General Recursive Functions in the Number-Theoretic Formal System

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The purpose of the present paper is to give a definition of general recursive functions formalized in the number-theoretic formal system and to show the calculability of those functions which we shall call simply 'recursive function' in this paper.

1. The number-theoretic formal system

As the *number-theoretic formal system*, in this paper, we use one formalized in Gentzen [2], which will be briefly explained in the following.

1. 1. 'Formula'

As *individual* and *function symbols* we use only 0 (zero) and '(successor function), respectively. Accordingly, by *term* we mean either one of the forms 0, 0', 0'' etc.—they are called *numerals*—or α , α' , α'' etc., where α is an arbitrary free variable. As *predicate symbols* we use =(equals) and any other predicate symbols which are not only decidable ones, as in Gentzen [2], but also undecidable ones, provided that they are defined for all non-negative integers. We use only *formulae* constructed by the ordinary formation rule of formula in the restricted predicate calculus from the above terms, those predicate symbols, the bound variables and the logical symbols: $\neg$ (not), $\&$ (and), $\vee$ (or), $\supset$ (implies), $\forall$ (for all) and $\exists$ (there exists).

When $\forall x \mathfrak{F}(x)$ or $\exists x \mathfrak{F}(x)$ is a formula of such form, $\mathfrak{F}(t)$ means the result of substituting an arbitrary term t for the bound variable x

throughout $\mathfrak{F}(\mathfrak{x})$.

We shall use Greek small letters to represent finite sequences of zero or more terms or bound variables, when we wish to indicate sets of terms or bound variables without naming them individually. For example, if τ stands for a sequence of terms $t_1, t_2, \dots, t_n$ and ξ stands for a sequence of bound variables $x_1, x_2, \dots, x_n$, then $\mathfrak{F}(\tau)$ means $\mathfrak{F}(t_1, t_2, \dots, t_n)$ and $\forall \xi \mathfrak{F}(\xi)$ means $\forall x_1 \forall x_2 \dots \forall x_n \mathfrak{F}(x_1, x_2, \dots, x_n)$ ($n=0, 1, 2, \dots$).

1. 2. 'Sequent'

A *sequent* is a formal expression of the form

$$\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_m \rightarrow \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$$

where $m, n \geq 0$ and $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_m, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$ are arbitrary formulae. The part $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_m$ is called the *antecedent*, and $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$ the *succedent* of the sequent.

We shall use Greek capitals, such as $\Gamma, \Delta, \Theta, \Lambda$ and so on, to represent finite sequences of zero or more formulae with separating formal commas included, when we wish to indicate sequences of formulae in antecedent or succedent without naming the formulae individually.

1. 3. 'Rules of inference'

As *rules of inference* we use ones obtained from those for Gentzen's 'Kalkül *LK*', which are represented as the 'Schlußfiguren-Schemata', by adjoining the following

Rule of inference for mathematical induction:

$$\frac{\mathfrak{F}(a), \Gamma \rightarrow \Theta, \mathfrak{F}(a')}{\mathfrak{F}(0), \Gamma \rightarrow \Theta, \mathfrak{F}(t)}$$

where $\mathfrak{F}(a)$ is an arbitrary formula, t is an arbitrary term, Γ and Θ are arbitrary finite sequences of formulae, a is an arbitrary free variable not contained in Γ and Θ , and $\mathfrak{F}(0)$, $\mathfrak{F}(t)$ and $\mathfrak{F}(a')$ are the results of substituting 0 , t and a' for a throughout $\mathfrak{F}(a)$, respectively.

The *LK*-inferences are divided into the *structural* and the *logical* inferences. The logical inferences constitute introductions of a logical symbol, but sometimes in the antecedent, and sometimes in the succedent. We shall denote, for example, an inference for the 'introduction of $\forall$ in

succedent' briefly by ' $\forall$ -succedent'. The logical inferences for the introductions of $\neg$, $\&$, $\vee$, $\supset$ are called the *propositional* inferences, and of $\forall$ and $\exists$ the *predicate* inferences.

1. 4. 'Proof'

As the (*formal*) *proof*, we use only the Gentzen's 'stammbaumförmige Herleitung' which has one lowermost sequent—the *endsequent*—and some uppermost sequents of the form

$$\mathfrak{D} \rightarrow \mathfrak{D}$$

where $\mathfrak{D}$ is an arbitrary formula.

