
\]
(15)
where, \(\tau_{max1}\) or \(\tau_{max2}\) is the maximum stress for shear transfer region.

Fig. 6 Slider model in shear direction.

Fig. 7 \(\tau_{max}\) and \(\tau_{max2}\) criterion.
In case of $\sigma_p^2<\sigma_f^2$,

$$\tau_{\text{max}}=\frac{1}{2}\left[\frac{\left|\left(1-\sigma_f/\rho^2\right)/\sigma_f\right|}{2}\right]$$

This relationship indicates not only enhanced shear strength due to normal stress but also contraction of the failure surface due to normal strain development.

Dashpot element follows the Newtonian law as well as in the normal direction.

$$\tau_{c2} = c_4 \frac{d\delta_n}{dt}$$

The viscosity of the visco-plastic component is reduced with COD development as well as in the normal direction. The viscosity coefficient decreases linearly with visco-plastic strain in the normal direction.

$$c_4 = \begin{cases} c_4 \left(1 - \frac{\omega_p}{\omega_c}\right) & (\omega_p < \omega_c) \\ 0 & (\omega_p \geq \omega_c) \end{cases}$$

where, $c_{4i}$ is initial value of viscosity coefficient of visco-plastic component in shear direction.

### 3.3.3 Relationships among the components

Each strain and stress component has the following relationships.

$$\gamma = \gamma_s + \gamma_p + \gamma_{vp}$$

$$\tau = \tau_{c1} = \tau_{c2} + \tau_{c1} = \tau_{c1} + \tau_{c2} + \tau_{c2}$$

### 3.4 Characteristics of the four-component combined model

Characteristics of the four-component combined model are examined. First, sustained displacement is applied to the model in the normal and shear directions in the mid-stream of softening branch. The obtained stress-strain responses are shown in Figs. 8-(a) and (b). The tensile and shear stresses decrease over time. In other words, relaxation occurs.

Next, a gradual increasing-strain cyclic load is applied to the model in the normal and shear directions. The obtained stress-strain responses are shown in Figs. 8-(c) and (d). The stress after unloading-reloading in the mid-stream of softening branch does not recover to the initial stress level after unloading. That is, under cyclic loading, stress decrement occurs as well as under sustained displacement.

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**Fig. 8** Behavior of single four-components combined mechanical model.
The other remarkable point is that no softening occurs in compression because all deformational components have no compression failures in normal direction, as shown in Eqs. (1), (2), and (3) but only tension failure as shown in Fig. 5. That is, macroscopic failure is the accumulation of tensile and shear fractures in the meso level (see Fig. 9), as shown by Nagai et al. (2004) in their static analysis.

4. Material constants

4.1 Macroscopic material properties

Kosaka et al. (1975) proposed the following relationships between compressive strength and the water to cement ratio (Eq. (20)), the elastic modulus (Eq. (21)) and the static tensile strength (Eq. (22)).

\[ f_{cm} = 21.3C/W - 10.6 \] (20)

\[ E_m = 10^6[7.7\ln(f_{cm}) - 5.5] \] (21)

\[ f_{tp} = 1.4\ln(f_{cm}) - 1.5 \] (22)

where, \( f_{cm} \) is static compressive strength of mortar [MPa], \( C \) is unit quantity of cement [kg/m^3], \( W \) is unit quantity of water [kg/m^3], \( E_m \) is static elastic modulus of mortar [MPa], and \( f_{tp} \) is pure static tensile strength of mortar [MPa].

Ayano (1993) proposed the following linear equation for the basic creep strain of concrete at low stress levels \((\sigma < 0.5)\).

\[ \varepsilon_{bc} = \sigma \cdot \varepsilon_{bc}'(1 - \exp(-0.09\theta^{0.6})) \] (23)

\[ \varepsilon_{bc}' = 15(C + W)^{2}(W/C)^{1.2}([\ln(\tau')]^{-0.67} \times 10^{-10}) \] (23)

where, \( \varepsilon_{bc} \) is basic creep strain of concrete, \( \sigma \) is stress [MPa], \( \theta \) is time [days] from the start of loading, \( \tau' \) is curing time [days], and \( \varepsilon_{bc}' \) is limit of basic creep strain per unit stress [MPa].

