Non-constructive Proofs of a Metamathematical Theorem Concerning the Consistency of Analysis and its Extension

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In this paper we shall treat the following theorem, and to which we shall give three proofs :

THEOREM. *Let A be a formula in the predicate calculus of 1st order. If A is provable in the type-logic, that is, in the predicate calculus of ω th order, then it is also provable in the predicate calculus of 1st order.*

First, S. Seki obtained a proof of the theorem by applying Gödel's *completeness theorem* [1]. In the next, S. Maehara obtained the second proof by using the so-called *Lindenbaum algebra* [2]. And, T. Nishimura obtained the third proof and an extension of the theorem by utilizing Schütte's idea [3].

This theorem means : If a system of axioms is consistent in the predicate calculus of 1st order, then it is also consistent in the type logic. On the other hand, it is well-known that, in the predicate calculus of 1st order, the natural number theory of some kind is consistent [4]. Therefore, our theorem implies that a considerably large part of analysis is also consistent. But it should be remarked that none of our proofs is finitary.

Prof. G. Takeuti extended Gentzen's *LK* [5] to a type-logic, and called his system *GLC*. He calls the next statement the *fundamental conjecture* of *GLC* [6]:

G. Gentzen's *cut-elimination theorem* holds in *GLC*. That is, if a sequent in *GLC* is provable, then it is also provable without cut.

It should be interesting and important to know whether or not the conjecture is affirmative. But it is remained open up to now.

Our theorem implies that a special case of the conjecture is yes. That is, if a sequent in *LK* is provable in *GLC*, then by our theorem it should be also provable in *LK*. Therefore it should be provable without cut by Gentzen's cut-elimination theorem [5].

This paper is divided into three chapters, each of which contains one of our proofs of the theorem. They are written by S. Seki, S. Maehara and T. Nishimura respectively.

Chapter I. The First Proof (by S. SEKI)

§ 1. *TLK*.

At first, we shall formulate a type-logic in Gentzen's style. The *symbols* which occur in the system are divided into *logical symbols* and *variables*. The logical symbols are the *negation sign* $\neg$, the *implication sign* $\supset$, the *disjunction sign* $\vee$, the *conjunction sign* $\wedge$, the *universal quantifier* $\forall$ and the *existential quantifier* $\exists$.

The variables are divided into *free variables* and *bound variables*. To each of them a non-negative integer is associated and said to be its *type*. Semantically, the variables of type 0, type 1, type 2,... are treated respectively as ranging over a domain of individuals, a domain of predicates, a domain of predicates of predicates and so on.

The free variables of type τ are denoted by

$$a^\tau, b^\tau, \dots, a_1^\tau, a_2^\tau, \dots,$$

and the bound variables of type τ are denoted by

$$x^\tau, y^\tau, \dots, x_1^\tau, x_2^\tau, \dots.$$

In this chapter we shall use only the variables having one argument-place. And we shall use no function-symbols. But we can treat also the case where those symbols occur in the same way.

Among expressions (i. e. finite concatenations of symbols) we distinguish formulas. The simplest, so-called *primitive formulas* are expressions of the form :

$$a^{\tau+1}b^{\tau}.$$

Formulas are built from simpler ones by means of logical symbols. For example, the following expressions are formulas :

$$\forall x^5(a^6x^5 \vee \neg a^6x^5), \quad a^2b^1 \supset \exists y^2(y^2b^1).$$

A formula containing a variable a^{τ} is often denoted by $A(a^{\tau})$. An expression obtained from $A(a^{\tau})$ by replacing a^{τ} by the symbol $*$ is called a *variety of type $\tau+1$* . A free variables of type 0 is called a *variety of type 0*.

Let $B(b^{\tau})$ be a formula. If we replace b^{τ} in $B(b^{\tau})$ by a variety V of type τ in the usual way, then we obtain a formula, which is denoted by $B(V)$.

