The Set-theoretic Multiverse: A Natural Context for Set Theory

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Abstract

Set theorists often take their subject as constituting a foundation for the rest of mathematics, in the sense that other abstract mathematical objects can be construed fundamentally as sets. In this way, they regard the set-theoretic universe as the universe of all mathematics. And although many set-theorists affirm the Platonic view that there is just one universe of all sets, nevertheless the most powerful set-theoretic tools developed over the past half century are actually methods of constructing alternative universes. With forcing and other methods, we can now produce diverse models of ZFC set theory having precise, exacting features. The fundamental object of study in set theory has thus become the model of set theory, and the subject consequently begins to exhibit a category-theoretic second-order nature. We have a multiverse of set-theoretic worlds, connected by forcing and large cardinal embeddings like constellations in a dark sky. In this article, I will discuss a few emerging developments illustrating this second-order nature. The work engages pleasantly with various philosophical views on the nature of mathematical existence.

This article is based on a talk I gave at the conference in honor of the retirement of Yuzuru Kakuda in Kobe, Japan, March 7, 2009. I would like to begin by expressing my gratitude to Kakuda-sensei and the rest of the logic group in Kobe for the opportunities provided to me to participate in logic in Japan. In particular, my time as a JSPS Fellow in the logic group at Kobe University in 1998 was a formative experience. I was part of a vibrant research group in Kobe; I enjoyed Japanese life, learned to speak a little Japanese and made many friends. Mathematically, it was a productive time, and after years away how pleasant it is for me to see that ideas planted at that time, small seedlings then, have grown into tall slender trees. I will begin with a general discussion in order to set the context for the main ideas of the article, which come in Sections §2 and §3. This survey is intended to introduce the

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area by describing the principal concepts and preliminary results, mostly adapted from [4], [5] and [1], but with the main proofs omitted. This survey has substantial overlap with material in [2], a similar but somewhat more technical survey aimed at a different audience.

1. Introduction

Set theory is the study of sets, particularly the transfinite, with a focus on well-founded transfinite recursion. The subject began in earnest with Cantor in the late 19th century and became mature in the mid 20th century, with the rise of forcing and other sophisticated techniques. Set theory today is vast. We have independence proofs, large cardinals, forcing, infinite combinatorics, set theory of the continuum, descriptive set theory and on and on and on. And while set theory is preoccupied with its own questions and problems, it also happens to serve as an ontological foundation for the rest of mathematics, in the sense that mathematical objects can be viewed fundamentally as sets. On this view, everything in mathematics is a set. Surely, being precise in mathematics often amounts to specifying an object or construction in set theory. For example, a *topological space* is an underlying set together with its open sets, a collection of sets having a certain property. A *group* is a set with a binary operation having certain properties. A binary operation is a function, and a function is a set of ordered pairs with a certain feature. The real numbers are Dedekind cuts in the rational numbers, that is, upper bounded sets of rational numbers closed downward and having no largest element. Even the rational numbers and the natural numbers can be viewed as special kinds of sets.

On this view, one imagines all mathematical arguments as taking place within set theory. Set theory consequently speaks to or with other mathematical subjects, particularly on foundational matters. After all, if the analysts are attempting to carry out a certain construction, and the set theorists prove that it is consistent with the axioms of set theory that it cannot be done, then the analysts should give up hope of succeeding, since their construction can be taken as occurring within set theory. More positively, if the set theorists were to prove that it is consistent that the construction can succeed, then the analysts should be encouraged, knowing that they are on the right track, or at least on an interesting track, and explore whether their method always succeeds or whether success depends on the set-theoretic background.

The fundamental axioms of set theory are the Zermelo-Fraenkel ZFC axioms with the Axiom of Choice, which concern set existence. The ZFC axioms were chosen to include all of the most general set construction principles used in the foundation of mathematics in set theory. Thus, one can view all these mathematical arguments as taking place within set theory, using the ZFC axioms.

A *model of set theory* is a mathematical structure in which the ZFC axioms are true. Since the set-theoretic axioms ground essentially all mathematical arguments,

each such model is a complete mathematical world, in which nearly any mathematician could be at home. Every model of ZFC has its own versions of all the mathematical objects that the typical mathematician deals with: the natural numbers, the real numbers, Hilbert spaces and so on. Thus, all of classical mathematics can be carried out inside any one of these models of set theory. Although these models of set theory may appear indistinguishable from a classical perspective, since each is sufficient to carry out the classical arguments, nevertheless they can in fact be distinguished in their set-theoretic features. A dominant theme of set-theoretic research in the past fifty years is the construction of more and more models of set theory. By means of forcing, ultrapowers, constructibility, definability and more general inner model theory, we have constructed a vast collection of diverse models of set theory, each a mathematical world of its own.

There is a strong analogy with geometry here. More than a century ago mathematicians were shocked to discover an abundance of alternative geometrical worlds. We now have Euclidean geometry and non-Euclidean geometry, with non-Euclidean geometry coming in several flavors. Surely, this situation was profound for geometry. The situation in set theory, I argue, is more profound, because set theory serves as a foundation for all mathematics. We have discovered the multiverse of alternative mathematical worlds, with broad agreement to be sure on basic axioms, but with definite disagreements beyond this as to what is mathematically true. The choice of which universe to live in can have consequences for one's mathematical truths, even for questions in classical mathematics. Indeed, even questions in elementary arithmetic, such as whether there is a natural number whose factors have a certain pattern, can have different answers in different models of set theory.