The limit of basic creep strain per unit stress \( \varepsilon_{bc}' \) is independent of the loading condition and time because it is one of the material properties. Neville proposed the following coefficient, which represents the effect of the aggregate volume on creep strain.

\[ \varepsilon_{c'} = \varepsilon_{c'} = (1 - g)^{1.9} \] (24)

where, \( \varepsilon_c \) is creep strain of concrete, \( \varepsilon_{c'} \) is creep strain of cement paste, and \( g \) is volumetric ratio of aggregate.

The average aggregate volume ratio in Ayano’s experiment (1993) is 0.69. Therefore, the correction coefficient of Eq. (21) for mortar becomes as follows.

\[ \varepsilon_{bc}' = \varepsilon_{bc}' = (1 - S / \rho)^{1.9} / (1 - 0.69)^{1.9} \] (25)

\[ = (1 - S / \rho)^{1.9} / 0.108 \]

where, \( \varepsilon_{bc}' \) is limit of basic creep strain of mortar per unit stress [MPa], \( S \) is aggregate volume of mortar [kg/m^3], and \( \rho \) is density of aggregate [kg/m^3].

Therefore, the limit of the basic creep of mortar per unit stress is given by following equation.

\[ \varepsilon_{bc}' = \varepsilon_{bc}' = \frac{(1 - S / \rho)^{1.9}}{0.108} \] (25)

Since the basic creep strain under low stress levels is represented by the visco-elastic component, the inverse of \( \varepsilon_{bc}' \), which is the stress per unit basic creep strain, corresponds to the macroscopic elastic modulus of the visco-elastic component.

\[ E_{cr} = \frac{1}{\varepsilon_{bc}' \rho} \] (26)

where, \( E_{cr} \) is elastic modulus of the visco-elastic component [MPa].

In this study, the macroscopic material properties are calculated from the mix proportion and curing condition of the mortar using the flowchart shown in Fig. 10. Both static and creep Poisson’s ratio are assumed as 0.2.

4.2 Mesoscopic material properties

In the analysis, due to the unique characteristics of the RBSM, the values of the material properties at the meso level that are assigned to the elements differ from the material properties of the object analyzed at the macroscopic level. This is because the rigid body spring network is not a continuum. In this study, the material
properties of the elements were determined in such a way as to give the correct macroscopic properties. For this purpose, Nagai et al. (2004) carried out elastic analysis and proposed equations that relate the mesoscopic and macroscopic elastic modulus and Poisson’s ratio, which are apparent response as a result of more micro behavior as well as the macroscopic properties.

\[ v_{\text{elem}} = 20\nu^3 - 13.8\nu^2 + 3.8\nu \quad (0 \leq \nu < 0.3) \]
\[ \bar{E}_{\text{elem}} = (-8\nu_{\text{elem}}^3 + 1.2\nu_{\text{elem}}^2 - 0.2\nu_{\text{elem}} + 1)E_{\text{m}} \]
\[ \nu_{\text{crelem}} = 20\nu_{\text{cre}}^3 - 13.8\nu_{\text{cre}}^2 + 3.8\nu_{\text{cre}} \]

where, \( \nu \) and \( \nu_{\text{cre}} \) are the macroscopic static Poisson’s ratio and creep Poisson’s ratio, \( \nu_{\text{elem}} \) and \( \nu_{\text{crelem}} \) are the mesoscopic static Poisson’s ratio and creep Poisson’s ratio, and \( \bar{E}_{\text{elem}} \) is average value of mesoscopic static elastic modulus.

The tensile strength at the mesoscale cannot be determined by elastic analysis but a method using a different technique that is described later.

To introduce the material heterogeneity of mortar, variation is given to the material properties using following probability density function (see Fig. 11) in this study.

\[ f(x_{\text{elem}}) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left\{-\frac{(x_{\text{elem}} - \mu_{\text{elem}})^2}{2\sigma^2}\right\} \]
\[ \sigma = -0.2f_{\text{tp}} + 1.5 \]

where, \( x_{\text{elem}} \) is distributed material property, and \( \mu_{\text{elem}} \) is average material property at the meso level.

