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研究論文

223. An Analysis of Dynamic Characteristics of Plutonium- and/or Thorium-recycled Reactor, (I)

Reactor kinetic equations

プルトニウム, トリウムリサイクル系における
原子炉動特性の解析, (I)

原子炉動特性方程式

By Kunihiko UEMATSU

Starting from the neutron balance equation, the reactor kinetic equations which are generally applicable to a plutonium- and/or thorium-recycled reactor are derived. The use of the newly defined effective fraction of delayed neutrons, β_{eff} , with these reactor kinetic equations simplifies the analysis of the dynamic behavior of a plutonium- and/or thorium-recycled reactor.

I. INTRODUCTION

It is important to clarify the transient behavior of the reactor from the view point of both reactor safety and control—especially for the plutonium recycle case, which usually has a positive overall temperature coefficient of reactivity. The usual reactor kinetic equations applicable to the monospecied fissile material are well known, however, the reactor kinetic equations which are applicable to the plutonium and/or thorium recycle cases have not been well defined. These kinetic equations for plutonium and/or thorium recycle are very important in view of a steadily increasing possibility of plutonium and/or thorium recycle in power reactors. The author derived the kinetic equations applicable to both recycle cases starting from the neutron balance equation, and defined the effective fraction of delayed neutrons, β_{eff} , for

the case of plutonium and/or thorium recycle.

II. DERIVATION OF BASIC EQUATIONS

The neutron flux of a homogeneous bare reactor is given by the neutron balance equation

$$\frac{1}{v} \frac{\partial \phi(r, t)}{\partial t} = \text{div } D \text{ grad } \phi(r, t) - \Sigma_a^F \phi(r, t) + S(r, t), \quad (1)$$

where

$$\Sigma_a^F = \Sigma_a^{25} + \Sigma_a^{28} + \Sigma_a^{40} + \Sigma_a^{40} + \Sigma_a^{41} + \Sigma_a^{42} + \Sigma_a^{23} + \Sigma_a^{03}. \quad (2)**$$

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** The superfixes, 25, 28, etc. of equations through this paper set such that the first digit denotes to the last digit of the atomic number and the second digit denotes to the last digit of the atomic weight. Therefore, 25 means $^{235}\text{U}_{92}$.

The source term is

$$\begin{aligned}
 S(r, t) = & (1 - \sum_i \beta_i \gamma_i) \varepsilon_{25} \eta_{25} P_{25} \sum_a^{25} \phi(r, t) \\
 & + (1 - \sum_j \beta_j \gamma_j) \varepsilon_{49} \eta_{49} P_{49} \sum_a^{49} \phi(r, t) \\
 & + (1 - \sum_k \beta_k \gamma_k) \varepsilon_{41} \eta_{41} P_{41} \sum_a^{41} \phi(r, t) \\
 & + (1 - \sum_m \beta_m \gamma_m) \varepsilon_{23} \eta_{23} P_{23} \sum_a^{23} \phi(r, t) \\
 & + \sum_i P_i \lambda_i C_i(r, t) \\
 & + \sum_j P_j \lambda_j C_j(r, t) \\
 & + \sum_k P_k \lambda_k C_k(r, t) \\
 & + \sum_m P_m \lambda_m C_m(r, t) + S_0(r, t) \quad (3)
 \end{aligned}$$

$i, j, k, m = 1, \dots, 6$

where

$$\begin{aligned}
 \eta_{25} &= \frac{\nu^{25} \sum_f^{25}}{\sum_a^{25}} \\
 \eta_{49} &= \frac{\nu^{49} \sum_f^{49}}{\sum_a^{49}} \\
 \eta_{41} &= \frac{\nu^{41} \sum_f^{41}}{\sum_a^{41}} \\
 \eta_{23} &= \frac{\nu^{23} \sum_f^{23}}{\sum_a^{23}} \quad (4)
 \end{aligned}$$