So the models of set theory can have different truth values even for some of the most important set-theoretic questions. For example, the *Continuum Hypothesis* CH is the assertion that every set of real numbers is either countable or equinumerous with the entire set of reals; in other words, there are no infinities between the cardinality of the natural numbers and the cardinality of the reals. This question, studied intensely by Cantor but not answered until a half-century later, was perhaps the most prominent set-theoretic question in the early days of set theory. Gödel finally provided a model of ZFC in which the Continuum Hypothesis holds, and Cohen provided one in which it fails. This means that the Continuum Hypothesis cannot be settled on the basis of the fundamental axioms—it is *independent* of ZFC. Set theorists have powerful methods to construct such models, the most important of which is likely the method of *forcing*, introduced by Cohen in 1963. Forcing has been used to construct thousands of different models of set theory, often built to exhibit precise, exacting features.

As set theory has matured, the fundamental object of study has become not any particular set or set construction method, but rather: the model of set theory. Just

as group theorists study groups and ring theorists study rings, the set theorists study the models of set theory. We seek models having certain features or a certain relation to other models. As a result, set theory now exhibits a category-theoretic nature. What we have is a vast cosmos of models of set theory, each its own mathematical universe, connected by forcing extensions and large cardinal embeddings, like constellations in a dark sky. The thesis of this article is that, as a result, set theory now exhibits an essential second-order nature.

I would like here to discuss two emerging developments exhibiting this second-order nature of the set-theoretic enterprise. The first is the modal logic of forcing, upward-oriented from a model of set theory to its forcing extensions. The second is set-theoretic geology, downward-oriented from a model of set theory down to its ground models. This analysis engages pleasantly with various philosophical views on the nature of mathematical existence. In particular, the two perspectives are unified by and find motivation in a multiverse view of set theory, the philosophical view that there are many set-theoretic worlds.

Before proceeding, let me briefly discuss some of these philosophical positions on the nature of mathematical existence. The orthodox view among set theorists has been the philosophical position known as *Mathematical Platonism*. This is the view that there is a single universe of set theory, the real mathematical universe, and our task as set theorists is to understand it. Platonists view the set-theoretic universe as having a real mathematical existence, built up in the cumulative hierarchy in the way that Cantor described. For a Platonist, the fact that set-theoretic questions such as CH are independent of ZFC does not settle the question of whether they may have a definitive answer for the actual universe.

One paradox for this view, to my mind, is the fact that the most powerful tools in set theory are most naturally viewed as methods of constructing alternative universes. That is, we build new models of set theory from existing models, via forcing and ultrapowers. With forcing, we are like Galileo viewing the moons of Jupiter, catching a glimpse of other worlds and gaining a small understanding of what life would be like in them. The Platonists reply that no, forcing is a kind of illusion—it is just about consistency, or it is merely informing us about what is *not* provable, or it is just about countable models—and that it does not give us any access to actual other worlds. This response reminds me of those who, in Gallileo's day, disbelieving his telescope, claimed it was an illusion and looked on the lenses to see what images were painted there.

So let me contrast the Platonist view with what I call the *Multiverse View*. This is the philosophical position accepting these alternative universes as fully existing mathematically. This is a form of realism, not formalism, since it takes these alternative universes as fully real, rejecting only the uniqueness of the mathematical universe. The philosophical view here is that there are various distinct concepts of set, each giv-

ing rise to a corresponding universe of set theory, and that we can fruitfully compare these concepts of set in order to deepen our set-theoretic understanding. Although the multiverse position is fundamentally a philosophical position, an ontological claim about the nature of mathematical existence, and the analysis of the multiverse view will necessarily be a philosophical analysis, it turns out nevertheless that many of the various set concepts that have arisen are closely enough related to one another that they can be fruitfully compared within a single set-theoretic context. For example, we can compare a model of set theory with its version of the constructible universe, its ultrapowers, its ground models and its forcing extensions, and this comparison can be fully formalized within ZFC. Thus, the free-form philosophical idea that there are alternative mathematical universes becomes a purely mathematical enterprise investigating the detailed nature of those comparisons, and it is these mathematical projects I shall discuss in sections 2 and 3. The multiverse philosophical view has both inspired and guided the mathematical research in these areas, both in suggesting the questions and also some solution methods.

Before continuing, let me discuss the method of forcing in somewhat more detail. Forcing (Cohen 1963) is a principal method of building models of set theory, used initially to prove the independence of Axiom of Choice and the Continuum Hypothesis. There was a subsequent explosion of applications, and we now have thousands of forcing constructions and an enormous variety of models of set theory built by forcing. The basic situation with forcing is that one has an initial model V , the ground model, and builds the forcing extension $V[G]$, a larger model, by adjoining a new ideal object G , the generic filter.

$$V \subseteq V[G]$$

The situation is analogous to the field-extension construction in algebra, where one adjoins a new root of a polynomial over the base field, and then closes under the algebraic operations to obtain the field extension. With forcing, the extension $V[G]$ is similarly obtained algebraically, and everything in $V[G]$ is easily constructed from objects in V and the new ideal object G . The forcing extension $V[G]$ is closely related to the ground model V , but exhibits new truths in a way that can be carefully controlled. In the case of the Continuum Hypothesis, Cohen designed a forcing notion so that the generic object G in effect added an enormous number of new real numbers, so many so that the Continuum Hypothesis must fail in $V[G]$.