We shall denote finite sequences of formulas by

$$\Gamma, \Delta, \Theta, \dots$$

These sequences may be empty. Expressions of the form

$$\Gamma \rightarrow \Delta$$

are called *sequents*. *Proof-figures* are defined in the same way as in Gentzen's *LK*. However the *inference-schemata* with respect to the quantifiers should be modified as follows :

$$\begin{array}{cc} \frac{A(V), \Gamma \rightarrow \Delta}{\forall x^{\tau} A(x^{\tau}), \Gamma \rightarrow \Delta} & \frac{\Gamma \rightarrow \Delta, A(a^{\tau})}{\Gamma \rightarrow \Delta, \forall x^{\tau} A(x^{\tau})} \\ \left(\begin{array}{l} V \text{ may be an arbitrary} \\ \text{variety of type } \tau. \end{array} \right) & \left(\begin{array}{l} a^{\tau} \text{ may not be contained} \\ \text{in the lower sequent.} \end{array} \right) \\ \frac{A(a^{\tau}), \Gamma \rightarrow \Delta}{\exists x^{\tau} A(x^{\tau}), \Gamma \rightarrow \Delta} & \frac{\Gamma \rightarrow \Delta, A(V)}{\Gamma \rightarrow \Delta, \exists x^{\tau} A(x^{\tau})} \\ \left(\begin{array}{l} a^{\tau} \text{ may not be contained} \\ \text{in the lower sequent.} \end{array} \right) & \left(\begin{array}{l} V \text{ may be an arbitrary} \\ \text{variety of type } \tau. \end{array} \right) \end{array}$$

We shall call this system of logic *TLK*. Of course, *TLK* is an extension

of Gentzen's *LK*.

§ 2. Logical Truth.

Let U be a non-empty set. We shall define a sequence of sets recursively as follows:

$$\begin{cases} U_0 = U \\ U_{n+1} = 2^{U_n} \text{ (power set)} \end{cases}$$

We shall call the set

$$\bar{U} = \bigcup_{n=0}^{\infty} U_n$$

the *closure* of U , and elements of $\bar{U}$ the *mathematical entities* with respect to U .

Let A be a formula (in *TLK*). We assume that all the free variables contained in A have been arranged as follows:

$$(1) \quad a_1^{\tau_1}, a_2^{\tau_2}, \dots, a_n^{\tau_n}.$$

We consider a system R formed by a non-empty set U , and by a sequence

$$E_1, E_2, \dots, E_n$$

of certain mathematical entities with respect to U , with the same number of terms as the sequence (1). The mathematical nature of each E_i depends on the logical character of the corresponding variable $a_i^{\tau_i}$. Thus, for example, if $\tau_i = 0$, then E_i should be an element of $U_0 = U$. If $\tau_i = 1$, then E_i should be an element of $U_1 = 2^U$, i. e. a subset of U ; and so on. Such a system

$$R = \{ U; E_1, E_2, \dots, E_n \}$$

is called a *realization* of A .

We assume it to be clear under what conditions a formula A is said to *hold in* R . Roughly speaking, this means that A turns to be true if

(α) all the bound variables of type τ occurring in A are assumed to range over the set U_τ ,

(β) the logical symbols are interpreted in the usual way,

(γ) each of $a_i^{\tau_i}$ is understood to denote the corresponding entity E_i .

A formula A is said to be *logically true* if it holds in its every reali-

zation. A sequent: $A_1, \dots, A_r \rightarrow B_1, \dots, B_s$ ($r+s>0$) is said to be *logically true*, if the formula

$$\neg A_1 \vee \dots \vee \neg A_r \vee B_1 \vee \dots \vee B_s$$

is logically true. The sequent $\rightarrow$ is said to be not logically true.

An important property of the notion of logical truth is stated in the following well-known theorem:

GÖDEL'S COMPLETENESS THEOREM. *Let S be a sequent in LK . Then, S is provable in LK , if and only if it is logically true.*

§ 3. PROOF OF THE THEOREM.

Now we shall prove the theorem, which is stated in the preface of this paper.

In this section, we shall formulate the theorem as follows:

THEOREM. *If a sequent in LK is provable in TLK , then it is also provable in LK .*

To prove this theorem, it is sufficient to prove the following

LEMMA. *If a sequent is provable in TLK , then it is logically true.*

Assume that the lemma has been proved, and assume that a sequent S in LK is provable in TLK . Then, by the lemma, it should be logically true. Hence, by Gödel's completeness theorem, it should be provable in LK . The proof of the theorem is thus completed.

PROOF OF LEMMA. For a proof-figure P , we call the number of sequents contained in P the *length* of P . Let S be a sequent in TLK , and assume that S is provable in TLK . The minimum of the lengths of proof-figures, whose end-sequents are S , is called the *rank* of S .

We shall prove the theorem by mathematical induction with respect to the rank of S .

BASIS (Case where the rank is 1). In this case, the proof-figure for S is S itself, and S should be of the form

$$A \rightarrow A.$$

Of course, the formula $\neg A \vee A$ holds in its any realization by the law of excluded middle.