$\varepsilon \equiv \varepsilon_{25} = \varepsilon_{49} = \varepsilon_{41} = \varepsilon_{23}$: Fast fission factor
 $P \equiv P_{25} = P_{49} = P_{41} = P_{23}$: Fission to thermal
 nonleakage probability

$\beta_{i,j,k,m}$: Fraction of delayed neutron of i -th, j -th, k -th and m -th group of ^{235}U , ^{239}Pu , ^{241}Pu and ^{233}U fission, respectively

$\gamma_{i,j,k,m}$: Utilization of delayed neutron of i -th, j -th, k -th and m -th group of ^{235}U , ^{239}Pu , ^{241}Pu and ^{233}U fission, respectively

$\lambda_{i,j,k,m}$: Decay constant of delayed neutron emitters of i -th, j -th, k -th and m -th group of ^{235}U , ^{239}Pu , ^{241}Pu and ^{233}U fission, respectively

$P_{i,j,k,m}$: Nonleakage probability of delayed neutron of i -th, j -th, k -th and m -th group of ^{235}U , ^{239}Pu , ^{241}Pu and ^{233}U fission, respectively

$C_{i,j,k,m}$: Concentration of delayed neutron precursor of i -th, j -th, k -th and m -th group of ^{235}U , ^{239}Pu , ^{241}Pu and ^{233}U fission, respectively

$S_0(r, t)$: Extraneous neutron source.

The rate of formation of delayed neutron emitters is given by

$$\frac{dC_i(r, t)}{dt} = \varepsilon \eta_{25} \beta_i \sum_a^{25} \phi(r, t) - \lambda_i C_i(r, t)$$

$$\frac{dC_j(r, t)}{dt} = \varepsilon \eta_{49} \beta_j \sum_a^{49} \phi(r, t) - \lambda_j C_j(r, t)$$

$$\frac{dC_k(r, t)}{dt} = \varepsilon \eta_{41} \beta_k \sum_a^{41} \phi(r, t) - \lambda_k C_k(r, t)$$

$$\frac{dC_m(r, t)}{dt} = \varepsilon \eta_{23} \beta_m \sum_a^{23} \phi(r, t) - \lambda_m C_m(r, t) \quad (5)$$

$i, j, k, m = 1, \dots, 6$.

Because the first mode of the neutron flux is nearly always predominant, it can be assumed to be separable into time- and space-dependent functions :

$$\phi(r, t) = \phi(r) \phi(t). \quad (6)$$

It is also assumed that the extraneous source and the delayed neutron precursor have space dependence on the flux :

$$S_0(r, t) = \phi(r) S_0(t)$$

$$C_{i,j,k,m}(r, t) = \phi(r) C_{i,j,k,m}(t), \quad (7)$$

and that the space-dependent flux $\phi(r)$ satisfies the steady-state equation of the critical reactor :

$$\begin{aligned}
 \text{div } D \text{ grad } \phi(r) + \varepsilon \eta_{25} P \sum_a^{25} \phi(r) + \varepsilon \eta_{49} P \sum_a^{49} \phi(r) \\
 + \varepsilon \eta_{41} P \sum_a^{41} \phi(r) + \varepsilon \eta_{23} P \sum_a^{23} \phi(r) - \sum_a^F \phi(r) = 0
 \end{aligned} \quad (8)$$

or

$$\nabla^2 \phi(r) + B^2 \phi(r) = 0. \quad (9)$$

Using the effective neutron lifetime

$$l = \frac{1}{v \sum_a^F (1 + L^2 B^2)}, \quad (10)$$

$L^2 = D / \sum_a^F$ and Eqs. (1), (2), (6), (7), (8), (9) and (10), the neutron density equation becomes,

