A Difficulty of Local Truth-Value Assignment in Contextual Approach

Katsuaki HIGASHI*

Abstract

In order to deal with Kochen and Specker's no-go result, van Fraassen invoked 'splitting' of non-maximal observable depending on 'context'. In his contextual approach there are different physical quantities corresponding to a non-maximal operator. Such physical quantities are different, so those values are not required to coincide. However, Heywood and Redhead showed the impossibility of local truth-value assignment in contextual approach. Their argument is restricted to finite dimensional and non-relativistic case. In this paper we shall show a similar no-go result in more general framework (von Neumann algebra). Also, our result will be applied to *Algebraic quantum field theory*.

Key words: Kochen and Specker, Context, Von Neumann Algebra.

1. Introduction

Kochen and Specker (1967) showed that there is no value assignment (to all observables) which satisfies the following assumptions. (a) Each observable has a sharp value which belongs to the spectrum of its representing operator. (b) Possessed values of observables are functionally related in the same way as their representing operators. When the dimension of Hilbert space is finite, this result is equivalent to the fact that there is no mapping $\mu : \mathcal{P}(\mathcal{H}) \rightarrow \{0, 1\}$ satisfying the following conditions.¹

1. $\mu(I) = 1$ where I is the identity operator.
2. $\mu(P + Q) = \mu(P) + \mu(Q)$ for any orthogonal projections P, Q .

Also, the same no-go result is shown as a corollary of Gleason's theorem (1957). Suppose there exists a mapping μ satisfying the above conditions. Then using Gleason's theorem, we have the consequence that there exists a density operator W such that $\mu(P) = \text{Tr}(WP)$ for any projection $P \in \mathcal{P}(\mathcal{H})$. However, such a density operator does not exist.

* Tokyo University of Marine Science and Technology
E-mail: higashi@i-ml.com

¹ For detailed argument, see Redhead, 1987, Chapter 5.

To this no-go result Bell (1966) said “It was tacitly assumed that measurement of an observable must yield the same value independently of what other measurements may be made simultaneously (see Bell, 2004, p.9). Bell thought as follows. Let X , Y , and Z denote self-adjoint operators. Suppose Y and Z are maximal. And suppose X , Y , and Z are in a functional relation $X = f(Y) = g(Z)$ (where $[Y, Z] \neq 0$ and f, g are Borel functions). As we have seen, the above no-go result assumes that possessed values of observables are functionally related in the same way as their representing operators. However, we can not measure two observables represented by Y and Z simultaneously. There is no a priori reason to believe that the value applying f to Y ’s measured value is the same as the value applying g to Z ’s measured value. Those values are under different ‘contexts’.

Van Fraassen (1973) interpreted realistically Bell’s idea. He thinks that there are different physical quantities corresponding to X in each context (like Y and Z). In his contextual approach X is splitted into X_Y and X_Z . X_Y and X_Z represent different physical quantities, so those values are not required to coincide.

Heywood and Redhead (1983) consider whether truth-value assignment satisfying locality requirement exists. They showed the impossibility of local truth-value assignment in contextual approach.² But their argument is restricted to finite dimensional and non-relativistic case. In this paper we shall show a similar no-go result in more general framework (von Neumann algebra). Also, our result will be applied to *Algebraic quantum field theory*.

2. Mathematical Preliminaries

2.1. Lattice Theory

Definition 1. The partially ordered set $(\mathcal{L}, \leq)$ is called a lattice if for any two elements $x, y \in \mathcal{L}$ there exists the least upper bound denoted by $x \vee y$ and the greatest lower bound denoted by $x \wedge y$.

In what follows every lattice will be assumed to have the least element denoted by 0 and the greatest element denoted by 1.

Definition 2. The lattice $(\mathcal{L}, \leq)$ is called a complete lattice if any subset of $\mathcal{L}$ has a greatest lower bound and a least upper bound. The lattice $(\mathcal{L}, \leq)$ is called σ -complete lattice if any countable subset of $\mathcal{L}$ has a greatest lower bound and a least upper bound.

Definition 3. A lattice $(\mathcal{L}, \leq)$ with 0 and 1 is called orthocomplemented if there is a mapping $\mathcal{L} \ni x \mapsto x^\perp \in \mathcal{L}$ satisfying the following conditions:

1. $x \vee x^\perp = 1$ and $x \wedge x^\perp = 0$ for every $x \in \mathcal{L}$,

² We can see the same result in Redhead (1987).

2. $x \leq y \Rightarrow y^\perp \leq x^\perp$ for every $x, y \in \mathcal{L}$,
3. $(x^\perp)^\perp = x$ for every $x \in \mathcal{L}$.

Definition 4. An orthocomplemented lattice is called an orthomodular lattice if the following relation holds.

If $x \leq y$ ($x, y \in \mathcal{L}$) then $y = x \vee (y \wedge x^\perp)$.

