On a forgotten model of lightning stroke

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Abstract In the present paper a model is described of the lightning discharge that was developed in the USSR by Stekolknikov in 1940s. This model is practically unknown to specialists in the atmospheric electricity. We describe its parameters and compare these with the data commonly used in the literature. Computations show that SM remains up to date and is consistent with the modern concepts.

Keywords: lightning discharge, current distribution, spectrum of the current moment.

1. Introduction
Many publications on the physics of lightning stokes stemmed from the paper by Bruce and Golde (1941). Unfortunately, specialists in the field of atmospheric electricity are unfamiliar with publications by I.S. Stekolknikov made in the Soviet Union practically during the same years (Stekolknikov, 1941, 1943). Although his approaches have much in common, there are interesting distinctions in the model details. The goal of this paper is to present an old but a rather effective model and to compare the description of lightning strokes suggested by Stekolknikov (1941, 1943) with that widely applied in engineering models of lightning strokes. We will not concentrate on a detailed description of typical models and relevant bibliography, which might be found in extensive literature (e.g., Ogawa, 1985, 1995, Brook and Ogawa, 1977, MacGorman and Rust, 1998, Rakov and Uman, 2003).

2. Common engineering model
The term ‘engineering model’ is used for the models describing the effective parameters of lightning stroke, which are necessary for computation of its fields rather than describing complicated physical processes in the discharge (Watt, 1967, Rakov and Uman, 2003). Since Bruce and Golde (1941), the current at the stroke base is introduced as a linear combination of exponential terms:

\[ I(t) = \sum_k I_k \exp \left(-\frac{t}{\tau_k}\right), \quad t \geq 0 \]  

(1)

Here \( I_k \) and \( \tau_k \) are the amplitudes and the time constants of individual terms, and the time moment \( t = 0 \) corresponds to the stroke initiation.

Typical parameters of model strokes are listed in Table 1 and relevant temporal variations of stroke current are plotted in Fig. 1. We use four accepted models suggested correspondingly by Bruce and Golde (1941) – black curve, Hepburn (1957) – blue line, Williams (1958) – brown plot, and Jones (1970) – green curve.

In addition to Eq. (1), one has to describe the current wave motion along the stroke channel when computing the electromagnetic field from a lightning stroke. As a rule, the exponentially decreasing velocity is used (Jones, 1970):

\[ V(t) = V_0 \exp \left(-\frac{t}{\tau'}\right). \]  

(2)

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This velocity is often a constant when obtained in the modern transmission line models of the stroke (Rakov and Uman, 2003). The distance passed by the current wave is readily found from the time dependence Eq. (2):

\[ L(t) = \int_0^t v(t) \, dt = V_0 t_v \left[ 1 - \exp \left( \frac{t}{t_v} \right) \right] \]  

(3)

![Graph showing waveforms of current at the stroke base for different stroke models.](image)

Table 1. Typical parameters of a lightning stroke used in classical models.

<table>
<thead>
<tr>
<th>Author</th>
<th>(-I_1)</th>
<th>(-I_2)</th>
<th>(-I_3)</th>
<th>(-I_4)</th>
<th>(t_1)</th>
<th>(t_2)</th>
<th>(t_3)</th>
<th>(t_4)</th>
<th>(V_0 \times 10^3)</th>
<th>(t_v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bruce and Golde</td>
<td>28.4</td>
<td>28.4</td>
<td>&quot;-&quot;</td>
<td>&quot;-&quot;</td>
<td>2.18</td>
<td>2.2</td>
<td>&quot;-&quot;</td>
<td>&quot;-&quot;</td>
<td>8</td>
<td>33.3</td>
</tr>
<tr>
<td>Hepburn</td>
<td>25</td>
<td>20</td>
<td>5</td>
<td>&quot;-&quot;</td>
<td>2</td>
<td>20</td>
<td>1430</td>
<td>&quot;-&quot;</td>
<td>3.5</td>
<td>180</td>
</tr>
<tr>
<td>Williams</td>
<td>16.8</td>
<td>15.34</td>
<td>1</td>
<td>0.45</td>
<td>1.7</td>
<td>33</td>
<td>500</td>
<td>6.8</td>
<td>8</td>
<td>50</td>
</tr>
<tr>
<td>Jones</td>
<td>28.45</td>
<td>23</td>
<td>5</td>
<td>0.45</td>
<td>1.66</td>
<td>33.3</td>
<td>500</td>
<td>6.8</td>
<td>8</td>
<td>50</td>
</tr>
</tbody>
</table>

