Rudin’s Lemma and Reverse Mathematics†

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Abstract

Domain theory formalizes the intuitive ideas of approximation and convergence in a very general way, and provides a fundamental tool in the study of computing theory and computability theory. In search of complete lattices on which the Laswon topology is Hausdorff, Gierz, Lawson and Stralka introduced in [3] quasicontinuous lattices, which inherit many good properties of domains. Gierz, et al. pointed in [3] that Rudin’s Lemma for finding a “cross-section” of certain descending family of sets plays a central role in the development of the whole theory of quasicontinuous lattices. In this paper, we study Rudin’s Lemma from reverse mathematics point of view and prove that the Rudin’s Lemma is equivalent to ACA₀ over RCA₀.

1. Introduction

Domain theory formalizes the intuitive ideas of approximation and convergence in a very general way. It provides a fundamental tool in the study of computing theory and computability theory. The idea was proposed independently by Dana Scott in paper [6] and Yuri Ershov in paper [1] around 1970s. Domain theory studies various topologies on partially ordered sets, such as Alexandrov topology, Scott topology, and Lawson topology, etc. A domain refers to a continuous dcpo (directly complete partial orders).

In their work of generalizing the theory of continuous dcpos to general ordered structures, and also in search of complete lattices on which the Laswon topology is Hausdorff, Gierz, Lawson and Stralka introduced in [3] quasicontinuous lattices,

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where many good properties of domains remain valid. In this theory, one of the milestone results is Rudin’s Lemma, a “lemma” enables us to find a “cross-section” of certain descending family of sets.

**Rudin’s Lemma**[2]: Given a poset $(P, \preceq)$, and $\{F_i\}_{i \in I}$ a $\sqsubseteq$-directed family of nonempty finite subsets of $P$, there is a $\preceq$-directed subset $D \subseteq \bigcup_{i \in I} F_i$ that meets all $F_i$.

Rudin’s Lemma implies that the Scott topology on quasicontinuous domain is locally compact and sober, and Lawson topology on quasicontinuous domain is regular and Hausdorff.

In this paper, we study Rudin’s Lemma from reverse mathematics point of view and prove that the Rudin’s Lemma is equivalent to ACA$_0$ over RCA$_0$.

We organize the paper as follows. In Section 2, we introduce basic definitions about domain theory and reverse mathematics. We will show in this section that RCA$_0$ can prove the equivalence between chain complete and directed complete. In Section 3, we prove the equivalence between Rudin’s Lemma and ACA$_0$ over RCA$_0$. In Section 4, we study a stronger version of Rudin’s Lemma, and prove that this strengthened version is equivalent to ACA$_0$ over RCA$_0$.

### 2. Preliminaries

Let $(X, \preceq)$ be a partially ordered set, and $\mathcal{P}_f(X)$ be the collection of all finite subsets of X. For $A \subseteq X$, denote $\uparrow A = \{x \in X : (\exists a \in A)[a \preceq x]\}$, the upper set of A.

**Definition 1.** For $A, B \in \mathcal{P}_f(X)$, define

$$A \sqsubseteq B \iff \uparrow B \subseteq \uparrow A.$$  

$\sqsubseteq$ on $\mathcal{P}_f(X)$ is called the Smyth preorder.

It is obvious that $A \sqsubseteq B \iff (\forall b \in B) (\exists a \in A) [a \preceq b]$.

**Definition 2.** For an ordered set $(X, \preceq)$, a subset $D$ of $X$ is directed if $\forall x, y \in D$, there exists $z \in D$ such that $x, y \preceq z$.

So, in Rudin’s Lemma , “$\{F_i\}_{i \in I}$ is a $\sqsubseteq$-directed family” means for all $F_i$ and $F_j$, there exists $F_k$ such that $F_i, F_j \sqsubseteq F_k$.