As seen in Fig. 11, the distribution varies according to the value of \( f_{\text{tp}} \). This is expressed by stating that higher-strength mortar is a more homogeneous material than lower-strength mortar. The same distribution is applied to the elastic modulus and the viscosity coefficients that will be described later. Heterogeneity is given by not only material property distribution but also random geometry in the discrete model. The effect of the random geometry is adequately small if the element size is 2-3 [mm²] (Nagai et al. 2004). Since the purpose of the analysis is to simulate whole behavior of cubic specimen whose thickness is adequately large in comparison with the element size, variation in the thickness direction is not taken into account.

### 4.3 Relationship between material properties and model constants

The general theory of RBSM and the experimental data of mortar loading test describe the relationship between the material properties and the model constants.

#### 4.3.1 Elastic component

The elastic component governs the initial elastic modulus of mortar. In this study, the initial elastic modulus is assumed to be time-independent. Based on general theory of RBSM under the plane stress condition, the elastic spring constants \( k_1 \) and \( k_3 \) are related to the mesoscopic elastic modulus using the following equation.

\[ k_1 = \frac{E_{\text{elem}}}{1 - \nu_{\text{elem}}^2} \]
\[ k_3 = \frac{E_{\text{elem}}}{1 + \nu_{\text{elem}}} \]

#### 4.3.2 Visco-elastic component

The visco-elastic spring constants \( k_2 \) and \( k_4 \) are related to the mesoscopic elastic modulus \( E_{\text{cre}} \) as in the elastic component.

\[ k_2 = \frac{E_{\text{crelem}}}{1 - \nu_{\text{crelem}}^2} \]
\[ k_4 = \frac{E_{\text{crelem}}}{1 + \nu_{\text{crelem}}} \]

where, \( E_{\text{crelem}} \) and \( \nu_{\text{crelem}} \) are the mesoscopic elastic modulus and the Poisson’s ratio of the visco-elastic component (creep Poisson’s ratio).
From Eqs. (21) and (24), the following equation is derived, which gives the basic creep strain of mortar.

$$\varepsilon_{crem} = \sigma \cdot \varepsilon_{0.09} \left[1 - \exp\left(-0.09t^{0.6}\right)\right]$$  \hspace{1cm} (32)

The average viscosity coefficients of the dashpot are $\tau_1 = 5.175 \times 10^{10}$ and $\tau_2 = 2.3 \times 10^{10}$ [MPa*sec] when the analytical result agrees with Eq. (32) and the creep Poisson’s ratio $\nu_{crem}$ becomes constant at 0.2.

### 4.3.3 Plastic component

The tensile strength and limit COD for transferring stress govern the behavior under extremely higher loading rates. In this study, loading rate which is higher than static loading is not targeted. Therefore, static loading corresponds to the upper limit of loading rate in the analysis. In other words, static loading is the highest loading rate in applicable range of the model. Thus, comparing with the experimental data on static behavior of mortar, the tensile strength and limit COD for transferring stress can be determined. Tanigawa et al. carried out static compressive loading test of mortar for different water to cement ratios (1977). $f_{i1} = 1.02f_{pu}$ and $\omega_{u1} = 0.03$ [mm] are determined so that the analytical result agrees with peak point stress and strain of the mortar obtained in the Tanigawa’s experiment.

Residual COD indicator $\sigma_{i1}$ governs the residual COD when the mortar is unloaded. In this study, it is simply assumed as $\sigma_{i1} = f_{i1}/5$. The development of a more representative value for actual materials is a task for the future.

### 4.3.4 Visco-plastic component

Since stress transfer and the re-contact phenomenon between crack surfaces depends upon the aggregate volume and maximum size, the limit COD for transferring stress and the remaining COD indicator are assumed to be time-independent and have the values $\omega_{u2} = 0.03$ [mm] and $\sigma_{i2} = f_{i2}/5$, respectively, which are identical to the value of plastic component. The slider, rather than the dashpot in parallel, for the visco-plastic component governs the behavior under monotonic loading at an extremely slow strain rate. In this study, a 30% reduction in strength is assumed for mortar under such a slow rate, resulting in $f_{i2} = 0.9f_{pu}$.