INDUCTION STEP. Assume that the theorem is true when the rank is less than m , and assume that the rank of S is m , and P is the proof-figure for S whose length is m . Let (I) be the lowest inference of P . Then, by the hypothesis of the induction, the theorem should hold for upper sequents in (I). Now, we shall consider the case where (I) is of the form:

$$\frac{A, C_1, \dots, C_m \rightarrow D_1, \dots, D_n}{A \wedge B, C_1, \dots, C_m \rightarrow D_1, \dots, D_n} \begin{array}{l} \dots\dots\dots S' \\ \dots\dots\dots S \end{array}$$

Let R be a realization of the formula

$$(2) \quad \neg(A \wedge B) \vee \neg C_1 \vee \dots \vee \neg C_m \vee D_1 \vee \dots \vee D_n.$$

The formula

$$(3) \quad \neg A \vee \neg C_1 \vee \dots \vee \neg C_m \vee D_1 \vee \dots \vee D_n$$

should hold in R , because R is also a realization of (3). If at least one of $\neg C_1, \dots, \neg C_m, D_1, \dots, D_n$ holds in R , then clearly the formula (2) holds in R . Assume that none of $\neg C_1, \dots, \neg C_m, D_1, \dots, D_n$ holds in R . Since (3) holds in R , $\neg A$ should hold in R . Hence $\neg(A \wedge B)$ holds in R . This means that (2) holds in R . Thus, S should be logically true.

Next, we shall consider the case where (I) is of the form:

$$\frac{C_1, \dots, C_m \rightarrow D_1, \dots, D_n, A(a^r)}{C_1, \dots, C_m \rightarrow D_1, \dots, D_n, \forall x^r A(x^r)} \begin{array}{l} \dots\dots\dots S' \\ \dots\dots\dots S \end{array}$$

Let R be a realization of the formula

$$(4) \quad \neg C_1 \vee \dots \vee \neg C_m \vee D_1 \vee \dots \vee D_n \vee \forall x^r A(x^r).$$

If at least one of $\neg C_1, \dots, \neg C_m, D_1, \dots, D_n$ holds in R , then clearly the formula (4) holds in R . Assume that none of $\neg C_1, \dots, \neg C_m, D_1, \dots, D_n$ holds in R . Let R' be a realization of

$$(5) \quad \neg C_1 \vee \dots \vee \neg C_m \vee D_1 \vee \dots \vee D_n \vee A(a^r),$$

and assume that R' is an extension of R . Since S' is logically true by the hypothesis of the induction, $A(a^r)$ should hold in R' . Remembering that R' is an arbitrary extension of R , we see that (4) should hold in R . Therefore (4) should be logically true.

The cases where (I) is of another form can be treated similarly. Thus the lemma is proved.

REMARK 1. For any ordinal number α , we can construct a predicate calculus of α th order. Our theorem is also true for those systems.

REMARK 2. For any natural number $n > 1$, we can construct a formula in the predicate calculus of n th order such that satisfies the following conditions:

- (a) provable in the predicate calculus of $(n+1)$ th order,
- (b) not provable in the predicate calculus of n th order.

Chapter II. The Second Proof (by S. MAEHARA)

In this chapter, we shall show that our theorem can be proved in almost the same way as in Rasiowa [2].

§1. We shall use two formal systems $\mathbf{S}$ and $\mathbf{S}_0$. $\mathbf{S}$ is a system for the predicate calculus of order ω , and $\mathbf{S}_0$ is a system for the predicate calculus of first order.

As *formulas*, in $\mathbf{S}$ and in $\mathbf{S}_0$, we shall use ones explained in §1 of Chapter I, and no varieties nor sequents will be used.

1. POSTULATES FOR THE FORMAL SYSTEM $\mathbf{S}$:

1. 1. Axioms:

1) *Identically true formulas*, each of which is the result of substituting some formulas, in our sense, for propositional variables throughout a provable formula in the propositional calculus.

2) Formulas of the form

$$\forall x^r A(x^r) \supset A(a^r), \quad A(a^r) \supset \exists x^r A(x^r)$$

or

$$\exists x^{r+1} \forall y^r [(A(y^r) \supset x^{r+1} y^r) \wedge (x^{r+1} y^r \supset A(y^r))].$$

1. 2. Rules of inference:

$$\frac{A \quad A \supset B}{B},$$

$$\frac{B \supset A(a^r)}{B \supset \forall x^r A(x^r)}, \quad \frac{A(a^r) \supset B}{\exists x^r A(x^r) \supset B}$$

In each of the last two rules, the free variable a^r shall not occur in $A(x^r)$ and in B .