A *proof* of a sequent is a proof which has the sequent as the endsequent. A sequent is said to be *provable*, if there exists a proof of the sequent.

1. 5. 'Calculi *LK* and *LJ*'

A proof containing no inference for mathematical induction is called an *LK-proof*. A sequent is said to be *LK-provable*, if there exists an *LK-proof* of the sequent.

A sequent not containing more than one formula in the succedent is called an *LJ-sequent*. An *LK-proof* containing only *LJ*-sequents is called an *LJ-proof*. A sequent is said to be *LJ-provable*, if there exists an *LJ-proof* of the sequent, accordingly an *LJ-provable* sequent is an *LJ-sequent*.

Gentzen [1] proved two following theorems, simultaneously formulating the Calculi *LK* and *LJ*.

1. 51. NORMAL FORM THEOREM. *For any LK- or LJ-provable sequent, there exists an LK- or LJ-proof of the sequent which contains no cut, respectively.*

1. 52. EXTENDED NORMAL FORM THEOREM ON *LK*. *For any LK-provable sequent containing only prenex formulae, there exists an LK-proof of the sequent which contains no cut and in which there is an occurrence of a sequent $\mathfrak{S}$ such that no quantifiers occur in $\mathfrak{S}$, and the part of the proof from $\mathfrak{S}$ to the endsequent consists solely of structural and predicate inferences.*

1. 6. The result proved by Gentzen [2] can be stated as follows:

CONSISTENCY THEOREM. *Let Γ_0 be a finite sequence which contains only prenex closed formulae containing no existential quantifiers. If the sequent*

$$\Gamma_0 \rightarrow$$

is provable, then the sequent is *LK*-provable.

2. Normal axiom system

A finite sequence of closed formulae is called an *axiom system*, and each of the closed formulae is called an *axiom* of the axiom system. A formula $\mathfrak{A}$ is said to be *provable from* an axiom system Γ_0 , if the sequent

$$\Gamma_0 \rightarrow \mathfrak{A}$$

is provable. An axiom system is said to be *contradictory*, if there exists a formula $\mathfrak{A}$ such that $\mathfrak{A}$ and $\neg \mathfrak{A}$ are both provable from the axiom system. An axiom system Γ_0 is contradictory, if and only if the sequent

$$\Gamma_0 \rightarrow$$

is provable. An axiom system is said to be *consistent*, if it is not contradictory.

By Γ_a we denote the following axiom system :

$$\forall x(x=x),$$

$$\forall x \forall y(x=y \supset y=x),$$

$$\forall x \forall y \forall z\{(x=y \ \& \ y=z) \supset x=z\},$$

$$\forall x \forall y(x=y \supset x'=y'),$$

$$\forall x \neg(x'=0),$$

$$\forall x \forall y(x'=y' \supset x=y).$$

By Gentzen's Consistency theorem and Extended normal form theorem on *LK*, it can be immediately proved that Γ_a is consistent.

An axiom system Γ_0 is said to be *normal*, if the following conditions are fulfilled :

- 1) Γ_0 contains all of the axioms of Γ_a .
- 2) Each of the axioms of Γ_0 , except the axioms of Γ_a , has the form

$$\forall \xi \mathfrak{P}(\xi) \quad \text{or} \quad \forall \xi(\mathfrak{C}(\xi) \supset \mathfrak{P}(\xi))$$

where $\mathfrak{P}(\xi)$ contains no logical symbols and $\mathfrak{C}(\xi)$ contains no logical symbols except $\&$.

3. Equality axiom

Let $\mathfrak{P}$ be a predicate symbol. Then the formula of the form

$$\forall \xi \forall \eta \{ (\xi = \eta \ \& \ \mathfrak{P}(\xi)) \supset \mathfrak{P}(\eta) \}$$

is called the *equality axiom* for $\mathfrak{P}$, where $\xi = \eta$ means

$$\xi_1 = \eta_1 \ \& \ \xi_2 = \eta_2 \ \& \ \cdots \ \& \ \xi_n = \eta_n,$$

if ξ and η are sequences of n bound variables $\xi_1, \xi_2, \dots, \xi_n$, and $\eta_1, \eta_2, \dots, \eta_n$, respectively.