The set of all projection operators $\mathcal{P}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a complete orthomodular lattice.

Definition 5. An orthocomplemented lattice $(\mathcal{L}, \leq)$ is called an Boolean lattice if the following distributive laws hold.

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for every $x, y, z \in \mathcal{L}$.

The set of all spectral projections $\{P_E^X | E \in \mathbf{B}(\mathbf{R})\}$ (where X is a self-adjoint operator and $\mathbf{B}(\mathbf{R})$ is one dimensional Borel field) is a complete Boolean lattice.

Definition 6. Let $\mathcal{A}$ and $\mathcal{B}$ be a σ -complete Boolean lattice. A mapping $r : \mathcal{A} \rightarrow \mathcal{B}$ is called a Boolean σ -homomorphism if the following relations hold.

1. $r(a^\perp) = (r(a))^\perp$ for every $a \in \mathcal{A}$.
2. $r(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} r(a_i)$ where $\{a_i\}_{i \in I} \subseteq \mathcal{A}$ and I is countable.

A injective σ -homomorphism is called σ -isomorphism.

2.2. Von Neumann algebra

Definition 7. Let $\mathcal{R}$ be a $*$ -closed subalgebra of $\mathcal{B}(\mathcal{H})$ (the set consisting of all bounded linear operators on $\mathcal{H}$). $\mathcal{R}$ is a von Neumann algebra if the following conditions hold.

1. $I \in \mathcal{R}$.
2. If $\{X_\alpha\} (\subset \mathcal{R})$ converges strongly to $X \in \mathcal{B}(\mathcal{H})$ (i.e. $\|(X - X_\alpha)\phi\| \rightarrow 0$ for any $\phi \in \mathcal{H}$) then $X \in \mathcal{R}$.

A von Neumann algebra $\mathcal{R}$ such that for every $X, Y \in \mathcal{R}$ $XY = YX$ is called abelian.

$\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.

Definition 8. Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$. The set

$$\mathcal{S}' \equiv \{X \in \mathcal{B}(\mathcal{H}) | \forall Y \in \mathcal{S} (XY = YX)\}$$

is called the commutant of $\mathcal{S}$. The double commutant of $\mathcal{S}$, denoted $\mathcal{S}''$, is the commutant of $\mathcal{S}'$.

Fact 1. If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is $*$ -closed then $\mathcal{S}'$ and $\mathcal{S}'' \equiv (\mathcal{S}')'$ are von Neumann algebras. $\mathcal{S}''$ is the smallest von Neumann algebra containing $\mathcal{S}$.

Suppose $\mathcal{R}$ is a von Neumann algebra and a self-adjoint operator X is in $\mathcal{R}$. $\{X\}''$ is the smallest von Neumann algebra containing X and an abelian von Neumann subalgebra of $\mathcal{R}$.

Fact 2. Let $\mathcal{P}(\mathcal{R})$ denote the set of all projections in a von Neumann algebra $\mathcal{R}$. $\mathcal{P}(\mathcal{R})$ is a complete orthomodular lattice. If $\mathcal{R}$ is abelian then $\mathcal{P}(\mathcal{R})$ is a complete Boolean lattice.³

Therefore, for any self-adjoint operator $X \in \mathcal{B}(\mathcal{H})$ $\mathcal{P}(\{X\}'')$ is a complete Boolean lattice.

Definition 9. A linear functional ρ on a von Neumann algebra $\mathcal{R}$ is called a state if ρ satisfies the following conditions.

1. $\rho(X^*X) \geq 0$ for any $X \in \mathcal{R}$.
2. $\rho(I) = 1$ where $I \in \mathcal{R}$ is the Identity operator.

A state ρ on $\mathcal{R}$ is called a normal state if there is a density operator W such that $\rho(X) = \text{tr}(WX)$ for any $X \in \mathcal{R}$.

3. Boolean representation in contextual approach

3.1. A corollary of generalized Gleason's theorem

Gleason's theorem (1957) for $\mathcal{P}(\mathcal{H})$ was generalized to $\mathcal{P}(\mathcal{R})$.⁴ In algebraic quantum field theory a von Neumann algebra $\mathcal{R}(O)$ is associated to each bounded open set O in Minkowski space $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$. Generalized Gleason's theorem is applicable to $\mathcal{P}(\mathcal{R}(O))$.

Definition 10. A mapping $\mu : \mathcal{P}(\mathcal{R}) \rightarrow \{0, 1\}$ is called a finitely additive truth-value assignment if the following conditions hold.

1. $\mu(I) = 1$ where I is the identity operator.
2. $\mu(P + Q) = \mu(P) + \mu(Q)$ for any orthogonal projections $P, Q \in \mathcal{P}(\mathcal{R})$.

Kitajima (2006) showed the following proposition using generalized Gleason's

³ For the proofs see Rédei (1998 p.82).