To be a bit more precise about how forcing works, suppose that V is a model of set theory and $\mathbb{P}$ a partial order in V . Suppose $G \subseteq \mathbb{P}$ is a V -generic filter, meaning that $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ in V . If the initial model is countable, such a G can be built directly, but in the general case, the hypothesis that there is such a filter G can be proved consistent with the elementary diagram of V (relativized to a predicate for the ground model), even when V is a proper class. The forcing

extension $V[G]$ adjoins G to V by closing under elementary set-building operations. Every object in $V[G]$ has a name in V , and is built directly from its name and G . The forcing relation $p \Vdash \varphi$ holds when every V -generic G with $p \in G$ has $V[G] \models \varphi$. The fundamental facts of forcing are (1) The forcing extension $V[G]$ satisfies ZFC; (2) If $V[G] \models \varphi$, then $p \Vdash \varphi$ for some $p \in G$; and (3) The forcing relation $p \Vdash \varphi$ is definable in V .

In the early days of forcing, it was often used to prove relative consistency results. Many early theorems had the form $\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \Psi)$, where Ψ was some particular interesting set-theoretic assertion. The proofs proceeded by starting in a model of set theory and obtaining Ψ in the forcing extension. The contemporary view is that such a proof method provides much more information about the possibility of Ψ than merely the relative consistency implication $\text{Con}(\text{ZFC}) \implies \text{Con}(\text{ZFC} + \Psi)$. Indeed, we now regard the most sophisticated uses of forcing as those which use forcing not only to obtain a certain theory in the forcing extension, but which also analyze in detail the relationship between the two models. How are set-theoretic properties affected by forcing? Let me give an example of this phenomenon.

Theorem 1 (Y. Kakuda 1981, indep. M. Magidor). *Suppose that $V \subseteq V[G]$ is a κ -c.c. forcing extension. An ideal on κ is precipitous in V if and only if the ideal it generates in $V[G]$ is precipitous in $V[G]$.*

Thus, we gain an understanding not just that a certain ideal can be precipitous in $V[G]$, but of exactly how that new ideal is constituted from the ideals we already had.

Let me now turn to the two emerging developments I mentioned earlier, the modal logic of forcing and set-theoretic geology, which exhibit the second-order nature of set theory and engage with various philosophical views on the nature of mathematical existence.

2. Looking upward: the modal logic of forcing

Let us consider now somewhat more abstractly a model of set theory V and its relation to all its forcing extensions $V[G]$, considering at first only extensions by set forcing. It seems very natural to introduce the idea that a statement φ in the language of set theory is *forceable* or *possible* if there is some forcing extension $V[G]$ in which φ is true. This is, of course, a modal possibility notion, so we will write $\Diamond \varphi$ for the assertion that φ is forceable. The natural dual notion is that φ is *necessary*, written $\Box \varphi$, when φ holds in all forcing extensions $V[G]$. There is, of course, a natural Kripke model lurking here, whose possible worlds are the models of set theory and whose accessibility relation is the relation of a model to its forcing extensions. Many set theorists habitually operate within this Kripke model, even if

they would not describe their activities this way, for whenever it is convenient and for whatever purpose they say, “let $G \subseteq \mathbb{P}$ be V -generic,” and make the move to the forcing extension $V[G]$. This amounts to traveling about in this Kripke model.

The modal assertions $\Diamond \varphi$ and $\Box \varphi$ are expressible, of course, in the language of set theory.

$$\begin{aligned}\Diamond \varphi &\iff \exists \mathbb{P} \exists p \in \mathbb{P} \, p \Vdash_{\mathbb{P}} \varphi \\ \Box \varphi &\iff \forall \mathbb{P} \forall p \in \mathbb{P} \, p \Vdash_{\mathbb{P}} \varphi\end{aligned}$$

The fundamental facts of forcing that I mentioned earlier, that everything true in the extension is forced by some condition and that the forcing relation is definable in the ground model, express precisely the sense in which the ground model has access to the truths of the forcing extension. It follows that both $\Diamond \varphi$ and $\Box \varphi$ are expressible in the language of set theory. And while $\Diamond$ and $\Box$ are therefore eliminable, we nevertheless retain them, for we are interested in what principles these operators must obey.

Many common elementary modal assertions, it is easy to see, are valid under this forcing interpretation. To be precise, let me define that a modal assertion $\varphi(p_0, \dots, p_n)$, in the language of propositional modal logic, is a *valid principle of forcing* if for any set-theoretic assertions $\psi_0, \dots, \psi_n$ the corresponding substitution instance $\varphi(\psi_0, \dots, \psi_n)$ holds. For example, it is easy to see that the following elementary modal assertions are valid principles of forcing.

K	$\Box(\varphi \implies \psi) \implies (\Box \varphi \implies \Box \psi)$
Dual	$\Box \neg \varphi \iff \neg \Diamond \varphi$
S	$\Box \varphi \implies \varphi$
4	$\Box \varphi \implies \Box \Box \varphi$
.2	$\Diamond \Box \varphi \implies \Box \Diamond \varphi$

Since these assertions axiomatize the modal theory known as S4.2, it follows that:

Theorem 2. *Every S4.2 modal assertion is a valid principle of forcing.*

The fundamental question is:

Question 3. *What are the valid principles of forcing?*

An answer will be provided by Theorem 4. As a tentative first step, let me mention that it is an enjoyable elementary exercise in forcing, which I encourage the reader to undertake, to show that none of the following modal assertions is a valid principle of forcing in every model of set theory.