THEOREM 1. *Let Γ_0 be an axiom system which contains only prenex formulae containing no existential quantifiers and which contains all of the axioms of Γ_a , and $\mathfrak{P}$ be a predicate symbol. If Γ_0 is consistent, then the axiom system*

$$\forall \xi \forall \eta \{ (\xi = \eta \ \& \ \mathfrak{P}(\xi)) \supset \mathfrak{P}(\eta) \}, \Gamma_0$$

is consistent.

PROOF. Let us assume that the latter axiom system is contradictory, i. e. the sequent

$$\forall \xi \forall \eta \{ (\xi = \eta \ \& \ \mathfrak{P}(\xi)) \supset \mathfrak{P}(\eta) \}, \Gamma_0 \rightarrow$$

is provable, then it is *LK*-provable (Consistency theorem), accordingly there exists an *LK*-proof $\mathbf{P}$ of it such that the following conditions are fulfilled (Extended normal form theorem on *LK*):

- 1) $\mathbf{P}$ contains no cut.
- 2) $\mathbf{P}$ contains no free variable.
- 3) In $\mathbf{P}$, there is an occurrence of a sequent $\mathfrak{S}$ such that no quantifiers occur in $\mathfrak{S}$, and the part of $\mathbf{P}$ from $\mathfrak{S}$ to the endsequent consists solely of structural and predicate inferences.

The succedent of $\mathfrak{S}$ is empty, and the formulae contained in the antecedent of $\mathfrak{S}$ are all provable from Γ_0 , then the sequent

$$\Gamma_0 \rightarrow$$

is provable, i. e. Γ_0 is contradictory, q. e. d.

4. Recursive function

In the present formal system each function of n variables defined for

all non-negative integers ($n=0, 1, 2, \dots$), except the successor function, is represented by a predicate $\mathfrak{P}(\xi, \eta)$ of $n+1$ arguments, in which ξ means n independent variables and η means the dependent variable of the function. But for the sake of formulating that function in the formal system, of course, it is necessary that a consistent axiom system from which the formulae

$$\forall \xi \exists \eta \mathfrak{P}(\xi, \eta) \quad \text{and} \quad \forall \xi \forall \eta \forall \delta \{ (\mathfrak{P}(\xi, \eta) \ \& \ \mathfrak{P}(\xi, \delta)) \supset y=z \}$$

are both provable can be founded. The function defined in the above way is nothing but the function denoted by the expression $\eta \mathfrak{P}(\xi, \eta)$ in the usual symbolism.

Let $\mathfrak{P}$ be a predicate symbol of $n+1$ argument places. If there exists a consistent *normal* axiom system from which the formulae

$$\forall \xi \exists \eta \mathfrak{P}(\xi, \eta) \quad \text{and} \quad \forall \xi \forall \eta \forall \delta \{ (\mathfrak{P}(\xi, \eta) \ \& \ \mathfrak{P}(\xi, \delta)) \supset \eta=\delta \}$$

are provable, then the function $\eta \mathfrak{P}(\xi, \eta)$ is called a *recursive function* of n variables.

4. 1. THEOREM 2. Let Γ_0 be a consistent normal axiom system,

$$\Gamma_0 \rightarrow \forall \xi \exists \eta \mathfrak{F}(\xi, \eta)$$

and

$$\Gamma_0 \rightarrow \forall \xi \forall \eta \forall \delta \{ (\mathfrak{F}(\xi, \eta) \ \& \ \mathfrak{F}(\xi, \delta)) \supset \eta=\delta \}$$

be provable sequents, and $\mathfrak{P}$ be a predicate symbol not contained in Γ_0 and $\mathfrak{F}(\xi, \eta)$. If $\mathfrak{F}(\xi, \eta)$ contains no free variables nor logical symbols except $\&$, $\vee$, $\exists$, then there exists a consistent normal axiom system Γ_1 such that Γ_1 contains all of the axioms of Γ_0 and the sequent

$$\Gamma_1 \rightarrow \forall \xi \forall \eta (\mathfrak{P}(\xi, \eta) \supset \mathfrak{F}(\xi, \eta))$$

is provable.