⁴ See, e.g., Hamhalter (1993) Chapter 5.

theorem and a mathematical fact in algebraic quantum field theory (Baumgartel, 1995, Corollary 1.11.6).

Proposition 1. (*See Kitajima, 2006, Theorem 5.2 and the next paragraph.*) For any bounded open set O in Minkowski space $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$ there is no finitely additive truth-value assignment on $\mathcal{P}(\mathcal{R}(O))$.

3.2. Boolean representation of $\mathcal{P}(\mathcal{R})$ in contextual approach

In reference to Maczynski (1971) we formulate contextual approach as follows.

Definition 11. Let $\mathcal{R}$ be a von Neumann algebra. A family of abelian von Neumann subalgebras of $\mathcal{R}$ is called a context-set and its element is called a context.

Definition 12. Suppose $\mathcal{R}$ is a von Neumann algebra and $\mathcal{F}$ is a context set. $(\mathcal{B}, \{r_{\mathcal{V}}\}_{\mathcal{V} \in \mathcal{F}})$ is called a Boolean representation of $\cup_{\mathcal{V} \in \mathcal{F}} \mathcal{P}(\mathcal{V})$ under the context-set $\mathcal{F}$ if the following conditions hold.

1. $\mathcal{B}$ is a σ -complete Boolean lattice.
2. A mapping $r_{\mathcal{V}} : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{B}$ is a σ -isomorphism for each $\mathcal{V} \in \mathcal{F}$.
3. For $\mathcal{V}, \mathcal{W} \in \mathcal{F}$ if $\mathcal{V} \subseteq \mathcal{W}$ then $r_{\mathcal{V}} = r_{\mathcal{W}} \circ h_{\mathcal{V}\mathcal{W}}$ (where $h_{\mathcal{V}\mathcal{W}} : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{W})$ is an inclusion mapping defined as $h_{\mathcal{V}\mathcal{W}}(P) \equiv P$ for each $P \in \mathcal{P}(\mathcal{V})$).
4. $\cup_{\mathcal{V} \in \mathcal{F}} r_{\mathcal{V}}(\mathcal{P}(\mathcal{V}))$ σ -generates $\mathcal{B}$.

The aims of the second and third conditions are as follows respectively. Suppose X and Y ($\in \mathcal{V}$) are self-adjoint operators and $X = f(Y)$ where f is a Borel function. Then for the spectral projections of X and Y , $P_E^X = P_{f^{-1}(E)}^Y$ where $E \in \mathbf{B}(\mathbf{R})$. Therefore, if the second condition holds then $r_{\mathcal{V}}(P_E^X) = r_{\mathcal{V}}(P_{f^{-1}(E)}^Y)$ for each $\mathcal{V} \in \mathcal{F}$. Functional relations of physical quantities are preserved in each context.

Let X and Y ($\in \mathcal{V}$) be self-adjoint operators such that $X = f(Y)$ where f is a Borel function. Suppose $\mathcal{V}$ and $\mathcal{W}$ are contexts to which X and Y respectively belong (i.e. $X \in \mathcal{V}$ and $Y \in \mathcal{W}$), furthermore, $\mathcal{V} \subseteq \mathcal{W}$. If the third condition holds then for any Borel set $E \in \mathbf{B}(\mathbf{R})$

$$r_{\mathcal{V}}(P_E^X) = r_{\mathcal{W}}(P_E^X) = r_{\mathcal{W}}(P_{f^{-1}(E)}^Y).$$

By this condition we can exclude uninteresting models from Boolean representations. For example, the model in which all abelian von Neumann subalgebras generated by any self-adjoint operator $X \in \mathcal{R}$ are contexts is excluded.

Maczynski showed “the set of all maximal observables (on a Hilbert space) is Boolean representable” (Maczynski, 1971, Theorem 4.4). In a similar way we can show the following proposition.

Proposition 2. (*Easy generalization based on Maczynski’s theorem and proof*)

Let $\mathcal{R}$ be a von Neumann algebra. Suppose $\mathcal{F}$ is the family of all maximal abelian von Neumann subalgebras of $\mathcal{R}$. Then there exists a Boolean representation of $\cup_{\mathcal{V} \in \mathcal{F}} \mathcal{P}(\mathcal{V})$ under the context-set $\mathcal{F}$.

We shall see an overview of proof. Suppose $\{\mathcal{B}_t\}_{t \in T}$ is a family of σ -complete Boolean lattices. $(\mathcal{B}, \{f_t\}_{t \in T})$ is called a free σ -product of $\{\mathcal{B}_t\}_{t \in T}$ if the following conditions hold. (The below definition is owed to Dwinger, 1967, p.318.)