The current moment of a lightning discharge is necessary for computing the electromagnetic fields (Ogawa, 1995, Nickolaenko and Hayakawa, 2002). It is found from the following relation:

\[ M_C(t) = \int_0^t i(x,t) \, dx = V_0 t_v \sum_{k=1}^{n} \frac{I_k t_k}{t_v} \left[ \exp \left( -\frac{t}{t_k} \right) - \exp \left( -\frac{t}{t_v} \right) \right] \quad t > 0. \]  

(4)
The charge moment \( M_Q(t) \) and the radiation moment \( M_R(t) \) of a stroke are found by differentiation or integration of Eq. (4) with respect to \( t \). One obtains the complex spectrum of the current moment after applying the Fourier transform to Eq. (4):

\[
M_R(\omega) = \int_{-\infty}^{\infty} M_R(t) \exp(-i\omega t) dt = Y_0 \sum_k \frac{I_k}{\omega - \omega_k} \left\{ \frac{1}{\omega_k + i\omega} - \frac{1}{\omega + i\omega} \right\},
\]

here we denote \( \omega_k = (t_k)^{-1} \) and \( \omega = (t_T)^{-1} \).

Spectra of the current moment \( M_Q(\omega) \) and of radiation moment \( M_R(\omega) \) are obtained by multiplying or dividing Eq. (5) by \( i\omega \). An obvious advantage of traditional description of a lightning stroke is its simplicity: formulas are easily transformed from the time to the frequency domain. The spectra must be computed only numerically in the other models.

Parameters of classical engineering models are arranged so that the current peak occurs in 5 – 10 µs after the stroke initiation. The current decreases exponentially afterwards. Individual model strokes have much in common, but have slightly different amplitudes and the rise/decline times.

3. Stekolnikov model (SM)

Figure 1 also depicts the time dependence of the stroke current following from the Stekolnikov model (in what follows SM) (Stekolnikov, 1941, 1943). We briefly describe this model. A discharge begins with the stepped leader process, and the leader forms a plasma column carrying an electric charge. This charge is downloaded by the main stroke (the term ‘return stroke’ is used nowadays). The main stroke is a current wave moving upward along the ionized channel. The first distinction from the common description is that initial velocity of the current wave is low in SM. It increases during the course of motion and reaches the values of 1/3 – 1/2 of the light velocity (10^7 – 1.4×10^8 m/s). We use the final value of \( V_k = 10^7 \) m/s (Stekolnikov, 1941, 1943). It takes the time ranging from a few milliseconds to a few tens of milliseconds for the current wave to reach the cloud (a few kilometer distance). The second distinction of the SM is that the stroke current is introduced as a function of the coordinate \( x \) along the stroke channel:

\[
I(x) = Q(x)\psi(x).
\]

Here \( x \) is the coordinate increasing from the ground surface; \( I(x) \) is the stroke current as a function of the \( x \) coordinate; \( Q(x) \) is the electric charge density accumulated by the leader process, and \( V(x) \) is the velocity of the main stroke current wave. The following charge distribution is assumed:

\[
Q(x) = Q_0 \exp(-ax),
\]

where \( Q_0 \) is the ‘initial’ density (nearby the ground surface), and \( a \) is the scale height of the vertical charge distribution.

As we already noted, the instant velocity of the tip of the current wave \( V(x) \) grows with the altitude \( x \). It is initially small owing to a transition from the velocity of the streamer wave to that of the main discharge (Stekolnikov, 1943). The following relation is introduced:

\[
V(x) = V_k \left[ \eta - \exp(-bx) \right],
\]

where \( V_k \) is the final velocity and \( \eta \) is a constant that determines the initial value of \( V(x) \) at \( x = 0 \). The latter is equal to 1% of its final value, so that \( \eta = 1.01 \).