PROOF. We can assume, without loss of generality, that Γ_0 contains all of the equality axioms for the predicate symbols contained in $\mathfrak{F}(\xi, \eta)$ (Theorem 1), and that $\mathfrak{F}(\xi, \eta)$ has the form

$$\exists \eta_1 \mathfrak{G}_1(\xi, \eta, \eta_1) \vee \exists \eta_2 \mathfrak{G}_2(\xi, \eta, \eta_2) \vee \dots \vee \exists \eta_n \mathfrak{G}_n(\xi, \eta, \eta_n)$$

where $\mathfrak{G}_1(\xi, \eta, \eta_2)$, $\mathfrak{G}_2(\xi, \eta, \eta_2)$, $\dots$, $\mathfrak{G}_n(\xi, \eta, \eta_n)$ contain no logical symbols except $\&$. Let Γ_1 be the axiom system obtained from Γ_0 by adjoining the axioms

$$\forall \xi \forall \eta \forall \eta_i (\mathcal{C}_i(\xi, \eta, \eta_i) \supset \mathcal{P}(\xi, \eta)) \quad (i=1, 2, \dots, n)$$

and

$$\forall \xi \forall \eta \forall \delta \{ (\mathcal{P}(\xi, \eta) \ \& \ \mathcal{P}(\xi, \delta)) \supset \eta = \delta \}.$$

Then the axiom system Γ_1 is normal, the sequent

$$\Gamma_1 \rightarrow \forall \xi \forall \eta (\mathcal{P}(\xi, \eta) \supset \mathcal{C}(\xi, \eta))$$

is provable, and Γ_1 is consistent because the axiom system

$$\forall \xi \forall \eta (\mathcal{P}(\xi, \eta) \supset \mathcal{C}(\xi, \eta)), \Gamma_0$$

from which the formulae

$$\forall \xi \forall \eta \forall \eta_i (\mathcal{C}_i(\xi, \eta, \eta_i) \supset \mathcal{P}(\xi, \eta)) \quad (i=1, 2, \dots, n)$$

and

$$\forall \xi \forall \eta \forall \delta \{ (\mathcal{P}(\xi, \eta) \ \& \ \mathcal{P}(\xi, \delta)) \supset \eta = \delta \}$$

are all provable is consistent, q. e. d.

From this theorem we can see that the function obtained by substituting recursive functions for some arguments of a recursive function is recursive.

4. 2. THEOREM 3. Let Γ_0 be a consistent normal axiom system, $\mathcal{P}_1$ and $\mathcal{P}_2$ be predicate symbols, the formulae

$$\forall \xi \mathcal{A} \eta \mathcal{P}_1(\xi, \eta),$$

$$\forall \xi \forall \eta \forall \delta \{ (\mathcal{P}_1(\xi, \eta) \ \& \ \mathcal{P}_1(\xi, \delta)) \supset \eta = \delta \},$$

$$\forall \xi \forall \xi \forall \eta \mathcal{A} u \mathcal{P}_2(\xi, \xi, \eta, u),$$

$$\forall \xi \forall \xi \forall \eta \forall u \forall v \{ (\mathcal{P}_2(\xi, \xi, \eta, u) \ \& \ \mathcal{P}_2(\xi, \xi, \eta, v)) \supset u = v \}$$

be provable from Γ_0 , $\mathcal{P}$ be a predicate symbol which is distinct from $\mathcal{P}_1, \mathcal{P}_2$ and is not contained in Γ_0 , and Γ_1 be the normal axiom system obtained from Γ_0 by adjoining the axioms

$$\forall \xi \forall \eta (\mathcal{P}_1(\xi, \eta) \supset \mathcal{P}(\xi, 0, \eta)),$$

$$\forall \xi \forall \xi \forall \eta \forall u \{ (\mathcal{P}(\xi, \xi, \eta) \ \& \ \mathcal{P}_2(\xi, \xi, \eta, u)) \supset \mathcal{P}(\xi, \xi', u) \},$$

$$\forall \xi \forall \xi \forall \eta \forall \delta \{ (\mathcal{P}(\xi, \xi, \eta) \ \& \ \mathcal{P}(\xi, \xi, \delta)) \supset \eta = \delta \}.$$

Then, Γ_1 is consistent and

$$\Gamma_1 \rightarrow \forall \xi \forall \xi \mathcal{A} \eta \mathcal{P}(\xi, \xi, \eta)$$

is provable (Primitive recursion).