1. $\mathcal{B}$ is a σ -complete Boolean lattice.
2. $f_t : \mathcal{B}_t \rightarrow \mathcal{B}$ is a σ -isomorphism for each $t \in T$.
3. For any $(\mathcal{C}, \{g_t\}_{t \in T})$ (where $\mathcal{C}$ is a σ -complete Boolean algebra and $g_t : \mathcal{B}_t \rightarrow \mathcal{C}$ is a σ -homomorphism) there uniquely exists a σ -homomorphism $g : \mathcal{B} \rightarrow \mathcal{C}$ such that for any $t \in T$ $g_t = g \circ f_t$.

A free σ -product of $\{\mathcal{B}_t\}_{t \in T}$ uniquely exists up to isomorphism. (For proof see Sikorski, 1964, Section 38.)

As stated in *Proposition 2*, a context-set $\mathcal{F}$ is supposed to be a family of maximal abelian von Neumann subalgebras of $\mathcal{R}$. Then the free σ -product of $\{\mathcal{P}(\mathcal{V})\}_{\mathcal{V} \in \mathcal{F}}$ is a Boolean representation of $\cup_{\mathcal{V} \in \mathcal{F}} \mathcal{P}(\mathcal{V})$ under the context-set $\mathcal{F}$. Clearly the first and second conditions of *Definition 12* are satisfied. For the other conditions as follows. Suppose $\mathcal{V}, \mathcal{W} \in \mathcal{F}$ are contexts and $\mathcal{V} \subseteq \mathcal{W}$. $\mathcal{F}$ is a family of maximal algebra, so $\mathcal{V} = \mathcal{W}$. Therefore, the third condition is trivially satisfied. Also, in general, for any free σ -product $(\mathcal{B}, \{f_t\}_{t \in T})$, $\cup_{t \in T} f_t(\mathcal{B}_t)$ σ -generates $\mathcal{B}$. Therefore, the forth condition is satisfied.

3.3. Algebraic locality

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be von Neumann algebras such that the following conditions hold.

1. $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-abelian.
2. $\mathcal{R}_1 \subseteq \mathcal{R}_2'$.
3. *Schlieder Property*: For any $X \in \mathcal{R}_1, Y \in \mathcal{R}_2$ if $X \neq 0$ and $Y \neq 0$ then $XY \neq 0$.

Let us suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are the set of all maximal abelian von Neumann subalgebras of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. We call $\mathcal{F}_1$ and $\mathcal{F}_2$ **local context-sets**. Let $\mathcal{V}_1 (\in \mathcal{F}_1)$ and $\mathcal{V}_2 (\in \mathcal{F}_2)$ be contexts. Then $(\mathcal{V}_1 \cup \mathcal{V}_2)''$ is an abelian von Neumann subalgebra of $(\mathcal{R}_1 \cup \mathcal{R}_2)''$. We call $\mathcal{F}_{12} \equiv \{(\mathcal{V}_1 \cup \mathcal{V}_2)'' : \mathcal{V}_1 \in \mathcal{F}_1, \mathcal{V}_2 \in \mathcal{F}_2\}$ a **global context-set**.

The following problem is interesting.

Problem 1. Let the context-set $\mathcal{F}$ be the union of local and global context-sets (i.e. $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_{12}$). Is there a Boolean representation of $\cup_{\mathcal{V} \in \mathcal{F}} \mathcal{P}(\mathcal{V})$ under the context-set $\mathcal{F}$.

We do not know general answers to this problem.⁵ We would like to add a comment and leave the problem open.

- Suppose that there exists a Boolean representation of $\cup_{\mathcal{V} \in \mathcal{F}} \mathcal{P}(\mathcal{V})$ under the above context-set. We denote the Boolean representation by $(\mathcal{B}, \{r_{\mathcal{V}_1}\}_{\mathcal{V}_1 \in \mathcal{F}_1} \cup \{r_{\mathcal{V}_2}\}_{\mathcal{V}_2 \in \mathcal{F}_2} \cup \{r_{(\mathcal{V}_1 \cup \mathcal{V}_2)''}\}_{(\mathcal{V}_1 \cup \mathcal{V}_2)'' \in \mathcal{F}_{12}})$. For any $\mathcal{V}_1 \in \mathcal{F}_1$ and $\mathcal{V}_2 \in \mathcal{F}_2$, the inclusion relation $\mathcal{V}_1 \subseteq (\mathcal{V}_1 \cup \mathcal{V}_2)''$ holds. Therefore, from the third condition in *Definition 12* $r_{\mathcal{V}_1} = r_{(\mathcal{V}_1 \cup \mathcal{V}_2)''} \circ h_{\mathcal{V}_1(\mathcal{V}_1 \cup \mathcal{V}_2)''}$. So, for any $P \in \mathcal{P}(\mathcal{V}_1)$, the equation $r_{\mathcal{V}_1}(P) = r_{(\mathcal{V}_1 \cup \mathcal{V}_2)''}(P)$ holds. In other word, for any $P \in \mathcal{P}(\mathcal{V}_1)$, even if $\mathcal{V}_2 \neq \mathcal{W}_2$ ($\mathcal{V}_2, \mathcal{W}_2 \in \mathcal{F}_2$),

$$r_{(\mathcal{V}_1 \cup \mathcal{V}_2)''}(P) = r_{(\mathcal{V}_1 \cup \mathcal{W}_2)''}(P).$$