5	$\Diamond \Box \varphi \implies \varphi$
M	$\Box \Diamond \varphi \implies \Diamond \Box \varphi$
W5	$\Diamond \Box \varphi \implies (\varphi \implies \Box \varphi)$
.3	$\Diamond \varphi \wedge \Diamond \psi \implies (\Diamond(\varphi \wedge \Diamond \psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond \varphi))$
Dm	$\Box(\Box(\varphi \implies \Box \varphi) \implies \varphi) \implies (\Diamond \Box \varphi \implies \varphi)$
Grz	$\Box(\Box(\varphi \implies \Box \varphi) \implies \varphi) \implies \varphi$
Löb	$\Box(\Box \varphi \implies \varphi) \implies \Box \varphi$
H	$\varphi \implies \Box(\Diamond \varphi \implies \varphi)$

As a hint for this exercise, let me mention that several of the assertions above are invalid in every model of set theory, with $\varphi = \text{CH}$ (or its negation) being a counterexample. The others are invalid in Gödel's constructible universe L (and other models), with counterexamples built from such assertions as $V \neq L$, $\omega_1^L < \omega_1$, CH, or Boolean combinations of these. The axioms above correspond to a hierarchy of modal theories:

The forcing interpretation of the modal operators $\Box \varphi$ and $\Diamond \varphi$ was introduced in [4], in connection with the forcing axiom called the Maximality Principle MP, which was fruitfully cast in these modal terms¹. Specifically, having the concept of a set-theoretic assertion φ being forceable or being necessary, we define that φ is *forceably necessary* if it is forceable that φ is necessary, that is, if we can force φ in such a way that it remains true in all further forcing extensions. The Maximality Principle is the scheme asserting that every forceably necessary statement is already true. In our

¹ See [10] for an earlier independent account, without the modal interpretation, of a version of MP.

modal notation, this is simply the assertion $\Diamond \Box \varphi \implies \varphi$, which happens to be the modal axiom known as S5. Thus, the Maximality Principle is simply the assertion that S5 is a valid principle of forcing. Elementary modal reasoning shows that MP is equivalent, as a scheme, to the scheme asserting that every forceably necessary statement is not only true, but also necessary, expressed by $\Diamond \Box \varphi \implies \Box \varphi$. In [4], it was proved that if there is a model of ZFC, then there is a model of ZFC + MP. Although this was a forcing argument, it involved a certain interesting non-forcing hiccup in the choice of ground model, and it is not true that every model of ZFC has a forcing extension that is a model of MP. Indeed, if ZFC is consistent, then there is a model of ZFC having no extension of any kind with the same ordinals that is a model of MP. The original forcing axioms, from Martin's Axiom onwards, have often been cast and were originally conceived (according to my conversations with Tony Martin) as asserting that a lot of forcing has already occurred. The Maximality Principle makes this idea universal, by asserting that any statement that could be forced necessary is already necessary.

But of course, the Maximality Principle does not hold in all models of set theory, so the question remains: What are the valid principles of forcing? The following theorem, the main theorem of [5], provides an answer.

Theorem 4 (Hamkins, Löwe [5]). *If ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly those in the modal theory S4.2.*

Let me mention a few concepts from the proof. We have already observed above that S4.2 is valid for forcing. The difficult part of the theorem, of course, is to show that there are no other validities. In other words, given $\text{S4.2} \not\vdash \varphi$, we must provide set-theoretic assertions ψ_i such that $\varphi(\psi_0, \dots, \psi_n)$ fails in some model of set theory. To accomplish this, two attractively simple concepts turn out to be key. Specifically, we define that a statement φ of set theory is a *switch* if both φ and $\neg\varphi$ are necessarily possible. Thus, a switch is a statement φ whose truth value can always be turned on or off by further forcing. In contrast, φ is a *button* if φ is (necessarily) possibly necessary. These are the statements that can be forced true in such a way that they remain true in all further forcing extensions. The idea here is that once you push a button, you cannot unpush it. The Maximality Principle, for example, is equivalent to the assertion that every button has already been pushed. Although buttons and switches may appear at first to be very special kinds of statements, it is nevertheless the case in set theory that every statement is either a button, a switch, or the negation of a button. (After all, if you can't always switch φ on and off, then it will either get stuck on or stuck off, and product forcing shows these possibilities to be mutually exclusive.) A family of buttons and switches is *independent*, if the buttons are not yet pushed and (necessarily) each of the buttons and switches can be controlled without affecting the others. For example, the statements $s_m =$ “the

Generalized Continuum Hypothesis (GCH) holds at $\aleph_{\omega+m}$ ” (for $m > 0$) form an independent family of switches, since any combination of such statements is forceable, and if $V = L$, one can augment these with an independent family of buttons.

The proof of Theorem 4 rests in part on a detailed understanding of the modal logic **S4.2** and its complete sets of Kripke frames. A *Kripke model* is a collection of propositional *worlds* (essentially a truth table row, assigning propositional variables to true and false), with an underlying accessibility relation called the *frame*. A statement is possible or necessary at a world, accordingly as it is true in some or all accessible worlds, respectively. Every Kripke model built on a frame that is a directed partial pre-order will satisfy the **S4.2** axioms of modal logic, and in fact the finite directed partial pre-orders are *complete* for **S4.2** in the sense that the statements true in all Kripke models built on such frames are exactly the statements provable from **S4.2**. An improved version of this, proved in [5], is that the finite pre-lattices, and even the finite pre-Boolean algebras, are complete for **S4.2**. The following lemma, a central technical claim at the core of [5], shows that any model of set theory with an independent family of buttons and switches is able to simulate any given Kripke model built on a finite pre-lattice frame.