**Definition 3.** For an ordered set $(X, \preceq)$,

1. $X$ is directed complete, if for all directed subsets $D \subseteq X$, the least upper bound of $D$ exists in $X$. 

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2. \(X\) is chain complete, if for all chains \(\{x_i\}_{i \in \Lambda} \subseteq X\), the least upper bound of the chain exists in \(X\).

In [5], Markowsky proved that assuming well-ordering principle, directed completeness and chain-completeness are equivalent. We will have a look at this equivalence from reverse mathematics point of view and show that this it can be proved in RCA\(_0\).

Reverse mathematics is a program devoted to determine the strength of mathematical theorems by calibrating the precise set existence axioms necessary and sufficient to carry out their proofs in second-order arithmetic.

**Definition 4.** (Simpson [7]) The axioms of second order arithmetic consist of the universal closures of the following \(L_2\)-formulas:

(1) basic axioms:
\[
\begin{align*}
n + 1 & \neq 0, \\
m + 1 = n + 1 & \rightarrow m = n, \\
m + 0 & = m, \\
m + (n + 1) & = (m + n) + 1, \\
m \cdot 0 & = 0, \\
m \cdot (n + 1) & = (m \cdot n) + m, \\
\neg m & < 0, \\
m < n + 1 & \iff (m < n \lor m = n).
\end{align*}
\]

(2) induction axiom:
\[
(0 \in X \land \forall n (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n (n \in X).
\]

(3) comprehension scheme:
\[
\exists X \forall n (n \in X \leftrightarrow \phi(n)),
\]
where \(\phi(n)\) is any formula of \(L_2\) in which \(X\) does not occur freely.

(4) By second order arithmetic, denoted as \(\mathbb{Z}_2\), we mean the formal system in the language \(L_2\) which are deducible from those axioms by means of the usual logical axioms and rules of inference.

By a subsystem of \(\mathbb{Z}_2\), we mean a formal system in the language \(L_2\) each of whose axioms is a theorem of \(\mathbb{Z}_2\). In this paper, we will be mainly focused on two subsystems: RCA\(_0\) and ACA\(_0\).

**Definition 5.**

1. The system RCA\(_0\) consists of basic axioms together with schemes of \(\Sigma^1_1\) induction and \(\Delta^0_1\) comprehension.
2. The system $\text{ACA}_0$ consists of basic axioms and the induction axiom together with the arithmetic comprehension.

$\text{RCA}_0$ is a kind of formalized recursive mathematics, as the development of ordinary mathematics within $\text{RCA}_0$ corresponds to the positive content of “computable mathematics”. In reverse mathematics, $\text{RCA}_0$ also plays a role of a weak base theory on which many equivalences between mathematical statements and stronger subsystems are established. Here we prove in $\text{RCA}_0$ two basic facts about partial orders.

**Proposition 1.** ($\text{RCA}_0$) Let $(X, \preceq)$ be a countable poset. Then $X$ is directed complete if and only if $X$ is chain complete.

**Proof.** One direction is trivial. Since any chain must be directed subset, then directed complete implies chain complete.

For another direction, let $X$ be chain complete, $D \subseteq X$ be a directed subset. Let $D = \{d_0, d_1, \ldots, d_n, \ldots\}$. Define a chain $E \subseteq D$ as following:

- $e_0 = d_0$;
- $e_1 \in D$ such that $e_0, d_1 \preceq e_1$;
- $e_2 \in D$ such that $e_1, d_2 \preceq e_2$;
- $\vdots$
- $e_n \in D$ such that $e_{n-1}, d_n \preceq e_n$;
- $\vdots$

$e_n$ above exists because $D$ is directed. The chain $E = \{e_0, e_1, \ldots, e_n, \ldots\}$ exists since $d_n \in E \iff \exists m \leq n[d_n = e_m]$.

It is easy to check that $\sup D = \sup E$. Since $X$ is chain-complete, $\sup E$ exists, and hence $\sup D$ exists.