Similarly, for any $P \in \mathcal{P}(\mathcal{V}_2)$, even if $\mathcal{V}_1 \neq \mathcal{W}_1$ ($\mathcal{V}_1, \mathcal{W}_1 \in \mathcal{F}_1$),

$$r_{(\mathcal{V}_1 \cup \mathcal{V}_2)''}(P) = r_{(\mathcal{W}_1 \cup \mathcal{V}_2)''}(P).$$

We call this relation **algebraic locality**. The above problem is whether the algebraic locality holds.

4. The impossibility of local truth-value assignment

Even if algebraic locality holds, that does not mean the existence of the local truth-value assignments which reproduce quantum predictions. We shall show this fact in *Proposition 5*.

4.1. Hardy-Type non-locality argument

Let $\mathcal{H}_1^2 \otimes \mathcal{H}_2^2$ denote the tensor product Hilbert space of two-dimensional Hilbert spaces $\mathcal{H}_1^2$ and $\mathcal{H}_2^2$. Suppose $|\psi\rangle \in \mathcal{H}_1^2 \otimes \mathcal{H}_2^2$ is a state vector of composite system. Then there exist orthonormal bases $\{|+\rangle_1, |-\rangle_1\}$ for $\mathcal{H}_1^2$, and $\{|+\rangle_2, |-\rangle_2\}$ for $\mathcal{H}_2^2$ such that $|\psi\rangle = \alpha|+\rangle_1|+\rangle_2 - \beta|-\rangle_1|-\rangle_2$ where α and β are non-negative real numbers satisfying $\alpha^2 + \beta^2 = 1$ (by Schmidt Decomposition). Hardy showed the following proposition.

Proposition 3. (*Hardy*) Suppose that using Schmidt decomposition, a state vector $|\psi\rangle \in \mathcal{H}_1^2 \otimes \mathcal{H}_2^2$ is $|\psi\rangle = \alpha|+\rangle_1|+\rangle_2 - \beta|-\rangle_1|-\rangle_2$ (where $\alpha^2 + \beta^2 = 1$). If $\alpha, \beta \neq 0$ and $\alpha \neq \beta$ then there exist four projections $P_1, P_2 \in \mathcal{P}(\mathcal{H}_1^2)$ and $Q_1, Q_2 \in \mathcal{P}(\mathcal{H}_2^2)$ satisfying the following relations.

$$\langle \psi | P_1 \otimes Q_1 | \psi \rangle = 0$$

⁵ Demopoulos (1980, Section 5, ADDENDA) gave a affirmative answer to this problem in the case of $\mathcal{B}(\mathcal{H}_1^N) \otimes \mathcal{B}(\mathcal{H}_2^N)$ (where $\mathcal{H}^N$ is a $N(\leq \infty)$ dimensional Hilbert space).

$$\begin{aligned}
\langle \psi | P_2 \otimes (I - Q_1) | \psi \rangle &= 0 \\
\langle \psi | (I - P_1) \otimes Q_2 | \psi \rangle &= 0 \\
\langle \psi | P_2 \otimes Q_2 | \psi \rangle &\geq 0
\end{aligned}$$

In von Neumann algebra the next proposition holds.

Proposition 4. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be von Neumann algebras consist of bounded operators on the same Hilbert space. And suppose those von Neumann algebras satisfy the following conditions.*

1. $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-abelian.
2. $\mathcal{R}_1 \subseteq \mathcal{R}_2'$.
3. **Schlieder Property:** For any $X \in \mathcal{R}_1, Y \in \mathcal{R}_2$ if $X \neq 0$ and $Y \neq 0$ then $XY \neq 0$.

Then there exist projections $P_1, P_2 \in \mathcal{R}_1; Q_1, Q_2 \in \mathcal{R}_2$ and a normal state ρ on $(\mathcal{R}_1 \cup \mathcal{R}_2)''$ such that the following relations hold.

$$\rho(P_1 Q_1) = 0 \quad (1)$$

$$\rho(P_2(I - Q_1)) = 0 \quad (2)$$

$$\rho((I - P_1)Q_2) = 0 \quad (3)$$

$$\rho(P_2 Q_2) \geq 0 \quad (4)$$

Proof. $\mathcal{R}_1$ is non-abelian, so there exist non-commuting projections $R, T \in \mathcal{R}_1$ such that $(I - R)TR \neq 0$. Then there exists a partial isometry $V (\neq 0) \in \mathcal{R}_1$ such that $(I - R)TR = V|(I - R)TR|$ (polar decomposition). Where the initial space and the end space of V is $\overline{\text{range}(((I - R)TR)^*)} = \overline{\text{range}(RT(I - R))} (\subseteq \text{range}(R))$ and $\overline{\text{range}((I - R)TR)} (\subseteq \text{range}(I - R))$, respectively. And V^*V and VV^* are non-zero orthogonal projections. Similarly, for $\mathcal{R}_2$ there exist a partial isometry $W \in \mathcal{R}_2$ and non-zero orthogonal projections W^*W and WW^* .