The dashpot in the visco-plastic component governs the rate of stress relaxation between the crack surfaces. In this study, the initial viscosity coefficient of dashpot is also varied using the probability density function shown in Fig. 11. The average value is simply assumed as $\tau_{i2} = \tau_{i4} = 50000$ [MPa*sec]. The development of a more representative value for actual materials is a task for the future.

### 5. Method for determining failure

The high stress creep and low cycle fatigue analyses in this study are load-controlled analyses. However, in load-controlled analysis, the failure point cannot be estimated because the applied load is forcibly applied to the boundary face and a peak load does not appear. Therefore, in this study, a new method of determining failure for load-controlled analyses was developed based on the idea that failure occurs when the material strength becomes less than the applied stress. To determine the failure, pseudo monotonic loading analysis controlled by displacement is conducted at a certain loading cycle interval for fatigue analysis or a certain time interval for creep analysis. That is, the analysis determines the failure state when the peak load obtained by pseudo monotonic loading analysis at a 40 [micro strain/sec] loading rate becomes less than the applied upper load or creep load, as shown in Fig. 12.

**Figure 13** is a flowchart of the method used to determine failure. First, all of the data in the analysis is stored at the point where the mortar is completely unloaded (point A and B in Fig. 12). Next, the pseudo monotonic loading analysis is executed and the peak load
obtained from the analysis, which corresponds to the residual strength at the point A, is compared to the applied upper or creep load. This process is done at every loading cycle in case of fatigue loading and 1 [mic] global strain interval in case of creep loading. If the peak load is larger than the upper or creep load (see the case of the pseudo monotonic loading starting from point A in Fig. 12), the determination is “no failure”, the data is recalled from storage, and fatigue or creep analysis is again conducted. If the peak load is less (see the case of the pseudo monotonic loading from point B), the determination is “failure” and the analysis ends.

6. Numerical analysis

6.1 Outline

Figure 14 shows the mortar specimen used for the analyses. The specimen is 75 [mm] x 150 [mm] in size, and contains 1800 (40 x 80) Voronoi diagram elements. The load is uniformly applied to the upper surface of the specimen using displacement or load control.

Table 1 lists the mix proportions and curing term for the mortar. Here, the unit weight of the fine aggregate is assumed as \( \rho = 2650 \) [kg/m\(^3\)]. The macroscopic material properties calculated by Eqs. (20), (21), (22) and (26) are \( f_{cm} = 32 \) [MPa], \( E_m = 21186 \) [MPa], \( f_y = 3.35 \) [MPa] and \( \epsilon'_b = 0.000228 \) [MPa\(^{-1}\)], respectively. Table 2 lists the condition of the seven analytical cases that were conducted in this study.

6.2 Static analysis

In this study, monotonic compressive loading at a global strain rate of 40 [micro strain/sec] is defined as static loading. The value of the applied load on the upper surface divided by the sectional area is called “global stress \( \sigma_G \),” while the upper surface displacement divided by specimen height is called “global strain \( \epsilon_G \).” In one analytical case (case “S” for static), the upper surface was subjected to forcible displacement in compression at a rate of 40 [micro strain/sec].

Figure 15 shows the global stress-strain relationship of mortar for case S. The obtained peak stress of 31.1 [MPa] is close to the target compressive strength of \( f_{cm} = 32 \) [MPa]. Fig. 16 shows the crack pattern resulting from 2650 [mic.] global strain. A single but significant crack in the lower-left area of the specimen led to the failure.

6.3 Gradual increasing-strain cyclic loading analysis

For the GIC (Gradual Increasing-strain Cyclic) analytical case, unloading-reloading was conducted at a 500 [mic] global strain interval and the same loading rate used in the case of static loading.

Figure 17 shows the global stress-strain relationship of the mortar for both the GIC (black line) and S (gray line) cases.
6.4 High-stress creep analysis
For the third analytical case, C90 (Creep 90% stress), the mortar specimen was subjected to creep stress equal to 90% of the static strength, 28.0 [MPa] (=0.9 x 31.1). In this analysis, the initial load was applied at a rate of 0.8 [MPa/sec] and kept constant at 28.0 [MPa] by means of load control.