Lemma 5. *If $W \models \text{ZFC}$ has sufficient independent buttons and switches, then for any Kripke model M on a finite pre-lattice frame, any $w \in M$, there is a translation of the propositional variables $p_i \mapsto \psi_i$ to set-theoretic assertions ψ_i , such that for any modal assertion $\varphi(p_1, \dots, p_n)$:*

$$(M, w) \models \varphi(p_1, \dots, p_n) \iff W \models \varphi(\psi_1, \dots, \psi_n).$$

Each ψ_i is a Boolean combination of the buttons and switches.

Consequently, if **S4.2** $\not\models \varphi$, then since we proved that there is a Kripke model M built on a finite pre-lattice frame in which φ fails, it follows that in any model of set theory W having independent buttons and switches, which we proved exist, the corresponding assertion $\varphi(\psi_1, \dots, \psi_n)$ fails. This exactly shows that φ is not a provably valid principle of forcing, as desired to prove Theorem 4. The proof is effective in the sense that if **S4.2** $\not\models \varphi$, then we are able explicitly to provide a model $W \models \text{ZFC}$ and the particular set-theoretic substitution instance $\varphi(\psi_1, \dots, \psi_n)$ which fails in W .

Although Theorem 4 tells us what are the ZFC-provably valid principles of forcing, it does not tell us that all models of ZFC exhibit only those validities. Indeed, we know that this isn’t the case, because we know there are models of the Maximality Principle, for which the modal theory **S5** is valid, and this is strictly stronger. So different models of set theory may exhibit different valid principles of forcing. For any $W \models \text{ZFC}$, consider the family Force^W of modal assertions φ that are valid for forcing over W . The proof of Theorem 4 can be adapted to show that

Theorem 6 (Hamkins, Löwe [5]). *If $W \models \text{ZFC}$, then $\text{S4.2} \subseteq \text{Force}^W \subseteq \text{S5}$.*

Furthermore, both of these endpoints occur, and so the theorem is optimal. Specifically, if W is a model of $V = L$, then $\text{Force}^W = \text{S4.2}$, and if W satisfies the Maximality Principle, then $\text{Force}^W = \text{S5}$.

Questions 7.

1. *Is there a model of ZFC whose valid principles of forcing form a theory other than S4.2 or S5?*
2. *If φ is valid in W , is it valid in all extensions of W ?*
3. *Equivalently, is Force^W normal?*
4. *Can a model of ZFC have an unpushed button, but not two independent buttons?*

The validity for forcing of many modal axioms can be re-cast in purely set-theoretic terms, in the button-and-switch manner. For example, a model $W \models \text{ZFC}$ has no unpushed buttons if and only if $\text{Force}^W = \text{S5}$, and W has independent buttons and switches if and only if $\text{Force}^W = \text{S4.2}$. Moving beyond this, if W has two semi-independent buttons (meaning that the first can be pushed without pushing the second), then W5 is invalid in W ; If W has two independent buttons, then .3 is invalid in W ; If W has an independent button and switch, then Dm is invalid in W ; And if W has long ratchets (sequences of buttons, such that each can be pushed without pushing the next and pushing any of them necessarily pushes all earlier buttons), also known as volume controls, then $\text{Force}^W \subseteq \text{S4.3}$.

When parameters are allowed into the scheme, large cardinals make a surprising entrance. By $\text{S5}(\mathbb{R})$ and similar notation below, we mean the scheme consisting of any S5 assertion using real parameters. For example, $\text{S5}(\mathbb{R})$ includes all statements of the form $\forall x \in \mathbb{R} (\Diamond \Box \varphi(x) \implies \Box \varphi(x))$.

Theorem 8. *The following are equiconsistent:*

1. $\text{S5}(\mathbb{R})$ is valid for forcing.
2. $\text{S4W5}(\mathbb{R})$ is valid for forcing.
3. $\text{Dm}(\mathbb{R})$ is valid for forcing.
4. *There is a stationary proper class of inaccessible cardinals.*

Theorem 9.

1. (Welch, Woodin) *If $\text{S5}(\mathbb{R})$ is valid in all forcing extensions (using the $\mathbb{R}$ of the extension), then $\text{AD}^{L(\mathbb{R})}$.*
2. (Woodin) *If $\text{AD}_{\mathbb{R}} + \Theta$ is regular, then it is consistent with ZFC that $\text{S5}(\mathbb{R})$ is valid in all forcing extensions.*

There are many directions for future work in this area. In addition to the questions above, it is natural to restrict the class of forcing to ccc forcing, or proper forcing

or any other natural class of forcing.