$\mathcal{R}_1 \subseteq \mathcal{R}_2'$, so V^*VW^*W and VV^*WW^* are projections. Furthermore, these projections are non-zero because of *Schlieder Property*.

Let α, β be positive real numbers such that $\alpha^2 + \beta^2 = 1$ and $\alpha \neq \beta$. And suppose ϕ is in $\text{range}(V^*VW^*W)$ ($\|\phi\| = 1$). Then $\psi \equiv \alpha\phi - \beta VW\phi$ is a vector state on $(\mathcal{R}_1 \cup \mathcal{R}_2)''$. Notice that two vectors ϕ and $VW\phi$ are orthonormal.

Projections $P_1, P_2 \in \mathcal{R}_1$ and $Q_1, Q_2 \in \mathcal{R}_2$ are defined as

$$\begin{aligned}
P_1 &\equiv \frac{1}{\alpha + \beta} (\beta^{1/2} V^* V + \alpha^{1/2} V) (\beta^{1/2} V^* V + \alpha^{1/2} V)^*, \\
P_2 &\equiv \frac{1}{\alpha^3 + \beta^3} (\beta^{3/2} V^* V - \alpha^{3/2} V) (\beta^{3/2} V^* V - \alpha^{3/2} V)^*,
\end{aligned}$$

$$Q_1 \equiv \frac{1}{\alpha + \beta} (\beta^{1/2} W^* W + \alpha^{1/2} W) (\beta^{1/2} W^* W + \alpha^{1/2} W)^*,$$

$$Q_2 \equiv \frac{1}{\alpha^3 + \beta^3} (\beta^{3/2} W^* W - \alpha^{3/2} W) (\beta^{3/2} W^* W - \alpha^{3/2} W)^*.$$

Then we have the following (in)equalities.

$$\begin{aligned} (\psi, P_1 Q_1 \psi) &= 0 \\ (\psi, Q_2 \psi) &= (\psi, P_1 Q_2 \psi) \\ (\psi, P_2 \psi) &= (\psi, P_2 Q_1 \psi) \\ (\psi, P_2 Q_2 \psi) &= \frac{(\beta^2 - \alpha^2)^2}{(\alpha^3 + \beta^3)^2} (\alpha\beta)^2 \geq 0 \end{aligned}$$

When we define a linear functional ρ_ψ on $(\mathcal{R}_1 \cup \mathcal{R}_2)''$ as $\rho_\psi \equiv (\psi, X\psi)$ for each $X \in (\mathcal{R}_1 \cup \mathcal{R}_2)''$, ρ_ψ is a normal state on $(\mathcal{R}_1 \cup \mathcal{R}_2)''$. So, the proposition is showed. $\square$

4.2. Deduction of contradiction

Using *Proposition 4* we shall see a difficulty of local truth-value assignment consistent with quantum predictions. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be von Neumann algebras satisfying the three conditions in *Proposition 4*. Suppose that context sets of $\mathcal{P}(\mathcal{R}_1)$ and $\mathcal{P}(\mathcal{R}_2)$ are the family of maximal abelian von Neumann subalgebras of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. And we define a following set which element is a pair of a projection and a global context.

$$\mathcal{S} \equiv \{ (P, (\mathcal{V}_1 \cup \mathcal{V}_2)'') \mid \mathcal{V}_1 \in \mathcal{F}_1, \mathcal{V}_2 \in \mathcal{F}_2, P \in \mathcal{P}(\mathcal{V}_1) \cup \mathcal{P}(\mathcal{V}_2) \}$$

If A mapping $\mu : \mathcal{S} \rightarrow \{1, 0\}$ is local truth-value assignment then the following two conditions should be satisfied.