Figures 18 and 19 show the obtained global stress-strain and the global strain-time relationship, respectively. The analysis finished when the determination of “failure,” marked by an X in these figures, was made, using the method shown in Fig. 12. Though the applied load was kept constant, the strain increased over time and reached the failure point at 514 [sec].

6.5 Low-cycle fatigue analysis
6.5.1 With constant stress level
Two low-cycle fatigue analyses with constant stress levels were conducted. In the F90 analytical case (Fatigue 90% stress), the upper and lower stresses were 90% of the static strengths of 28.0 [MPa] and 0 [MPa], respectively. In the F85 analytical case (Fatigue 85% stress), the upper and lower stresses were 85% of the static strengths of 26.435 [MPa] and 0 [MPa], respectively. In both cases, an initial load was applied at a rate of 0.8 [MPa/sec], while unloading-reloading was done at a rate of 0.01 [Hz].

Figures 20, 21, and 22 show the global stress-strain, the global strain-time relationship, and the crack pattern at failure in the case of F90, respectively. Figures 23 and
show the global stress-strain and the global strain-time, respectively, for the analytical case F85. In Fig. 23, not all of the unloading-reloading curves are drawn, and the digits in the figure represent the number of loading cycles. The analysis finished when the determination was of “failure”, marked by an X in these figures as well as in Figs. 18 and 19, was made. The number of loading cycles until failure in the cases of F90 and F85 were 28 [cycles] and 78 [cycles], respectively. Stiffness at unloading-reloading decreased as the global strain became larger, as was the case for GIC. Moreover, Fig. 23 shows that the area surrounded by the unloading-reloading curve increased and its nonlinearity became stronger as the number of loading cycles increased. These tendencies agree with the fatigue characteristics of concrete reported by Matsushita et al. (1979). Compared

6.5.2 With varying stress level

Also conducted were two low-cycle fatigue analyses in which the stress level was changed in midstream. In the first analysis, F8590 (Fatigue 85% to 90% stress), fatigue loading with stress equal to 85% of the static strength was applied up to halfway point of the fatigue life (78/2=39 [cycles]), and then fatigue loading with stress equal to 90% of the static strength was applied until failure occurred. In the second analysis, F9085 (Fatigue 90% to 85% stress), fatigue loading with stress equal to 90% of the static strength was applied up to halfway point of the fatigue life (28/2=14 [cycles]), and
then fatigue loading with stress equal to 85% of the static strength was applied until failure occurred.

Figures 25 and 26 show the global stress-strain relationships for F8590 and F9085, respectively. The notable points of these analyses are not only the reduction in stiffness and change in nonlinearity with the number of cycles as well as in the case of constant stress levels, but also the difference in fatigue life. In the case of F8590, failure occurred after 39 cycles of 85% fatigue loading and 25 cycles of 90% fatigue loading. In the case of F9085, however, failure occurred after 14 cycles of 90% fatigue loading and 26 cycles of 85% fatigue loading.

Miner’s law (cumulative damage law) is the most popular method for estimating fatigue life when the stress levels are not constant. According to Miner’s law, degree of cumulative damage under fatigue loading is given by following equation.

\[
D = \sum \frac{n_i}{N_i} \tag{33}
\]

where, \(D\) is cumulative damage index (when \(D=1\), failure occurs), \(n_i\) is number of loading cycles at each stress level, and \(N_i\) is fatigue life cycles at each stress level.

Using Eq. (33), cumulative damage index \(D\) in the analyses can be calculated as follows.

\[
D_{85\%90\%} = \frac{39}{78} + \frac{25}{28} = 1.39
\]

\[
D_{80\%85\%} = \frac{14}{28} + \frac{26}{78} = 0.83 \tag{34}
\]

Cumulative damage index \(D\) exceeds 1 when the stress level increases and becomes less than 1 when the stress level decreases. The result, which indicates that the Miner’s law estimation is on the safe side in the former case and on the dangerous side in the latter case, agrees with the concrete characteristics observed in the experiment conducted by Oh et al. (1991).