PROOF. The formulae

$$\forall \xi \mathcal{A} \eta \mathcal{P}(\xi, 0, \eta)$$

and

$$\forall \varepsilon (\forall \xi \exists \eta \mathfrak{P}(\xi, \varepsilon, \eta) \supset \forall \xi \exists \eta \mathfrak{P}(\xi, \varepsilon', \eta))$$

are both provable from Γ_1 . Hence

$$\forall \xi \forall \varepsilon \exists \eta \mathfrak{P}(\xi, \varepsilon, \eta)$$

is provable from Γ_1 .

In the following we assume, without loss of generality, that Γ_0 contains the equality axiom for $\mathfrak{P}_2$ (Theorem 1).

Let Γ_1 be contradictory, i. e. the sequent

$$\Gamma_1 \rightarrow$$

be provable, then the sequent is *LK*-provable (Consistency theorem), accordingly there exists an *LK*-proof $\mathbf{P}$ such that the following conditions are fulfilled (Extended normal form theorem on *LK*):

- 1) The endsequent of $\mathbf{P}$ is $\Gamma_1 \rightarrow$.
- 2) $\mathbf{P}$ contains no cut.
- 3) $\mathbf{P}$ contains no free variable.
- 4) In $\mathbf{P}$, there is an occurrence of a sequent $\mathfrak{S}$ such that no quantifiers occur in $\mathfrak{S}$, and the part of $\mathbf{P}$ from $\mathfrak{S}$ to the endsequent consists solely of structural and predicate inferences.

The succedent of $\mathfrak{S}$ is empty.

For any formula $\mathfrak{A}$ containing no quantifiers, we define a formula denoted by $\mathfrak{A}^*$, recursively, as follows:

- 1) If $\mathfrak{Q}$ is a predicate symbol distinct from $\mathfrak{P}$ and τ is a sequence of terms, $(\mathfrak{Q}(\tau))^*$ is $\mathfrak{Q}(\tau)$.
- 2) If t is a term and τ is a sequence of terms, $(\mathfrak{P}(\tau, 0, t))^*$ is $\mathfrak{P}_1(\tau, t)$.
- 3) If a is a free variable, t is a term and τ is a sequence of terms, $(\mathfrak{P}(\tau, a, t))^*$ is $\mathfrak{P}(\tau, a, t)$.
- 4) If ε and t are terms and τ is a sequence of terms, $(\mathfrak{P}(\tau, \varepsilon', t))^*$ is $\exists \eta ((\mathfrak{P}(\tau, \varepsilon, \eta))^* \& \mathfrak{P}_2(\tau, \varepsilon, \eta, t))$.
- 5) If $\mathfrak{A}$ is a formula, $(\neg \mathfrak{A})^*$ is $\neg \mathfrak{A}^*$.
- 6) If $\mathfrak{A}$ and $\mathfrak{B}$ are formulae, $(\mathfrak{A} \& \mathfrak{B})^*$, $(\mathfrak{A} \vee \mathfrak{B})^*$ and $(\mathfrak{A} \supset \mathfrak{B})^*$ are $\mathfrak{A}^* \& \mathfrak{B}^*$, $\mathfrak{A}^* \vee \mathfrak{B}^*$ and $\mathfrak{A}^* \supset \mathfrak{B}^*$, respectively.

Now, let $\mathfrak{S}$ be

$$\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n \rightarrow.$$

Then

$$\mathfrak{A}^*_1, \mathfrak{A}^*_2, \dots, \mathfrak{A}^*_n \rightarrow$$

is provable. On the other hand, the formulae $\mathfrak{A}^*_1, \mathfrak{A}^*_2, \dots, \mathfrak{A}^*_n$ are all provable from Γ_0 . Hence

$$\Gamma_0 \rightarrow$$

is provable, i. e. Γ_0 is contradictory, q. e. d.

4. 3. THEOREM 4. *Let Γ_0 be a consistent normal axiom system which contains the axioms*

$$\forall x \forall u (u = x \supset S(x, 0, u)),$$

$$\forall x \forall y \forall u \forall v \{ (S(x, y, u) \ \& \ v = u') \supset S(x, y', v) \},$$

$$\forall x \forall y \forall u \forall v \{ (S(x, y, v) \ \& \ S(x, y, v)) \supset u = v \}$$

and from Γ_0 the formulae

$$\forall \xi \forall \eta \exists u \mathfrak{P}_0(\xi, \eta, u),$$

$$\forall \xi \forall \eta \forall u \forall v \{ (\mathfrak{P}_0(\xi, \eta, u) \ \& \ \mathfrak{P}_0(\xi, \eta, v)) \supset u = v \},$$

$$\forall \xi \exists \eta \mathfrak{P}_0(\xi, \eta, 0)$$

be provable, where $\mathfrak{P}_0$ is a predicate symbol, and $\mathfrak{P}$ be a predicate symbol which is distinct from $\mathfrak{P}_0$ and is not contained in Γ_0 .