The main part of this paper is to prove the equivalence between Rudin’s Lemma and $\text{ACA}_0$. Below is an easy fact about finite directed sets provable within $\text{RCA}_0$.

**Proposition 2.** ($\text{RCA}_0$) Let $(X, \preceq)$ be a countable poset. $A \in \mathcal{P}_f(X)$, then $A$ is directed if and only if $A$ contains a greatest element of $A$.

**Proof.** Suppose $A = \{a_0, a_1, \ldots, a_n\}$. If $A$ is directed, let

$b_1 \in A$ such that $a_0, a_1 \preceq b_1$
b_2 \in A \text{ such that } b_1, a_2 \preceq b_2
\vdots
b_n \in A \text{ such that } b_{n-1}, a_n \preceq b_n

so b_n \in A \text{ is the greatest element of } A.

For the other direction, if } A \text{ contains a greatest element } b \text{ of } A, \text{ then certainly any two elements of } A \text{ are less than or equal to } b, \text{ and hence } A \text{ is directed.} \tag*{□}

In Proposition 2, the finiteness of } A \text{ is necessary for the forward direction. A simple counterexample is to let } X \text{ be } \mathbb{N} \text{ under inclusion, } A \text{ be the set of all finite subsets of } \mathbb{N}, \text{ then } A \text{ is directed but has no greatest element.}

The following are well-known facts about ACA_0.

**Proposition 3.** (Simpson [7]) The following are equivalent over RCA_0:

1. ACA_0;
2. For all one to one function } f : \mathbb{N} \to \mathbb{N}, \text{ there exists a set } X \subseteq \mathbb{N} \text{ such that}
   \[ \forall n (n \in X \iff \exists m (f(m) = n)); \]
3. König lemma: Every infinite, finitely branching tree } T \subseteq \mathbb{N}^{\mathbb{N}} \text{ has at least one infinite path.}

3. Rudin’s Lemma and ACA_0

In this section, we show the equivalence between Rudin’s lemma and ACA_0 over RCA_0.

**Theorem 1.** The following are equivalent over RCA_0:

1. ACA_0;
2. Rudin’s Lemma: Given a countable poset } (X, \preceq), \text{ and } \mathcal{F} = \langle F_i : i \in \mathbb{N} \rangle \text{ is a } \subseteq\text{-directed family of nonempty finite subsets of } X, \text{ then there exists a } \preceq\text{-directed subset } D \subseteq \bigcup_{i \in \mathbb{N}} F_i \text{ that meets all } F_i.

**Proof.** We first prove Rudin’s Lemma within ACA_0. Let } X \subseteq \mathbb{N}. \text{ We define a function } f : \mathbb{N} \to \mathbb{N} \text{ as following:

\begin{align*}
f(0) &= 0, \\
f(n+1) &= \mu k [F_{n+1} \subseteq F_k, \ F_{f(n)} \subseteq F_k]
\end{align*}

\text{f is well-defined since } \mathcal{F} \text{ is } \subseteq\text{-directed.}
Now, consider a new family $B = \langle B_i : i \in \mathbb{N} \rangle$ where

\[
\begin{align*}
B_0 &= F_{f(0)}, \\
B_1 &= F_1, \\
B_2 &= F_{f(1)}, \\
&\vdots \\
B_{2n} &= F_{f(n)}, \\
B_{2n+1} &= F_{n+1}, \\
&\vdots
\end{align*}
\]

It’s obvious that $B$ is also $\subseteq$-directed and contains all $F_i$, and that $\bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} F_i$. So it is sufficient to find $\preceq$-directed subset $D \subseteq \bigcup_{i \in \mathbb{N}} B_i$ that meets all $B_i$.

Define a tree $T \subseteq \omega^{<\omega}$ such that

$$\sigma \in T \iff \forall n < lh(\sigma) [\sigma(n) \in B_n].$$

$T$ is infinite, finitely branching tree because every $B_n$ is finite set.