Questions 10. *What are the valid modal principles of ccc forcing? Of proper forcing? Of class forcing? Of arbitrary extensions?*

Class forcing and arbitrary extensions involve the meta-mathematical complication that the corresponding possibility and necessitation operators are no longer first-order expressible. The work on Question 10 has been surprisingly difficult, even for what we expected would be the easier cases, and has led to some interesting, subtle questions in forcing combinatorics. For example, the question of whether there must be switches in the modal logic of collapse forcing (the class of all forcing $\text{Coll}(\omega, \delta)$ to collapse a cardinal δ to ω using finite conditions, and more generally also the Lévy collapse $\text{Coll}(\omega, <\delta)$) leads directly to the following question:

Question 11. *Can there be a model of set theory V that is elementarily equivalent to $V[G]$, whenever G is V -generic for the collapse of a cardinal δ to ω ?*

Such a model of set theory would be an extreme counterexample in having no switches at all for the class of collapse forcing, and would have valid principles of collapse forcing that are beyond S5, a hard upper bound for the other natural classes of forcing. Mitchell and Welch have given lower bounds with large values of $o(\kappa)$, but for the upper bound, an early suggestion of Mitchell to perform Radin forcing over a model of $o(\kappa) = \kappa^+$ has reportedly not worked out as hoped.

3. Looking downward: set-theoretic geology

The second topic I would like to discuss is what we have called set-theoretic geology. Forcing is ordinarily viewed as a method of constructing *outer* as opposed to *inner* models of set theory, for with forcing, one usually begins with a ground model V and builds the forcing extension $V[G]$ by adjoining G and constructing relative to V . Nevertheless, a simple switch in perspective allows us to use forcing to describe inner models as well. The idea is simply to consider forcing from the perspective of the forcing extension rather than the ground model and to look downward from the universe V for how it may have arisen by forcing. Given the set-theoretic universe V , we search for the possible ground models $W \subseteq V$ such that there is a W -generic filter $G \subseteq \mathbb{P} \in W$ such that $V = W[G]$. Such a perspective quickly leads one to look for deeper and deeper grounds, burrowing down to what we call bedrock models and deeper still, to what we call the mantle and the outer core. In this way, one arrives at set-theoretic geology. The topic is introduced in [1], which gives the initial results and numerous open questions, and the material here is adapted from that article.

Haim Gaifman has pointed out (with humor) that our terminology implicitly

takes a position on the philosophical question of whether forcing and the construction of models of set theory is a human activity or a natural one. If one understands the construction of models of set theory as a human activity, then the subject might be called set-theoretic *archeology*, for in this case we would be burrowing down to discover ancient human set-theoretic civilizations.

The topic of set-theoretic geology rests fundamentally on the following theorem, a shockingly recent result, considering the fundamental nature of the question it answers. Laver's proof of this theorem builds on work of mine [3] concerning the approximation and covering properties.

Theorem 12 (Laver [7], independently Woodin [11]). *Every model of set theory $V \models \text{ZFC}$ is a definable class in all of its set forcing extensions $V[G]$, using parameters in V .*

This theorem led Jonas Reitz and me to introduce the following hypothesis, which we take to be the beginning of set-theoretic geology.

Definition 13 (Hamkins, Reitz). The Ground Axiom GA is the assertion that the universe is not obtained by nontrivial set forcing over any inner model.

Although this assertion may appear at first to be second order, because of the quantification over ground models, in fact the Ground Axiom is expressible by a first order statement in the language of set theory.

Theorem 14 (Reitz [8, 9]). *The Ground Axiom is first order expressible in set theory.*

The Ground Axiom holds in many canonical models of set theory, such as L , $L[0^\#]$, $L[\mu]$ and many instances of K . Since these models all exhibit many highly regular structural features, it is very natural to inquire: To what extent are these regularity features consequences of the Ground Axiom? The answer, which Reitz provided in his dissertation, is that every model of ZFC has a class forcing extension, preserving any desired initial segment V_α (and mild in the sense that every new set is generic for set forcing), which is a model of the Ground Axiom. Thus, the Ground Axiom does not imply any of the usual combinatorial set-theoretic regularity features $\diamond$, GCH and so on. Reitz's method obtained the Ground Axiom by forcing very strong versions of $V = \text{HOD}$, and so his analysis did not settle the question of whether $\text{GA} \implies V = \text{HOD}$. In a three-generation collaboration, we settled that question with the following:

Theorem 15 (Hamkins, Reitz, Woodin [6]). *Every model of set theory has an extension which is a model of GA plus $V \neq \text{HOD}$.*

After some preparatory forcing, we use a class Silver iteration adding a Cohen

subset to every regular cardinal. The argument is flexible and robust, and leads us to expect the Ground Axiom after most any Easton support progressively closed class iteration.

Let me set some terminology. A transitive class W is a *ground* of V if $W \models \text{ZFC}$ and $V = W[G]$ is a forcing extension of W by set forcing $G \subseteq \mathbb{P} \in W$. The model W is a *bedrock* of V if it is a ground of V and there is no deeper ground inside W . Equivalently, W is a bedrock of V if it is a ground of V and satisfies the Ground Axiom.

Theorem 16 (Reitz [8]). *If there is a model of ZFC, then there is a model of ZFC having no bedrock.*

We don't know if a model can have more than one bedrock model.

Question 17. *Is the bedrock unique when it exists?*

The principal new concept is the following:

Definition 18. The *Mantle* M of V is the intersection of all grounds of V .

The Mantle is a first-order parameter-free definable transitive class containing all ordinals. Much of this is easy to see, once one realizes that there is a broad uniformity in the definition of the ground model in its forcing extensions. The basic situation is described by the following.