According to the failure concept shown in Fig. 12, fatigue failure is caused by degradation of the remaining material strength. In this study, therefore, it is considered that “remaining material strength” can be an index for “real damage” due to fatigue. Figure 27 shows crack distribution when the specimen is completely unloaded at \(D=0.5\) under 85% and 90% stress level. In the figure, crack width is defined as a summation of time-independent and time-dependent COD (\(\omega = \omega_0 + \omega_p\)). The cracks with width in range of \(0.002 < \omega < 0.005\ [mm]\), \(0.005 < \omega < 0.01\ [mm]\), and \(0.01 < \omega\ [mm]\) are drawn in gray, black, and bold line, respectively. More crack development can be seen in the case of 90% stress level than 85% stress level. Since remaining strength becomes less in concrete with more crack development, the remaining strength at the halfway of the fatigue life (\(D=0.5\)) becomes less with higher applied stress. On the other hands, at the failure (\(D=1\)), the remaining strength becomes larger with higher applied stress because the material’s strength decreases to the applied stress.

Based on the above discussion on the damage development, it can be understood that two hypotheses on damage in the Miner’s law would not agree with the actual damage development. One is the liner relationship between damage and number of loading cycles. According to the Miner’s law, damage is the same if cumulative damage index \(D\) is the same, however Fig. 27 shows that the damage is not proportional to number of loading cycles but damage at the halfway becomes less with lower stress levels. The other is the hypothesis of same damage at the fatigue failure point even under different stress levels. However, in the actual phenomenon, damage degree at the failure under a higher applied stress must be smaller than that of a lower applied stress. Therefore, damage accumulation curves, which is relationship between actual damage degree and number of loading cycles, can be schematically drawn for different stress levels as shown in Fig. 28. That is, damage becomes greater with higher stress levels during the early stage of fatigue life, however the damage becomes greater with lower stress levels near the end of fatigue life.

When fatigue loading is applied with a lower stress level up to the halfway of the fatigue life (\(D=0.5\)), the actual cumulative damage is assumed to be \(D_{A_L}\) (see Fig. 28). For the same actual damage, the damage index for a higher stress level is \(a\), meaning that the remaining damage index to reach the fatigue failure is 1-a. Therefore, the total damage index to the fatigue failure in the
case where a lower stress is applied first and followed by a high stress is as follows:

\[ D_{H \to L} = 0.5 + (1 - a) \quad (35) \]

On the other hand, when fatigue loading is applied with higher stress level up to the halfway of the fatigue life \( D = 0.5 \), the actual cumulative damage is assumed to be \( D_{A, H} \). For the same actual damage, the damage index is \( b \) for a lower stress level. Thus, in the case where a higher stress is applied first and followed by a lower stress, the total damage index to the fatigue failure is calculated as below:

\[ D_{H \to L} = 0.5 + (1 - b) \quad (36) \]

According to Fig. 28, there is a relationship of \( a < 0.5 < b \), resulting in the following relationship between \( D_{H \to L} \) and \( D_{L \to H} \):

\[ D_{H \to L} < 1 < D_{L \to H} \quad (37) \]

As shown in Eq. (37), difference in fatigue life between the actual one and the one predicted by Miner’s law can be explained by the difference in shape of actual damage accumulation curve for different stress levels.

To examine the Miner’s law more precisely, investigation for more complicated stress histories should be done, which requires experimental data.

6.6 Discussion on the failure mechanism in terms of a mesoscopic approach

To investigate the failure mechanism of mortar under creep and fatigue loading, the mesoscopic behavior was investigated. Figs. 29 and 30 show the stress-strain behavior of springs connected at points A, B, and C in Fig. 14, where visible crack occurred as shown in Fig. 22, in the cases of C90 and F90, respectively.

Figure 29 shows that the time at which softening starts differs for each location. Point A, B, and C began to soften at 12.4, 150, and 327 [sec], respectively. Since initial loading was completed at 35 [sec], only point A had softened by the time when the creep loading had...
started. After that, points B and C began to soften. This time-series process is caused by the characteristics of the constitutive model. That is, since the four-component combined model has a relaxation characteristic, as shown in Fig. 8, the stress decreases over time at point A (stress release). To resist the applied creep load, the stress at the other location (point B) increases (stress redistribution).

Point B began to soften at 150 [sec] because of this increasing stress. At this moment, the stress at point C was compression stress, but it became tension stress due to stress redistribution from point B to C. Subsequently, point C began to soften at 327 [sec].