Now, we construct a subtree $S \subseteq T$ such that for any infinite path $g$ of $S$, the nodes on $g$ forms a $\preceq$-directed set, i.e, $\{g(n) : n \in \mathbb{N}\}$ is $\preceq$-directed. $S$ is constructed as follows:

For $\sigma \in T$,

- $\emptyset \in S$;
- if $lh(\sigma) = 1$, then $\sigma \in S$;
- if $lh(\sigma) = 2n (n \geq 1)$, then $\sigma \in S$ $\iff$ $\forall \eta \subset \sigma [\eta \in S]$;
- if $lh(\sigma) = 2n + 1 (n \geq 1)$, then

$$\sigma \in S \iff (\forall \eta \subset \sigma [\eta \in S]) \land [\sigma(2n - 2), \sigma(2n - 1) \preceq \sigma(2n)].$$
From the last case, if $\sigma \in S$ and $lh(\sigma) > 2k$, then we have

$$\sigma(0), \sigma(1) \preceq \sigma(2),$$

$$\sigma(2), \sigma(3) \preceq \sigma(4),$$

$$\vdots$$

$$\sigma(2k-2), \sigma(2k-1) \preceq \sigma(2k),$$

so $\sigma(2k)$ is the greatest element among $\{\sigma(n) : n \leq 2k\}$, it implies $\{\sigma(n) : n \leq 2k\}$ is $\preceq$-directed by proposition 2.

Obviously, $S$ is a finitely branching tree. We have the following claims for $S$:

1. $\forall n \forall x \in B_{2n}$, there exists $\sigma \in S$ such that $lh(\sigma) = 2n + 1$ and $\sigma(2n) = x$.
2. $S$ is infinite tree.
3. For every infinite path $g$ of $S$, the set $\{g(n) : n \in \mathbb{N}\}$ is $\preceq$-directed.

By Claim 2, with König lemma (which is equivalent to ACA$_0$ by proposition 3), $S$ contains at least one infinite path $g$. By Claim 3, with $\Sigma^0_1$-comprehension,

$$D = \{x : \exists n[x = g(n)]\}$$

eexists and forms a $\preceq$-directed set. As $D$ meets all $B_i$, $D$ is the desired set.

We prove Claim 1 by arithmetical induction. Let

$$\phi(n) \iff \forall x[x \in B_{2n} \rightarrow \exists \sigma \in S[lh(\sigma) = 2n + 1 \land \sigma(2n) = x]]$$

It is obvious that $\phi(0)$ is true. Suppose $\phi(k)$ is true for all $k \leq n$. To show $\phi(n+1)$ is true, we consider for any $x \in B_{2n+2} = F_{f(n+1)}$, since $B_{2n+1} = F_{n+1} \subseteq F_{f(n+1)}$.
and $B_{2n} = F_{f(n)} \subseteq F_{f(n+1)}$, there is some $y \in B_{2n+1}$, and $z \in B_{2n}$ such that $y \leq x$ and $z \leq x$. By induction, $\phi(n)$ holds, there is an $\eta \in S$ such that $lh(\eta) = 2n + 1$ and $\eta(2n) = z$. By construction, we have $\eta \sim y \in S$, and hence $\eta \sim y \sim x \in S$. So we choose $\sigma = \eta \sim y \sim x$, and $\phi(n+1)$ holds.

Claim 2 follows directly from Claim 1.

To show Claim 3, suppose that $g$ is an infinite path of $S$. Let

$$D = \{x : \exists n [x = g(n)]\}.$$ 

We will prove $D$ is $\preceq$-directed. For any $x, y \in D$, then exists $l \in \mathbb{N}$ such that $x, y \in \{g(0), g(1), \ldots, g(l)\}$. Wolg, assume $l$ is even, then $g(l)$ is the greatest element among $\{g(0), g(1), \ldots, g(l)\}$, so $x, y \preceq g(l)$ and $D$ is directed.