Then there exists an another consistent normal axiom system which contains all of the axioms of Γ_0 , and from which the formulae

$$\forall \xi \exists \eta \mathfrak{P}(\xi, \eta),$$

$$\forall \xi \forall \eta \forall \delta \{ (\mathfrak{P}(\xi, \eta) \ \& \ \mathfrak{P}(\xi, \delta)) \supset \eta = \delta \},$$

$$\forall \xi \forall \eta (\mathfrak{P}(\xi, \eta) \supset \mathfrak{P}_0(\xi, \eta, 0)),$$

$$\forall \xi \forall \eta \forall \delta \{ (\mathfrak{P}(\xi, \eta) \ \& \ \mathfrak{P}_0(\xi, \delta, 0)) \supset \eta \leq \delta \}$$

are provable, where $\eta \leq \delta$ means $\exists x S(x, \eta, \delta)$.

The function $\eta \mathfrak{P}(\xi, \eta)$ defined in this theorem is nothing but the function denoted by the expression $\mu \eta \mathfrak{P}_0(\xi, \eta, 0)$ which means the least η such that $\mathfrak{P}_0(\xi, \eta, 0)$.

PROOF. We can assume, without loss of generality, that Γ_0 contains the following axioms (Theorem 3):

$$\forall x \forall u (u = 0 \supset P(x, 0, u)),$$

$$\forall x \forall y \forall u \forall v \{ (P(x, y, u) \ \& \ S(u, x, v)) \supset P(x, y', v) \},$$

$$\begin{aligned}
& \forall x \forall y \forall u \forall v \{ (P(x, y, u) \ \& \ P(x, y, v)) \supset u=v \}, \\
& \forall y (y=0 \supset \text{Sg}(0, y)), \\
& \forall x \forall y \forall z \{ (\text{Sg}(x, y) \ \& \ z=1) \supset \text{Sg}(x', z) \}, \\
& \forall x \forall y \forall z \{ (\text{Sg}(x, y) \ \& \ \text{Sg}(x, z)) \supset y=z \}.
\end{aligned}$$

Let $\mathfrak{F}(\xi, \eta, \delta, u)$ mean $\mathcal{A}u(\mathfrak{P}_0(\xi, \eta, u) \ \& \ P(\delta, u, u))$, then the formulae

$$\forall \xi \forall \eta \forall \delta \mathcal{A}u \mathfrak{F}(\xi, \eta, \delta, u)$$

and

$$\forall \xi \forall \eta \forall \delta \forall u \forall v \{ (\mathfrak{F}(\xi, \eta, \delta, u) \ \& \ \mathfrak{F}(\xi, \eta, \delta, v)) \supset u=v \}$$

are provable from Γ_0 . Hence, there exists a consistent normal axiom system Γ_1 which contains all of the axioms of Γ_0 and from which the formula

$$\forall \xi \forall \eta \forall \delta \forall u (\mathfrak{P}_1(\xi, \eta, \delta, u) \supset \mathfrak{F}(\xi, \eta, \delta, u))$$

is provable, where $\mathfrak{P}_1$ is a predicate symbol distinct from $\mathfrak{P}$, $\mathfrak{P}_0$ and not contained in Γ_0 (Theorem 2).

Let Γ_2 be the normal axiom system obtained from Γ_1 by adjoining the following axioms:

$$\begin{aligned}
& \forall \xi \forall u (u=1 \supset \mathfrak{P}_2(\xi, 0, u)), \\
& \forall \xi \forall \eta \forall \delta \forall u \{ (\mathfrak{P}_2(\xi, \eta, \delta) \ \& \ \mathfrak{P}_1(\xi, \eta, \delta, u)) \supset \mathfrak{P}_2(\xi, \eta', u) \}, \\
& \forall \xi \forall \eta \forall u \forall v \{ (\mathfrak{P}_2(\xi, \eta, u) \ \& \ \mathfrak{P}_2(\xi, \eta, v)) \supset u=v \},
\end{aligned}$$

where $\mathfrak{P}_2$ is a predicate symbol distinct from $\mathfrak{P}$, $\mathfrak{P}_0$, $\mathfrak{P}_1$ and not contained in Γ_1 . Then Γ_2 is consistent (Theorem 3).