A reasonable physical explanation was offered in SM for the velocity increase along the channel: the wave is accelerated by an increasing voltage between the charged cloud and the moving upward current wave, since the latter carries the zero potential of the ground. By substituting Eqs. (7) and (8) into Eq. (6), we arrive at the following spatial distribution of the current along the stroke channel:
\[ I(x) = A \exp(-ax) \left[ \frac{1}{\eta} \exp(-ax) - \exp(-cx) \right]. \] (9)

Here \( A = Q_0 V_s \), and \( c = a + b \). It follows from the above equations that distance \( x \) covered by the current wave is the following function of the time:

\[ x = \frac{1}{b} \ln \left( \frac{1}{\eta} \frac{\exp(V_s/b)(\eta - 1) + 1}{\eta} \right). \] (10)

Stekolnikov (1943) mentioned the following parameters of the stroke: \( V_s = 10^7 \) m/s, \( a = 0.003 \) m/l, and \( b = 0.03 \) m/l. The wave moving with the velocity (8) covers the distance from \( x_1 \) to \( x_2 \) in the time interval

\[ t_{21} = \frac{1}{V_s} \int_{x_1}^{x_2} \frac{dx}{\eta - \exp(-bx)} = \frac{1}{bV_s} \ln \left( \frac{\eta \exp(-bx_2) - 1}{\eta \exp(-bx_1) - 1} \right). \] (11)

For the time counted from the stroke onset (\( x_{l-0} = 0 \)), we have:

\[ t(x) = \frac{1}{bV_s} \ln \left( \frac{\eta \exp(-bx) - 1}{\eta - 1} \right). \] (12)

It follows from Eq. (8) that the current wave velocity reaches its final value when \( bx \gg 1 \). In other words, for a particular \( b = 0.03 \) m/l one can use the constant \( V_s \) rate starting from altitudes of about 100 m above the ground. From here, the motion is a uniform one, and \( x = tV_s \). Progress of the current wave is addressed below, here we note that the current at the stroke base \( I(t) \) is found by substituting the coordinate \( x(t) \) in the form (10) into Eq. (9), while the current moment is \( M_c(t) = I(t)x(t) \). Both \( I(t) \) and \( M_c(t) \) depend on the particular distributions (7) and (8).

By substituting specified parameters and supposing that \( Q_0 = 3 \times 10^3 \) C/m, we can compute temporal variations of the current at the stroke base shown in Fig.1. It is worth noting here that the complete charge downloaded to the ground is \( Q_\infty = \int_0^\infty Q_0 \exp(-ax) dx = \frac{Q_0}{a} = 1 \) C, the value is noticeably smaller than in modern models. The time passed from the stroke initiation is plotted along the abscissa in microseconds. The current amplitude is plotted on the ordinate in kA. Four typical engineering models are shown in Fig. 1 (Nickolaenko and Hayakawa, 2002): the black curve marked by squares presents the Bruce and Golde (1941) model, the blue line with open diamonds shows the model by Hepburn (1957), the Jones’s (1970) model is the green curve with triangles, and the brown plot depicts the Williams (1958) model. The original SM is presented in Fig.1 by the red curve with red stars.

Comparison of waveforms reveals two distinctions of SM. First, the current in this model has a maximum of approximately 20 kA at 23 \( \mu \)s from the stroke initiation, which is three – four times later than in common models. Perhaps, the deviation was conditioned by the particular experimental material used in the elaboration of SM. The second dissimilarity is a smooth current increase in SM, while the current onset is abrupt in the classical models: the discontinuity is found in the current derivative. Special models with smooth current variation were reported not so long ago (e.g. Heidler, 1985, Rakov and Uman, 2003). Thus, the ‘spatial’ approach applied by Stekolnikov (1941, 1943) results in a distinct advantage of better correspondence to the physical process in the lightning discharge.

Concerning Fig.1, it is quite clear that the modification of \( a \) and \( b \) parameters will alter the maximum position over the time axis. We show a modified dependence (the pink curve with open stars) corresponding to the increased constants: \( a = 0.01 \) and \( b = 0.1 \). Now, the peak occurs at 10 \( \mu \)s, which is close to common engineering models. A better fitting of the classical curves was not our goal, and it is clear that a matching process might be organized that changes all the model parameters. Curves in Fig.1 demonstrate that description of the lightning discharge applied by Stekolnikov (1941) agrees with the modern data, provided that numerical parameters are slightly modified.
Figure 2 depicts the temporal evolution of the SM stroke parameters. The black line in Fig. 2a presents the spatial distribution of electric charge accumulated in the leader channel. The charge is concentrated in the lower part of leader, close to the ground. The pink curve in Fig. 2a demonstrates the \( t(x) \) function (12). One may observe that motion becomes uniform at very low altitudes.