Theorem 19. *There is a parameterized family $\{W_r \mid r \in V\}$ of transitive classes such that*

1. *Every W_r is a ground of V and $r \in W_r$.*
2. *Every ground of V is W_r for some r .*
3. *The relation " $x \in W_r$ " is first order.*

The parameterized family $\{W_r \mid r \in V\}$ of grounds in Theorem 19 reduces second order properties about grounds to first order properties about their parameters in this family. For example, the Ground Axiom holds if and only if $\forall r W_r = V$. The model W_r is a bedrock if and only if $\forall s (W_s \subseteq W_r \implies W_s = W_r)$. The Mantle is defined by $M = \{x \mid \forall r (x \in W_r)\}$. Because of Theorem 19, each of these assertions is first order expressible in the language of set theory. The proof of Theorem 19 relies, of course, on the proof of Theorem 12, which in turn relied on some of my work in [3] concerning the approximation and cover properties.

When looking downward at the various grounds, it is very natural to inquire whether one can fruitfully intersect them. Let us define that the grounds are *downward directed* if for every r and s there is t such that $W_t \subseteq W_r \cap W_s$. The grounds are *downward set-directed* if for every A there is t with $W_t \subseteq \bigcap_{r \in A} W_r$. The question

of whether there can be distinct bedrock models in the universe is of course related to the question of whether there is a ground in their intersection:

Question 20. *Are the grounds downward directed? Are they downward set-directed?*

In every model for which we can determine the answer to these questions, the answers are yes. The importance of the questions is that in the situations where the answers are yes, the Mantle is well behaved.

Theorem 21.

1. *If the grounds are downward directed, then the Mantle is constant across the grounds, and $M \models \text{ZF}$.*
2. *If the grounds are downward set-directed, then $M \models \text{ZFC}$.*

We don't know whether the Mantle must always satisfy ZF or ZFC, and this is a current focus of study. It is natural, of course, to consider how the Mantle is affected by forcing. Since every ground model of V is a ground model of any forcing extension $V[G]$, it follows that the Mantle of $V[G]$ is contained within the Mantle of V . That is, the Mantle gets smaller (or at least no larger) as you perform more and more forcing. In the limit of this process, we arrive at:

Definition 22. The *generic mantle* of a model of set theory V is the intersection of all ground models of all set forcing extensions of V .

We will use the notation M to denote the Mantle and gM to denote the generic Mantle. The generic Mantle has proved in several ways to be a more robust version of the Mantle (although in truth we do not know them to differ). For example, for any model of ZFC, the generic Mantle is always a model at least of ZF, without any need for a directedness hypothesis. If the *generic grounds*, the ground models of the forcing extensions of V , are downward directed, then in fact the Mantle and the generic Mantle are the same. If the generic grounds are downward set directed, then the generic Mantle is a model of ZFC.

Question 23. *Can the mantle and the generic mantle differ?*

If one were to prove the downward-directedness hypothesis for grounds, then the answer would be no, and in our view this result would make the overall theory here very compelling. Alternatively, on the other side, one might hope to prove even that every model V of ZFC is the generic mantle of a model W of the Ground Axiom, so that the mantle of W is W , and this would provide a very attractive answer to Question 23.

Set-theoretic geology is naturally carried out in a context that includes all the forcing extensions of a model of set theory, all the grounds of these extensions, all

forcing extensions of these resulting grounds and so on. The *generic multiverse* of a model of set theory, introduced by Woodin [11], is the smallest family of models of set theory containing that model and closed under both forcing extensions and grounds. There are numerous philosophical motivations to study the generic multiverse. Indeed, Woodin introduced it specifically in order to criticize a certain multiverse view of truth, namely, truth as true in every model of the generic multiverse. Although I do not hold such a view of truth, nevertheless I want to investigate the fundamental features of the generic multiverse, a task I place at the foundation of any deep understanding of forcing. Surely the generic multiverse is the most natural and illuminating background context for the project of set-theoretic geology.

The generic Mantle gM is a parameter-free uniformly definable class, invariant by forcing, containing all ordinals and $\text{gM} \models \text{ZF}$. Because it is invariant by forcing, it follows that the generic Mantle gM is constant across the multiverse, and in fact, it follows that the generic Mantle gM is the intersection of the generic multiverse. On this view, the generic Mantle is a canonical, fundamental feature of the generic multiverse, deserving of intense study.

The class HOD is the class of hereditarily ordinal definable sets, the sets that are definable using ordinal parameters, and all their members are definable using ordinal parameters, and so on. Introduced classically, HOD is intensely studied, and known to be a transitive inner model of ZFC , containing all ordinals. Let me now define the *generic* HOD to be the intersection of all HOD s of all the forcing extensions.

$$\text{gHOD} = \bigcap_G \text{HOD}^{V[G]}$$

The generic HOD was originally introduced by Fuchs in an attempt to identify a very large canonical forcing-invariant class.

Theorem 24.

1. gHOD is constant across the generic multiverse.
2. The HOD s of all forcing extensions are downward set-directed.
3. Consequently, gHOD is locally realized and $\text{gHOD} \models \text{ZFC}$.
4. The following inclusions hold.

$$\begin{array}{c} \text{HOD} \\ \cup \\ \text{gHOD} \subseteq \text{gM} \subseteq M \end{array}$$

Question 25. *To what extent can we control and separate these classes?*

For the remainder of this article, I will sketch several answers to this question. First, we can control the classes to keep them all low.