$$\begin{aligned}
 \frac{dn(t)}{dt} = & -\frac{n(t)}{l} \\
 & + v \varepsilon \eta_{25} P (1 - \sum_i \beta_i \gamma_i) \sum_a^{25} n(t) \\
 & + v \varepsilon \eta_{49} P (1 - \sum_j \beta_j \gamma_j) \sum_a^{49} n(t) \\
 & + v \varepsilon \eta_{41} P (1 - \sum_k \beta_k \gamma_k) \sum_a^{41} n(t) \\
 & + v \varepsilon \eta_{23} P (1 - \sum_m \beta_m \gamma_m) \sum_a^{23} n(t) \\
 & + \sum_i P_i \lambda_i C_i(t) \\
 & + \sum_j P_j \lambda_j C_j(t) \\
 & + \sum_k P_k \lambda_k C_k(t) \\
 & + \sum_m P_m \lambda_m C_m(t) + S_0(t) \quad (11)
 \end{aligned}$$

$i, j, k, m = 1, \dots, 6$.

Defining the new effective multiplication factor as

$$k_{eff} = \frac{\varepsilon \bar{\eta} P}{1 + L^2 B^2}, \quad (12)$$

where

$$\bar{\eta} = \frac{\nu^{25} \sum_f^{25} + \nu^{49} \sum_f^{49} + \nu^{41} \sum_f^{41} + \nu^{23} \sum_f^{23}}{\sum_a^F}, \quad (13)$$

and combining Eqs. (11), (12) and (13),

$$\begin{aligned} \frac{dn(t)}{dt} = & -\frac{n(t)}{l} + \frac{k_{eff}(1 - \sum_i \beta_i \gamma_i)}{l} n(t) \frac{\eta_{25}}{\bar{\eta}} \frac{\sum_a^{25}}{\sum_a^F} \\ & + \frac{k_{eff}(1 - \sum_j \beta_j \gamma_j)}{l} n(t) \frac{\eta_{49}}{\bar{\eta}} \frac{\sum_a^{49}}{\sum_a^F} \\ & + \frac{k_{eff}(1 - \sum_k \beta_k \gamma_k)}{l} n(t) \frac{\eta_{41}}{\bar{\eta}} \frac{\sum_a^{41}}{\sum_a^F} \\ & + \frac{k_{eff}(1 - \sum_m \beta_m \gamma_m)}{l} n(t) \frac{\eta_{23}}{\bar{\eta}} \frac{\sum_a^{23}}{\sum_a^F} \\ & + \sum_i P_i \lambda_i C_i(t) + \sum_j P_j \lambda_j C_j(t) + \sum_k P_k \lambda_k C_k(t) \\ & + \sum_m P_m \lambda_m C_m(t) + S_0(t) \end{aligned} \quad (14)$$

$i, j, k, m = 1, \dots, 6.$

From Eqs. (5), (6), (7) and (12),

$$\begin{cases} \frac{dC_i(t)}{dt} = \frac{k_{eff} \beta_i}{Pl} n(t) \frac{\eta_{25}}{\bar{\eta}} \frac{\sum_a^{25}}{\sum_a^F} - \lambda_i C_i(t) \\ \frac{dC_j(t)}{dt} = \frac{k_{eff} \beta_j}{Pl} n(t) \frac{\eta_{49}}{\bar{\eta}} \frac{\sum_a^{49}}{\sum_a^F} - \lambda_j C_j(t) \\ \frac{dC_k(t)}{dt} = \frac{k_{eff} \beta_k}{Pl} n(t) \frac{\eta_{41}}{\bar{\eta}} \frac{\sum_a^{41}}{\sum_a^F} - \lambda_k C_k(t) \\ \frac{dC_m(t)}{dt} = \frac{k_{eff} \beta_m}{Pl} n(t) \frac{\eta_{23}}{\bar{\eta}} \frac{\sum_a^{23}}{\sum_a^F} - \lambda_m C_m(t) \end{cases} \quad (15)$$