- Finite additivity** 1. $\mu(I, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 1$ for any global context $(\mathcal{V}_1 \cup \mathcal{V}_2)''$.
 2. For any orthogonal projections $P, Q \in (\mathcal{V}_1 \cup \mathcal{V}_2)''$

$$\mu(P + Q, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = \mu(P, (\mathcal{V}_1 \cup \mathcal{V}_2)'') + \mu(Q, (\mathcal{V}_1 \cup \mathcal{V}_2)'')$$

Locality Suppose a projection P is an element of $\mathcal{V}_1$. For any local contexts $\mathcal{V}_2, \mathcal{W}_2 \in \mathcal{F}_2$

$$\mu(P, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = \mu(P, (\mathcal{V}_1 \cup \mathcal{W}_2)'').$$

Suppose a projection Q is an element of $\mathcal{V}_2$. For any local contexts $\mathcal{V}_1, \mathcal{W}_1 \in \mathcal{F}_1$

$$\mu(Q, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = \mu(Q, (\mathcal{W}_1 \cup \mathcal{V}_2)'').$$

Let $\mathcal{T}$ denote the set of all mappings $\mu : \mathcal{S} \rightarrow \{1, 0\}$ satisfying the above conditions. In relation to a normal state ρ on $(\mathcal{R}_1 \cup \mathcal{R}_2)''$, we require that $\mathcal{T}$ and its element μ satisfy the following two rules.

Rule of simultaneous valuation I Suppose projections P and Q are elements of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. If $\rho(PQ) = 0$ then for any $\mu \in \mathcal{T}$ and for any local context $\mathcal{V}_1$ (where $P \in \mathcal{V}_1$) and $\mathcal{V}_2$ (where $Q \in \mathcal{V}_2$), $\mu(P, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 0$ or $\mu(Q, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 0$.

Rule of simultaneous valuation II Suppose projections P and Q are elements of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. If $\rho(PQ) \geq 0$ then there exist $\mu \in \mathcal{T}$ and local context $\mathcal{V}_1$ (where $P \in \mathcal{V}_1$) and $\mathcal{V}_2$ (where $Q \in \mathcal{V}_2$) such that $\mu(P, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 1$ and $\mu(Q, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 1$.

We can show the next proposition.

Proposition 5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be von Neumann algebras satisfying the conditions in Proposition 4. Suppose ρ_ψ is a normal state constructed in the proof of Proposition 4. And suppose that $P_1, P_2 \in \mathcal{R}_1$ and $Q_1, Q_2 \in \mathcal{R}_2$ are projections constructed in the proof of Proposition 4. Then for the normal state ρ_ψ and the projections P_1, P_2, Q_1, Q_2 , a contradiction is derived from **Finite additivity**, **Locality**, and **Rules of simultaneous valuation I, II**.*

Proof. From the inequality (4) on page 52 and *Rule of simultaneous valuation II*, there exist $\mu \in \mathcal{T}$ and local context $\mathcal{V}_1$ (where $P_2 \in \mathcal{V}_1$) and $\mathcal{V}_2$ (where $Q_2 \in \mathcal{V}_2$) such that

$$\mu(P_2, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 1 \quad (5)$$

and

$$\mu(Q_2, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 1. \quad (6)$$

The context-set $\mathcal{F}_1$ is the family of all maximal abelian von Neumann subalgebras of $\mathcal{R}_1$, so there is a context $\mathcal{W}_1 (\in \mathcal{F}_1)$ such that $P_1, I - P_1 \in \mathcal{W}_1$. From (3) and *Rule of simultaneous valuation I*, $\mu(I - P_1, (\mathcal{W}_1 \cup \mathcal{V}_2)'') = 0$ or $\mu(Q_2, (\mathcal{W}_1 \cup \mathcal{V}_2)'') = 0$. If $\mu(Q_2, (\mathcal{W}_1 \cup \mathcal{V}_2)'') = 0$ then from *Locality* we have the equation $\mu(Q_2, (\mathcal{V}_1 \cup \mathcal{V}_2)'') = 0$. However, this contradicts the equation (6). Therefore, $\mu(I - P_1, (\mathcal{W}_1 \cup \mathcal{V}_2)'') = 0$. Then from *Finite additivity* we have the equation

$$\mu(P_1, (\mathcal{W}_1 \cup \mathcal{V}_2)'') = 1. \quad (7)$$

The context-set $\mathcal{F}_2$ is the family of all maximal abelian von Neumann subalgebras of $\mathcal{R}_2$, so there is a context $\mathcal{W}_2 (\in \mathcal{F}_2)$ such that $Q_1, I - Q_1 \in \mathcal{W}_2$. In the same way as the adjacent paragraph we have

$$\mu(Q_1, (\mathcal{V}_1 \cup \mathcal{W}_2)'') = 1. \quad (8)$$

From the equation (7), (8) and *Locality*,

$$\mu(P_1, (\mathcal{W}_1 \cup \mathcal{W}_2)'') = 1 \quad \text{and} \quad \mu(Q_1, (\mathcal{W}_1 \cup \mathcal{W}_2)'') = 1.$$

However, from (1) and *Rule of simultaneous valuation I*

$$\mu(P_1, (\mathcal{W}_1 \cup \mathcal{W}_2)'') = 0 \quad \text{or} \quad \mu(Q_1, (\mathcal{W}_1 \cup \mathcal{W}_2)'') = 0.$$

So, a contradiction is derived. $\square$

In *Proposition 5* we assumed that von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfied the following conditions.