A similar process can be applied under fatigue loading. According to Fig. 30, point A was already starting to soften during the first loading cycle. Since the four-component combined model has a stress decrement characteristic during unloading-reloading, as shown in Fig. 8, the stress was released at point A during the loading cycle. This stress release led to stress redistribution to point B. Subsequently, point B began to soften during the 8th cycle. Stress release at point B led to stress redistribution to point C. Subsequently, point C began to soften during the 23rd cycle.

The above process in meso-scale may explain the actual failure process of mortar as follows:

1. A micro crack occurs at point A under initial loading.
2. Stress at point A is redistributed to other areas.
3. Stress then increases and a new crack is generated at point B.
4. Repeating the above process, the crack propagates over time to point C. Finally, the crack extends to the bottom side of the specimen and reaches the failure point.

7. Conclusions

The following conclusions were obtained from the analyses of mortar under high stress creep and low cycle fatigue in which a Rigid Body Spring Model (RBSM) with meso-scale elements is extended to time-dependent problems.

1. Using the four-component combined mechanical model that was developed as a meso-scale time-dependent constitutive model of RBSM, the failure process of mortar under various time-dependent loads could be numerically simulated.
2. The analyses conducted in this study show that macroscopic creep and fatigue failure of mortar can be attributed to the accumulation of tensile and shear fractures at the meso-scale level, as in the case of static lading.
3. A new method was developed for determining the failure state in creep and fatigue analysis, in which pseudo monotonic loading analysis is conducted at a certain time or loading cycle interval and the obtained peak load is compared with the applied load.
4. The gradual increasing-strain cyclic loading analysis expressed well the experimental facts that stiffness at unloading-reloading decreases with strain development, and the reloading curve follows the envelope under static loading.
5. The high-stress creep analysis showed strain development and failure over time under even constant stress.
6. The low-cycle fatigue analysis with a constant stress level expressed well the experimental facts that stiffness at unloading-reloading decreases and nonlinearity becomes stronger as the number of loading cycles increases. Moreover, the crack pattern had a homogeneous distribution compared to the case of static loading.
7. The low-cycle fatigue analysis with varying stress level analytically expressed the experimental fact that Miner’s law estimation is on the safe side when the stress level increases and on the dangerous side when the stress level decreases. This characteristic can be explained by the difference of damage accumulation curve for different stress levels.
8. The investigation of the behavior of connected springs clarified the failure mechanism of mortar at the meso-scale level under creep and fatigue. That is, stress is released around initial micro cracking, leading to stress redistribution to another area. As a result, the crack propagates over time. This process occurs for many micro cracks and finally causes global failure.

The authors further this study to show the applicability of the proposed mesoscopic approach under various loading conditions and to extend it for the case of concrete. The study results will be reported in separate papers.

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Appendix: List of symbols
Macroscopic properties:
C=Unit quantity of cement [kg/m³]

E_{e}^{c} =Macroscopic elastic modulus of visco-elastic com-
ponent [MPa]

E_{s}^{c} =Macroscopic static elastic modulus of mortar [MPa]

\varepsilon_{h}^{c}=Limitation of basic creep strain of concrete per unit
stress [MPa⁻¹]

\varepsilon_{h}^{m}=Limitation of basic creep strain of mortar per unit
stress [MPa⁻¹]

\varepsilon_{c}=Creep strain of concrete

\varepsilon_{p}=Creep strain of cement paste

\varepsilon_{c}=Global strain

f_{c}=Static compressive strength of mortar [MPa]

f_{p}=Pure static tensile strength of mortar [MPa]

\rho=Volumetric ratio of aggregate

\rho=Density of aggregate [kg/m³]

\nu=Macroscopic static Poisson’s ratio

\nu=Macroscopic creep Poisson’s ratio

S=Aggregate volume of mortar [kg/m³]

\sigma_{c}=Global stress [MPa]

W=Unit quantity of water [kg/m³]