$i, j, k, m = 1, \dots, 6.$

Ordinarily we may take

$$\begin{cases} P_i = P_j = P_k = P_m = 1 \\ \gamma_i = \gamma_j = \gamma_k = \gamma_m = 1 \\ S_0(t) = 0. \end{cases} \quad (16)$$

Rewriting the variables of Eqs. (14) and (15),

$$\begin{cases} \sum_i \beta_i \gamma_i = \beta_I & \sum_i \beta_i \lambda_i = \lambda_I \beta_I \\ \sum_j \beta_j \gamma_j = \beta_J & \sum_j \beta_j \lambda_j = \lambda_J \beta_J \\ \sum_k \beta_k \gamma_k = \beta_K & \sum_k \beta_k \lambda_k = \lambda_K \beta_K \\ \sum_m \beta_m \gamma_m = \beta_M & \sum_m \beta_m \lambda_m = \lambda_M \beta_M \end{cases} \quad (17)$$

$k_{eff} = 1 + k_{ex}.$

The final results are as follows :

$$\begin{aligned} \frac{dn(t)}{dt} = & -\frac{n(t)}{l} \\ & + \frac{(1 + k_{ex})(1 - \beta_I)}{l} n(t) \frac{\eta_{25}}{\bar{\eta}} \frac{\sum_a^{25}}{\sum_a^F} \\ & + \frac{(1 + k_{ex})(1 - \beta_J)}{l} n(t) \frac{\eta_{49}}{\bar{\eta}} \frac{\sum_a^{49}}{\sum_a^F} \\ & + \frac{(1 + k_{ex})(1 - \beta_K)}{l} n(t) \frac{\eta_{41}}{\bar{\eta}} \frac{\sum_a^{41}}{\sum_a^F} \\ & + \frac{(1 + k_{ex})(1 - \beta_M)}{l} n(t) \frac{\eta_{23}}{\bar{\eta}} \frac{\sum_a^{23}}{\sum_a^F} \\ & + \sum_i \lambda_i C_i(t) + \sum_j \lambda_j C_j(t) \\ & + \sum_k \lambda_k C_k(t) + \sum_m \lambda_m C_m(t) \end{aligned} \quad (18)$$

$i, j, k, m = 1, \dots, 6,$

$$\begin{cases} \frac{dC_i(t)}{dt} = \frac{(1 + k_{ex}) \beta_I}{l} n(t) \frac{\eta_{25}}{\bar{\eta}} \frac{\sum_a^{25}}{\sum_a^F} \\ \frac{dC_j(t)}{dt} = \frac{(1 + k_{ex}) \beta_J}{l} n(t) \frac{\eta_{49}}{\bar{\eta}} \frac{\sum_a^{49}}{\sum_a^F} \\ \frac{dC_k(t)}{dt} = \frac{(1 + k_{ex}) \beta_K}{l} n(t) \frac{\eta_{41}}{\bar{\eta}} \frac{\sum_a^{41}}{\sum_a^F} \\ \frac{dC_m(t)}{dt} = \frac{(1 + k_{ex}) \beta_M}{l} n(t) \frac{\eta_{23}}{\bar{\eta}} \frac{\sum_a^{23}}{\sum_a^F} \end{cases} \quad (19)$$

$i, j, k, m = 1, \dots, 6.$

As can be seen easily, Eqs. (18) and (19) can be reduced to the ordinary reactor kinetic equations neglecting plutoniums and ^{233}U .

Certainly the six-group delayed neutrons for each of fissile atoms and four different fissile atoms make the reactor kinetic equations very complicated and make it hard to solve these equations at a high power of the reactor. However, we knew from the analysis of the ordinary reactor kinetic equations that the one group of delayed neutrons approximation agreed fairly well with the experimental results, besides reducing the complexity of the problems. It has been proved that the results of the analysis with the above-mentioned approximation are on the safe side.