We now prove the other direction. We assume Rudin’s Lemma, and let $f : \mathbb{N} \to \mathbb{N}$ be a one to one function. Consider $(\mathbb{N}, \preceq)$ with the standard order, and define

$$A_{(i,j)} = \begin{cases} \{2i\}, & \text{if } \exists k \leq j \text{ such that } f(k) = i; \\ \{1\}, & \text{otherwise.} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is a standard coding from $\mathbb{N}^2$ to $\mathbb{N}$. $A_{(i,j)}$ exists by $\Delta^0_1$-comprehension:

$$x \in A_{(i,j)} \iff (x = 2i \land \exists k \leq j[f(k) = i]) \lor (x = 1 \land \forall k \leq j[f(k) \neq i]).$$

$\langle A_{(i,j)} : i, j \in \mathbb{N} \rangle$ is $\preceq$-directed family because any two $A_{(i_1,j_1)}, A_{(i_2,j_2)}$ is $\preceq$-comparable. By Rudin’s Lemma, there exists $D \subseteq \bigcup_{i,j \in \mathbb{N}} A_{(i,j)}$ such that $D$ meets all $A_{(i,j)}$. Let $X = \{i \in \mathbb{N} : 2i \in D\}$: $X$ exists by $\Delta^0_1$-comprehension. Then $\forall n (n \in X \iff \exists m (f(m) = n))$, so ACA$_0$ holds. $\square$

4. Further Research

One may ask the following question: in Rudin’s Lemma, is it possible to find a directed subset $D$ of $\bigcup_{i \in I} F_i$ which intersects each $F_i$ at exactly one point? Heckmann and Keimel pointed out in [4] that in general, we cannot have a positive solution. For example, if the family contains the subsets $\{1\}, \{2\}$ and $\{1, 2\}$, then the directed set $D$ must intersect $\{1, 2\}$ at two points. However, we can give a positive solution to this question if $\{F_i\}_{i \in I}$ is a disjoint family. In the following, we prove that this variant is equivalent to ACA$_0$ over RCA$_0$.

Theorem 2. The following are equivalent over RCA$_0$:

1. ACA$_0$;
2. Given a countable poset $(X, \preceq)$, and $F = \langle F_i : i \in \mathbb{N} \rangle$ a $\preceq$-directed family...
of disjoint nonempty finite subsets of X, then there exists a \( \leq \)-directed subset \( E \subseteq \bigcup_{i \in \mathbb{N}} F_i \) that each \( E \cap F_i \) is a singleton.

**Proof.** We first assume ACA\(_0\). The proof of (2) is quite similar to the proof of Theorem 1, where \( f \) will be constructed. The following construction will ensure that each \( B_i \cap D \) is \( \leq \)-linearly ordered, and following this, we can select the greatest element as the unique witness. First, we define functions \( f: \mathbb{N} \to \mathbb{N} \) and \( g: \mathbb{N} \to \mathbb{N} \) simultaneously as follows:

\[
f(0) = 0, \quad g(0) = 1,
\]

\[
f(1) = \mu k[F_f(0), F_g(0) \sqsubseteq F_k],
\]

\[
g(1) = \mu k[k \notin \{f(0), g(0), f(1)\}],
\]

\[
\vdots
\]

\[
f(n + 1) = \mu k[F_f(n), F_g(n) \sqsubseteq F_k],
\]

\[
g(n + 1) = \mu k[k \notin \{f(0), g(0), \ldots, f(n), g(n), f(n + 1)\}],
\]

\[
\vdots
\]

Now, consider a new family \( B = \langle B_i : i \in \mathbb{N} \rangle \) where

\[
B_0 = F_f(0), \quad B_1 = F_g(0), \quad B_2 = F_f(1), \quad B_3 = F_g(1), \ldots,
\]

\[
B_{2n} = F_f(n), B_{2n+1} = F_g(n), \ldots.
\]

Obviously, \( B \) is also a \( \sqsubseteq \)-directed family and contains all \( F_i \), and \( \bigcup_{i \in \mathbb{N}} B_i = \bigcup_{i \in \mathbb{N}} F_i \).
So it is sufficient to find \( \leq \)-directed subset \( E \subseteq \bigcup_{i \in \mathbb{N}} B_i \) that meets each \( B_i \) at exactly one point, and hence meets each \( F_i \) also at exactly one point.