Mesoscopic properties:
c₁ = Viscosity coefficient of visco-elastic dashpot in normal direction [MPa*sec]
φ₁ = Average of viscosity coefficient of visco-elastic dashpot in normal direction [MPa*sec]
c₂ = Viscosity coefficient of visco-plastic dashpot in normal direction [MPa*sec/mm]
c₂ᵢ = Initial viscosity coefficient of visco-plastic dashpot in normal direction [MPa*sec/mm]
φ₂ᵢ = Average of initial viscosity coefficient of visco-plastic dashpot in normal direction [MPa*sec/mm]
c₃ = Viscosity coefficient of visco-elastic dashpot in shear direction [MPa*sec]
φ₃ = Average of viscosity coefficient of visco-elastic dashpot in shear direction [MPa*sec]
c₄ = Viscosity coefficient of visco-plastic dashpot in shear direction [MPa*sec/mm]
c₄ᵢ = Initial viscosity coefficient of visco-plastic dashpot in shear direction [MPa*sec/mm]
φ₄ᵢ = Average of initial viscosity coefficient of visco-plastic dashpot in shear direction [MPa*sec/mm]
δₚ = Time-independent CSD [mm]
δᵥ = Time-dependent CSD [mm]
Eₐ = Mesoscopic static elastic modulus [MPa]
Eᵥ = Average value of mesoscopic static elastic modulus [MPa]
Eᵥₑₐ = Mesoscopic elastic modulus of visco-elastic component [MPa]
Eᵥₑₐ = Average value of mesoscopic elastic modulus of visco-elastic component [MPa]
εₚ = Total strain in normal direction
εₑₚ = Elastic strain in normal direction
εᵥₑₚ = Visco-elastic strain in normal direction
εᵥₚ = Plastic strain in normal direction
εᵥₑₚ = Basic creep strain of concrete
f₁ = Tensile strength of plastic slider in normal direction [MPa]
F₁ᵢ = Average value of tensile strength of plastic slider in normal direction [MPa]
f₂ = Tensile strength of visco-plastic slider in normal direction [MPa]
F₂ᵢ = Average value of tensile strength of visco-plastic slider in normal direction [MPa]
γₚ = Total strain in shear direction
γₑₚ = Elastic strain in shear direction
γᵥₑₚ = Visco-elastic strain in shear direction
γᵥₚ = Plastic strain in shear direction
γᵥₑₚ = Visco-plastic strain in shear direction
h₁, h₂ = Length of perpendicular lines from the element computational point to the face [mm]
k₁ = Elastic modulus of elastic spring in normal direction [MPa]
k₂ = Elastic modulus of visco-elastic spring in normal direction [MPa]
kᵥₑₐ = Elastic modulus of elastic spring in shear direction [MPa]
kᵥₑₐ = Elastic modulus of visco-elastic spring in shear direction [MPa]
υᵥₑₐ = Mesoscopic creep Poisson’s ratio
υₑₐ = Mesoscopic static Poisson’s ratio
ωᵥₑₐ = Time-independent COD [mm]
ωₑₐ = Time-dependent COD [mm]
ωᵥₑₐ = Limit COD [crack opening deformation] to transfer stress of plastic slider [mm]
ωₑₐ = Limit COD [crack opening deformation] to transfer stress of visco-plastic slider [mm]
σₑₚ = Stress of elastic spring in normal direction [MPa]
σᵥₑₚ = Stress of visco-elastic spring in normal direction [MPa]
σᵥₑₐ = Stress of visco-plastic slider in normal direction [MPa]
σᵥₑₐ = Stress of visco-plastic dashpot in normal direction [MPa]
σᵥₑₐ = Residual COD indicator of plastic slider [MPa]
σᵥₑₐ = Residual COD indicator of visco-plastic slider [MPa]
τₑₚ = Stress of elastic spring in shear direction [MPa]
τᵥₑₚ = Stress of visco-elastic spring in shear direction [MPa]
τᵥₑₐ = Stress of visco-plastic slider in shear direction [MPa]
τᵥₑₐ = Stress of visco-plastic dashpot in shear direction [MPa]
τᵥₑₐ = Stress of visco-elastic dashpot in shear direction [MPa]
τᵥₑₐ = Stress of visco-plastic dashpot in shear direction [MPa]
τᵥₑₐ = Yielding strength of plastic slider in shear direction [MPa]
τᵥₑₐ = Yielding strength of visco-plastic slider in shear direction [MPa]

Others:
t = Time [sec or days]
t' = Curing terms [days]