Thus, from Eqs. (18) and (19), the reactor kinetic equations become

$$\begin{aligned} \frac{dn(t)}{dt} = & -\frac{n(t)}{l} + \frac{(1 + k_{ex})(1 - \beta_{eff})}{l} n(t) \\ & + \lambda_I C_I(t) + \lambda_J C_J(t) + \lambda_K C_K(t) \\ & + \lambda_M C_M(t), \end{aligned} \quad (20)$$

$$\begin{cases} \frac{dC_I(t)}{dt} = \frac{(1 + k_{ex}) \beta_I}{l} n(t) \frac{\eta_{25}}{\bar{\eta}} \frac{\sum_a^{25}}{\sum_a^F} - \lambda_I C_I(t) \\ \frac{dC_J(t)}{dt} = \frac{(1 + k_{ex}) \beta_J}{l} n(t) \frac{\eta_{49}}{\bar{\eta}} \frac{\sum_a^{49}}{\sum_a^F} - \lambda_J C_J(t) \end{cases} \quad (21)$$

$$\begin{cases} \frac{dC_K(t)}{dt} = \frac{(1+k_{ex})\beta_K n(t)}{l} \frac{\eta_{41}}{\eta} \frac{\sum_a^{41}}{\sum_a^F} - \lambda_K C_K(t) \\ \frac{dC_M(t)}{dt} = \frac{(1+k_{ex})\beta_M n(t)}{l} \frac{\eta_{23}}{\eta} \frac{\sum_a^{23}}{\sum_a^F} - \lambda_M C_M(t), \end{cases}$$

where

$$\begin{aligned} \beta_{eff} = & \beta_I \frac{\eta_{25}}{\eta} \frac{\sum_a^{25}}{\sum_a^F} + \beta_J \frac{\eta_{49}}{\eta} \frac{\sum_a^{49}}{\sum_a^F} \\ & + \beta_K \frac{\eta_{41}}{\eta} \frac{\sum_a^{41}}{\sum_a^F} + \beta_M \frac{\eta_{23}}{\eta} \frac{\sum_a^{23}}{\sum_a^F}. \end{aligned} \quad (22)$$

Therefore, the analysis of the dynamic behavior of the plutonium- and/or thorium-recycled reactor can be simplified with the use of the effective fraction of delayed neutrons, β_{eff} .

III. CONSTANT POWER REMOVAL MODEL

In the analysis of the transient behavior of the reactor, the model consisting of the constant power removal, one group delayed neutron, and the constant amount of delayed neutron precursor was used to avoid too much complication of the analysis. This model does not hold when the temperature increases significantly and is an unrealistic representation of actual reactor kinetics long after the introduction of a change in the reactivity. At least, the constant-power-removal model is good for a time short in comparison with the thermal relaxation time, and shows good agreement with experimental results.

The neutron density equation for one group delayed neutron and the constant amount of delayed neutron precursor is

$$\begin{aligned} \frac{dn}{dt} = & -\frac{n}{l} + \frac{(1+k_{ex})(1-\beta_{eff})n}{l} \\ & + \frac{n_0}{l} \beta_{eff}(1+k_{ex}). \end{aligned} \quad (23)$$

Assuming the cell model such that the coolant flows through the gap between fuel and moderator region, the temperature equations become

$$\begin{cases} \frac{dT_1}{dt} = a_1(n-n_0) \\ \frac{dT_2}{dt} = a_2(n-n_0), \end{cases} \quad (24)$$

where

T_1 : Fuel temperature
 T_2 : Moderator temperature
 $a_{1,2}$: Coefficient of power removal in the fuel and the moderator, respectively
 n_0 : Initial neutron density.

The reactivity equation is

$$k_{ex} = \Delta k - r_1(T_1 - T_1^0) + r_2(T_2 - T_2^0), \quad (25)$$

where

Δk : Excess reactivity induced from the outside cause
 r_1 : Fuel temperature coefficient of reactivity, assumed as a negative number
 r_2 : Moderator temperature coefficient of reactivity, assumed as a positive number.