Theorem 26 (Fuchs, Hamkins, Reitz). *If $V \models \text{ZFC}$, then there is a class forcing extension $V[G]$ in which*

$$V = M^{V[G]} = \text{gM}^{V[G]} = \text{gHOD}^{V[G]} = \text{HOD}^{V[G]}$$

In particular, as mentioned earlier, every model of ZFC is the mantle and generic mantle of another model of ZFC. It follows that we cannot expect to prove any regularity features about the mantle or the generic mantle, beyond what can be proved about an arbitrary model of ZFC. It also follows that the mantle of V is not necessarily a ground model of V , even when it is a model of ZFC. One can therefore iteratively take the mantle of the mantle and so on, and we have proved that this process can strictly continue. Indeed, by iteratively computing the Mantle of the Mantle and so on, what we call the *inner* Mantles, we might eventually arrive at a model of ZFC plus the Ground Axiom, where the process would naturally terminate. If this should occur, then we call this termination model the *outer core* of the original model. Generalizing the theorem above, Fuchs, Reitz and I have conjectured that every model of ZFC is the α^{th} inner Mantle of another model of ZFC, for arbitrary ordinals α (or even $\alpha = \text{ORD}$ or beyond).

There is an interesting philosophical view related to and perhaps refuted by this conjecture, namely, the philosophical view I call *Ancient Paradise*, holding that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If only we could sweep this accumulated material away, the view holds, then we should find an ancient paradise. The Mantle, of course, wipes away an entire strata of forcing, and the iterated inner Mantles sweep away additional layers. So the philosophical view would lead us to believe that in this way we would be getting close to a highly regular core. If the conjecture is correct, however, then what we should expect to find after sweeping such layers away even ORD many times is, not a highly regular ancient paradise, but rather something ordinary: an arbitrary model of set theory.

Let me sketch a central idea from the proof of Theorem 26. The initial idea goes back to McAloon (1971), who explained how to make sets definable by forcing. For an easy warm-up case, consider an arbitrary real $x \subseteq \omega$. This real may not happen to be definable in V , but it is an elementary exercise in product forcing to force the GCH to hold at $\aleph_n$ exactly when $x(n) = 1$. In the resulting forcing extension $V[G]$, therefore, the original real x is definable, without parameters. In a similar way, any set whatsoever can become definable in a forcing extension, and the proof of Theorem 26 proceeds by coding every ground model set in this way.

In our second answer to Question 25, we keep the Mantles low, while allowing HOD to inflate.

Theorem 27 (Fuchs, Hamkins, Reitz). *If $V \models \text{ZFC}$, then there is a class forcing extension $V[G]$ in which*

$$V = M^{V[G]} = \text{gM}^{V[G]} = \text{gHOD}^{V[G]} \quad \text{but} \quad \text{HOD}^{V[G]} = V[G]$$

For this theorem, our strategy is to balance the forces on M , gM , gHOD and HOD . We perform proper class forcing to $V[G]$ in such a way that every set in V will be coded unboundedly into the GCH pattern, and we also ensure that G itself is definable, but not so robustly, so that the gHOD will fall back down to V . Specifically, the proof uses many instances of *self-encoding forcing*, the set-sized forcing that adds a generic set, and then proceeds to code this set into the continuum function, and then codes these resulting new sets and so on, closing the iteration off by the next Beth fixed point. In the resulting extension, the generic filter becomes definable, but further forcing to collapse cardinals to ω effectively erases the coding, throwing those sets again out of HOD , and hence they were not in gHOD originally.

In a third possibility, we keep the HODs low, while allowing the Mantle to inflate, seeking $V[G]$ with $V = \text{HOD}^{V[G]} = \text{gHOD}^{V[G]}$ but $M^{V[G]} = V[G]$. Such a model $V[G]$ will of course be a model of the Ground Axiom plus $V \neq \text{HOD}$. By modifying the proof of the three-generation theorem (Theorem 15), we are able to obtain:

Theorem 28 (Fuchs, Hamkins, Reitz). *If $V \models \text{ZFC}$, then there is a class forcing extension $V[G]$ in which*

$$V = \text{HOD}^{V[G]} = \text{gHOD}^{V[G]} \quad \text{but} \quad M^{V[G]} = V[G]$$

We have not yet been able to compute the generic Mantle of this model. Our last combination is to push both the Mantles and the HODs high.

Theorem 29 (Fuchs, Hamkins, Reitz). *If $V \models \text{ZFC}$, then there is a class forcing extension $V[G]$ in which*

$$V[G] = \text{HOD}^{V[G]} = \text{gHOD}^{V[G]} = M^{V[G]} = \text{gM}^{V[G]}$$

This theorem is proved by Reitz's method of forcing every set to be coded into the GCH pattern. I would like to emphasize that in none of our theorems have we managed to separate the generic Mantle from either the Mantle or the generic HOD. We know that $\text{gHOD} \subseteq \text{gM} \subseteq M$, and we have separated the generic HOD from the Mantle in Theorem 28, so the model of this theorem does perform at least one of the desired separations, but as we have not been able to compute the generic Mantle of that model, we don't know which separation has occurred. Thus, I reiterate Question 23 in part, in the dual formulation.

Question 30. *Is the generic Mantle the same as the Mantle? Is the generic Mantle*

the same as the generic HOD?

Of course, not both answers can be yes. Let me close the article by inviting all researchers to attack this open question and the others I have mentioned. The research topics here are young and ripe for progress.

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