1. $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-abelian.
2. $\mathcal{R}_1 \subseteq \mathcal{R}_2'$.
3. **Schlieder Property:** For any $X \in \mathcal{R}_1, Y \in \mathcal{R}_2$ if $X \neq 0$ and $Y \neq 0$ then $XY \neq 0$.

These conditions are satisfied in AQFM (*Algebraic quantum field theory*). In that theory von Neumann algebras $\mathcal{R}(O)$ are associated to each bounded open set O in Minkowski space $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$. First, $\mathcal{R}(O)$ are non-abelian because $\mathcal{R}(O)$ are properly infinite.⁶ Second, in AQFT $\mathcal{R}(O_1) \subseteq (\mathcal{R}(O_2))'$ if O_1 and O_2 are spacelike separated.⁷ Finally, if O_1 and O_2 are strictly spacelike separated (i.e. there exists a neighborhood N of the origin such that for any $x \in N$, $O_1 + x$ and O_2 are spacelike separated), then from axiom of AQFT we can show that *Schlieder Property* holds for $\mathcal{R}(O_1)$ and $\mathcal{R}(O_2)$.⁸

5. Conclusion

Contextual approach dissolved Kochen and Specker's paradox by splitting observables. However, then, as Heywood and Redhead showed, another problem, i.e. non-locality, arised. Their proof was restricted to finite and non-relativistic case. In this paper we showed the similar no-go result in more general framework (von Neumann algebra). Our result was applied to *Algebraic quantum field theory*. In that theory, even if splitting observables, for any two von Neumann algebras over strictly spacelike separated regions there is no truth-value assignment satisfying **Finite additivity**, **Locality**, and **Rules of simultaneous valuation I, II**.

⁶ For proof, see, e.g., Baumgartel, 1995, p.47 (Corollary 1.11.6).

⁷ This is a requirement in axiom of AQFT. See, e.g., Baumgartel, 1995, p.10.

⁸ For proof, see Baumgartel, 1995, p.51 (Theorem 1.12.3).

References

- [1] Baumgartel, H. (1995) *Operatoralgebraic Methods in Quantum Field Theory*, Akademie Verlag.
- [2] Bell, J.S. (1966) “On the Problems of Hidden Variables in Quantum Mechanics”, *Review of Modern Physics* **38**: 447–475. Reprinted in Bell, J.S. (2004) *Speakable and Unspeakable in Quantum Mechanics* (Second Edition): 1–13, Cambridge.
- [3] Demopoulos, W. (1980) “Locality and the Algebraic Structure of Quantum Mechanics”, in Suppes, P. (ed.), *Studies in Foundations of Quantum Mechanics*: 119–144, East Lansing.
- [4] Dwingher, Ph. (1967) “Direct Limits of Partially Ordered Systems of Boolean Algebras”, *Indagationes Mathematicae* **29**: 317–325.
- [5] Gleason, A. (1957) “Measures on the Closed Subspaces of a Hilbert Space”, *Journal of Mathematics and Mechanics* **6**: 885–893. Reprinted in Hooker, C.A. (ed.) (1975) *The Logico-Algebraic Approach to Quantum Mechanics*: 123–133, Reidel.
- [6] Hamhalter, J. (2003) *Quantum Measure Theory*, Kluwer.
- [7] Hardy, L. (1993) “Nonlocality for Two Particles without Inequalities for almost all Entangled States”, *Physical Review Letters* **71**: 1665–1668.
- [8] Heywood, P. and Redhead, M. (1983): “Nonlocality and the Kochen-Specker Paradox”, *Foundations of Physics* **13**: 481–499.
- [9] Kitajima, Y. (2006) “On the Problem of Hidden Variables in Algebraic Quantum Field Theory”, *Annals of the Japan Association for Philosophy of Science* **15-1**: 25–38.
- [10] Kochen, S. and Specker, E. (1967) “The Problem of Hidden Variables in Quantum Mechanics”, *Journal of Mathematics and Mechanics* **17**: 59–87. Reprinted in Hooker, C.A. (ed.) (1975) *The Logico-Algebraic Approach to Quantum Mechanics*: 293–328, Reidel.
- [11] Maczynski, M.J. (1971) “Boolean Properties of Observables in Axiomatic Quantum Mechanics”, *Reports on Mathematical Physics* **2**: 135–150.
- [12] Rédei, M. (1998) *Quantum Logic in Algebraic Approach*, Kluwer.
- [13] Redhead, M. (1987) *Incompleteness, Nonlocality, and Realism*, Oxford.
- [14] Sikorski, R. (1964) *Boolean Algebras* (Second Edition), Springer.
- [15] Van Fraassen, B. (1973) “Semantic Analysis of Quantum Logic”, in Hooker, C. (ed.), *Contemporary Research in Foundations and Philosophy of Quantum Theory*: 80–113, Reidel.

(Received 2009.1.17; Accepted 2009.1.30)