For \( B \), we have the following properties: for any \( i, j \), either \( B_i = B_j \) or \( B_i \cap B_j = \emptyset \), but for each \( n \), \( B_{2n+1} \neq B_0, B_1, \ldots, B_{2n} \).

We now construct \( T \subseteq \mathbb{N}^{<\mathbb{N}} \) and \( S \subseteq T \) as Theorem 1. The same arguments help us find a directed set \( D \) which meets all \( B_i \). We then construct a set \( E \subseteq D \) such that:

\[
x \in E \iff (x \in D) \land (\forall i \forall y[x, y \in B_i \cap D \to y \preceq x])
\]

which means that we choose the greatest element from \( B_i \cap D \) as the witness. We have the following three claims.

1. For all \( i \), \( B_i \cap D \) is \( \leq \)-linearly ordered. So the greatest element of \( B_i \cap D \) exists in \( B_i \cap D \).
2. \( E \) intersects each \( B_i \) at exactly one point.
3. \( E \) is directed.
For Claim 1, let $D = \{h(n) : n \in \mathbb{N}\}$, where $h$ is an infinite path of tree $S$. Now fix $i$, we consider $B_i \cap D$. Assume $B_i$ repeats, like $B_i = B_{j_1} = B_{j_2} = B_{j_3} = \cdots$ with $i \leq j_1 \leq j_2 \leq j_3 \leq \cdots$, then by the construction, we have all $j_1, j_2, \cdots$ are even number. Hence $h(i) \leq h(j_1) \leq h(j_2) \leq h(j_3) \leq \cdots$. So $B_i \cap D$ is $\preceq$-linearly ordered. Because $B_i$ is finite, the greatest element of $B_i \cap D$ exists.

Claim 2 follows directly from Claim 1 since the greatest element is unique.

For Claim 3, note that for all $x, y \in E$, $x, y \in D$. As $D$ is directed, there exists some $z \in D$ such that $x, y \preceq z$. Assume $z \in B_i$ and $w$ is the greatest element in $B_i \cap D$. Then $z \preceq w$ and $w \in E$, and hence $w$ is an upper bound of $x$ and $y$ in $E$. Thus, $E$ is directed.

We now prove the other direction. Assume $f : \mathbb{N} \to \mathbb{N}$ is a one to one function. Consider $(\mathbb{N}, \preceq)$ with the standard order, and define

$$A_{(i,j)} = \begin{cases} \{5^{i+1}\} & \text{if } f(j) = i, \\ \{2^i \cdot 3^j\} & \text{otherwise.} \end{cases}$$

$A_{(i,j)}$ exists by $\Delta^0_1$-comprehension, and they are disjoint because $f$ is one to one. $\langle A_{(i,j)} : i, j \in \mathbb{N}\rangle$ is also a $\subseteq$-directed family because any two $A_{(i_1,j_1)}, A_{(i_2,j_2)}$ is $\subseteq$-comparable. By (2), there exists $E \subseteq \bigcup_{i,j \in \mathbb{N}} A_{(i,j)}$ such that $E$ meets all $A_{(i,j)}$. Let $X = \{i \in \mathbb{N} : 5^{i+1} \in E\}$, $X$ exists by $\Delta^0_1$-comprehension. Then $\forall n(n \in X \iff \exists m(f(m) = n))$, so ACA$_0$ holds. □

References


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