Now, let $\mathfrak{G}(\xi, \eta)$ mean

$$\mathcal{A}u(\text{Sg}(u, 1) \ \& \ \mathfrak{P}_2(\xi, \eta, u) \ \& \ \mathfrak{P}_2(\xi, \eta', 0)),$$

then

$$\begin{aligned}
& \forall \xi \mathcal{A}\eta \mathfrak{G}(\xi, \eta), \\
& \forall \xi \forall \eta \forall \delta \{ (\mathfrak{G}(\xi, \eta) \ \& \ \mathfrak{G}(\xi, \delta)) \supset \eta=\delta \}, \\
& \forall \xi \forall \eta (\mathfrak{G}(\xi, \eta) \supset \mathfrak{P}_0(\xi, \eta, 0)), \\
& \forall \xi \forall \eta \forall \delta \{ (\mathfrak{G}(\xi, \eta) \ \& \ \mathfrak{P}_0(\xi, \delta, 0)) \supset \eta \leq \delta \}
\end{aligned}$$

are all provable from Γ_2 . Hence, by Theorem 2, it is almost clear that there exists a consistent normal axiom system required in the Theorem, q. e. d.

5. Calculability of recursive functions

In the Proof of Theorem 5, we shall use the following Lemmas 1 and 2, the former of which is the immediate consequence from the result of § 1 Maehara [3], and the latter can be simply proved by Gentzen's Normal form theorem on *LJ*.

LEMMA 1. *If the endsequent of an LK-proof containing no $\supset$ -, $\supset$ - nor $\vee$ -succedent is an LJ-sequent, then the endsequent is LJ-provable.*

LEMMA 2. *Let Γ_0 be an axiom system which contains only prenex formulae containing no logical symbol $\vee$ nor $\exists$, and $\exists \eta \mathfrak{F}(\eta)$ be a closed formula of such form. If the sequent*

$$\Gamma_0 \rightarrow \exists \eta \mathfrak{F}(\eta)$$

is LJ-provable, then there exists a numeral n such that the sequent

$$\Gamma_0 \rightarrow \mathfrak{F}(n)$$

is LJ-provable.

THEOREM 5. *Let Γ_0 be a consistent normal axiom system, $\mathfrak{P}$ be a predicate symbol, and the formulae*

$$\forall \xi \exists \eta \mathfrak{P}(\xi, \eta)$$

and

$$\forall \xi \forall \eta \forall \delta \{ (\mathfrak{P}(\xi, \eta) \ \& \ \mathfrak{P}(\xi, \delta)) \supset \eta = \delta \}$$

be provable from Γ_0 . If ν is a sequence of arbitrary numerals, then there exists only one numeral n such that the sequent

$$\Gamma_0 \rightarrow \mathfrak{P}(\nu, n)$$

is LJ-provable.

PROOF. Let ν be a sequence of numerals, then the sequent

$$\Gamma_0 \rightarrow \exists \eta \mathfrak{P}(\nu, \eta)$$

is provable, accordingly the sequent

$$\forall \eta \supset \mathfrak{P}(\nu, \eta), \Gamma_0 \rightarrow$$

is provable. By Gentzen's Consistency theorem, the latter sequent is *LK*-provable, hence so is the former, accordingly there exists an *LK*-proof of the former sequent which contains no cut (Normal form theorem).

Then, that *LK*-proof contains no $\supset$ -, $\supset$ - nor $\vee$ -succedent, accordingly the

endsequent is LJ -provable (Lemma 1). Hence, there exists a numeral n such that the sequent

$$\Gamma_0 \rightarrow \mathfrak{P}(\nu, n)$$

is LJ -provable, because each of the axioms of Γ_0 contains no logical symbol $\vee$ nor $\mathcal{E}$ (Lemma 2).

The uniqueness of such a numeral n , from the consistency of Γ_0 , is almost clear, q. e. d.

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