Introducing the new variables,

$$x = \frac{n - n_0}{n_0} \quad (26)$$

$$y = \frac{T_1 - T_1^0}{a_1 n_0} \quad (27)$$

$$z = \frac{T_2 - T_2^0}{a_2 n_0}, \quad (28)$$

where T_1^0 and T_2^0 are the steady state temperature of the fuel and moderator, respectively. Eqs. (23) and (24) become

$$\frac{dx}{dt} = -\frac{x}{l} + \frac{x}{l} (1 - \beta_{eff})(1 + k_{ex}) + \frac{k_{ex}}{l} \quad (29)$$

$$\frac{dy}{dt} = x \quad (30)$$

$$\frac{dz}{dt} = x. \quad (31)$$

Fortunately, Eqs. (30) and (31) have the same functional form and the same initial conditions:

$$\begin{cases} \frac{dy}{dt} = \frac{dz}{dt} = 0 \\ y = z = 0, \end{cases} \quad \text{at } t = 0 \quad (32)$$

Therefore, Eqs. (30) and (31) can be expressed by only one equation, (30). Substituting the reactivity equation into Eq. (29) and rearranging it,

$$\frac{dx}{dt} = Ax - Bxy - Cy + D. \quad (33)$$

The constants A , B , C and D can be found in Table 1. Therefore, four simultaneous equations—Eqs. (23), (24) and (25)—become only

two simultaneous differential equations—Eqs. (30) and (33)—to be solved.

Table 1 Coefficient of reactor kinetic equations, recycle case (Constant power removal model)

$$A = \frac{1}{l} [\Delta k (1 - \beta_{eff}) - \beta_{eff}]$$

$$B = \frac{1}{l} (1 - \beta_{eff}) (r_1 a_1 n_0 - r_2 a_2 n_0)$$

$$C = \frac{1}{l} (r_1 a_1 n_0 - r_2 a_2 n_0)$$

$$D = \frac{\Delta k}{l}$$

IV. NEWTON'S LAW OF COOLING MODEL

The application of Newton's law of cooling is a more proper and realistic representation of the heat transfer characteristics of a reactor. Eq.(24) has to be modified for this model.

$$\frac{dT_1}{dt} = a_1(n - n_0) - b_1(T_1 - T_1^0) \quad (34)$$

$$\frac{dT_2}{dt} = a_2(n - n_0) - b_2(T_2 - T_2^0), \quad (35)$$

where

$b_{1,2}$: Inverse time constant of the temperature increase in the fuel and the moderator, respectively.

The reactor kinetic equations become

$$\frac{dx}{dt} = Ex - Fxy + Gxz - Hy + Kz + L \quad (36)$$

$$\frac{dy}{dt} = x - b_1 y \quad (37)$$

$$\frac{dz}{dt} = x - b_2 z. \quad (38)$$

Constants E, F, G, H, K and L can be found in Table 2.

Table 2 Coefficient of reactor kinetic equations, recycle case (Newton's law of cooling model)

$$E = \frac{1}{l} [\Delta k (1 - \beta_{eff}) - \beta_{eff}]$$

$$F = \frac{1}{l} (1 - \beta_{eff}) r_1 a_1 n_0$$

$$G = \frac{1}{l} (1 - \beta_{eff}) r_2 a_2 n_0$$

$$H = \frac{1}{l} r_1 a_1 n_0$$

$$K = \frac{1}{l} r_2 a_2 n_0$$

$$L = \frac{\Delta k}{l}$$

V. SUMMARY

Reactor kinetic equations generally applicable for a plutonium- and/or thorium-recycled reactor were developed. It was shown that the use of the effective fraction of the delayed neutrons, β_{eff} , which was defined in Eq.(22), and the assumption of the constant amount of delayed neutron precursors made it possible to reduce the complicated reactor kinetic equations of a plutonium- and/or thorium-recycled reactor to the simple ordinary reactor kinetic equations.

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