The time is plotted along the abscissa in microseconds in Fig. 2b and Fig. 2c. The green curve of Fig. 2b demonstrates temporal changes of the velocity of the current wave in megameters per second (1 Mm = \( 10^6 \) m = 1000 km). One may observe that velocity turns into the constant of \( 1.01 \times 10^7 \) m/s in \( \sim 20 \) \( \mu \)s after the stroke initiation. For greater times, the coordinate \( x(t) \) becomes the linear function of time. The brown plot shows the electric charge [in mC] downloaded to the ground by the stroke. The charge transfer is almost a constant at the beginning of the stroke. Such a behavior is conditioned by the concentration of charge in the lower part of the leader column and by relatively small initial velocity of the current wave. After the velocity increases (the green curve approaches the horizontal asymptote), the charge transfer reduces noticeably.

![Diagram](image)

**Fig. 2.** Major characteristics of SM: \( a \) the black line shows the spatial distribution of electric charge in the channel, and the pink curve shows the \( t(x) \) function (12); \( b \) the green curve demonstrates alterations of the current wave velocity in megameters per second (1 Mm = \( 10^6 \) m = 1000 km) and the brown curve shows the time dependence of the charge download to the ground; \( c \) the red curve depicts the current \( I(t) \) at the stroke base in kA and the blue curve shows the current moment \( M_C(t) \) in kA·km.

Temporal variations of the current at the stroke base is demonstrated by the red curve in Fig. 2c. The initial charge distribution and evolution of velocity form a characteristic pattern of the current \( I(t) \) function. The latter is equal to \( I(t) = Q(t)V(t) \), and therefore it depends on distributions (7) and (8). The instant current moment of the lightning stroke is found from:

\[
M_S(t) = I(t)x(t),
\]

where \( x(t) \) is defined by Eq. (10)

If we ignore the alterations of the current wave velocity and use its final constant value, the following simple relation is valid:

\[
M_S(t) = Bt[\eta \exp(-\alpha_1 t) - \exp(-\alpha_2 t)].
\]

Here \( M_S(t) \) denotes the current moment as a function of time in the framework of SM, \( B = Q_0V_k^2 \), \( \alpha_1 = aV_k \), and \( \alpha_2 = (a + b)V_k \). The spectrum of the current moment can be found by using the Fourier transform and by subsequent application of the identity: \( t \exp(-\alpha t) = \left \{ 0 \right \} \frac{d}{d\alpha} \left \{ \exp(-\alpha t) \right \} \). As a result, we obtain a relation that is formally very close to Eq. (5):
Equations (14) and (15) are valid when \( V(t) = V_k \). The exact formulas for the current moment, especially for its spectra, are too cumbersome. It is much simpler to compute initially the \( M_S(x) \) variations, and afterwards to recalculate the coordinate \( x \) into the time \( t \) by using Eq. (12). Thus we obtain the exact temporal variations \( M_S(t) \) shown in Fig. 2c in blue line. Finally, spectrum of the current moment \( M_S(\omega) \) is computed for SM by using the fast Fourier transform (FFT) procedure.

\[
M_S(\omega) = B \left[ \frac{\eta}{(\alpha_1 + i\omega)^2} - \frac{1}{(\alpha_2 + i\omega)^2} \right]
\]

Fig. 3. Amplitude spectra of Steklokov and modified SMs as compared with the spectra of common models.

4. Comparison of models and conclusion

Figure 3 compares amplitude spectra of commonly used models with that of SM. We can accomplish that the spectrum of SM is similar to the commonly used spectra. The current moment of SM is lower than that of common models by a factor of about 30. This is not a surprise since the postulated charge transfer in SM is only 1 C. The recent data use the charge transfers higher up to two orders in magnitude.

An important distinction is that radiation from a SM discharge is concentrated at lower frequencies. It is interesting to note that no continuing current was used, while such currents were introduced in Williams and Jones models. Enhanced low frequency radiation of SM might be helpful when modeling the parent discharges of the Q – bursts or of the slow tail atmospherics.

To conclude the paper we note that works by Steklokov have formulated an original model of lightning stroke, which is in accord with modern commonly used models. We can only regret that pre-war publications by this author have passed unnoticed in the field of atmospheric electricity. The present author expresses his hope that the present publication drives attention toward Steklokov’s interesting results.

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References


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