2. The formal system $\mathbf{S}_0$ is distinguished from $\mathbf{S}$ by only the following
 RESTRICTION ON BOUND VARIABLES: Bound variables used in the system $\mathbf{S}_0$ are ones of type 0 only.

3. It is easily proved that the system $\mathbf{S}$ is equivalent to *TLK* in the preceding chapter, and that $\mathbf{S}_0$ is essentially equivalent to the ordinary system for the predicate calculus of first order.

§ 2. Lindenbaum Algebra of $\mathbf{S}_0$. Let $|A_0|$ denote the set of all $\mathbf{S}_0$ -formulas B_0 such that $(A_0 \supset B_0) \wedge (B_0 \supset A_0)$ is $\mathbf{S}_0$ -provable. Then the *Lindenbaum algebra* L_0 of $\mathbf{S}_0$ is the Boolean lattice of all cosets $|A_0|$ with the operations

$$|A_0| \cap |B_0| = |A_0 \wedge B_0|, |A_0| \cup |B_0| = |A_0 \vee B_0| \text{ and } |A_0|^c = |\neg A_0|.$$

The following propositions are easy to prove.

PROPOSITION 1. If $|A_0| = 1$, where 1 is the unit element of L_0 , then A_0 is $\mathbf{S}_0$ -provable.

PROPOSITION 2. Let J_0 be the set of all free variables of type 0, then

$$|\forall x^0 A_0(x^0)| = \bigcap_{a^0 \in J_0} |A_0(a^0)| \text{ and } |\exists x^0 A_0(x^0)| = \bigcup_{a^0 \in J_0} |A_0(a^0)|.$$

§ 3. 1. Let L be the Boolean lattice which is the minimal complete extension of L_0 , and we define recursively the sequence of sets $\Phi_0, \Phi_1, \Phi_2, \dots$ as follows:

$$\begin{cases} \Phi_0 = J_0 (= \text{the sets of all free variables of type 0}), \\ \Phi_{r+1} = L^{\Phi_r} \quad (\text{set-theoretic power set}). \end{cases}$$

2. For each free variable a^r , we define the element a^{r*} of Φ_r recursively as follows:

$$\begin{aligned} a^{0*} &= a^0 \\ a^{r+1*}(\varphi^r) &= \begin{cases} |a^{r+1} b^r|, & \text{if } \varphi^r = b^{r*}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

3. Let $F(a^r, b^s, \dots)$ be an arbitrary $\mathbf{S}$ -formula containing no free variable except $a^r, b^s, \dots$. Then, for that formula, we define the mapping

$F^*(\varphi^\tau, \psi^\sigma, \dots)$ from $\Phi_\tau \times \Phi_\sigma \times \dots$ to L , recursively as follows:

- 1) Case where $F(a^{\tau+1}, b^\tau)$ is a formula of the form $a^{\tau+1}b^\tau$:

$$F^*(\varphi^{\tau+1}, \psi^\tau) = \varphi^{\tau+1}(\psi^\tau).$$

- 2) Case where $F = \neg A$, $F = A \wedge B$, $F = A \vee B$ or $F = A \supset B$:

$$F^* = A^{*c}, F^* = A^* \wedge B^* \quad F^* = A^* \vee B^* \quad \text{or} \quad F^* = A^{*c} \vee B^*,$$

respectively.

- 3) Case where $F(b^\sigma, \dots) = \forall x^\tau A(x^\tau, b^\sigma, \dots)$ or $F(b^\sigma, \dots) = \exists x^\tau A(x^\tau, b^\sigma, \dots)$:

$$F^*(\psi^\sigma, \dots) = \bigcap_{\varphi^\tau \in \Phi_\tau} A^*(\varphi^\tau, \psi^\sigma, \dots)$$

or

$$F^*(\psi^\sigma, \dots) = \bigcup_{\varphi^\tau \in \Phi_\tau} A^*(\varphi^\tau, \psi^\sigma, \dots),$$

respectively.

4. The following proposition is easily proved, by the help of Proposition 2.

PROPOSITION 3. *If $F_0(a^\tau, b^\sigma, \dots)$ is an $\mathbf{S}_0$ -formula containing no free variable except $a^\tau, b^\sigma, \dots$, then*

$$|F_0(a^\tau, b^\sigma, \dots)| = F_0^*(a^{\tau*}, b^{\sigma*}, \dots).$$

§ 4 PROPOSITION 4. *If an $\mathbf{S}$ -formula $F(a^\tau, b^\sigma, \dots)$, which contains no free variable except $a^\tau, b^\sigma, \dots$ is $\mathbf{S}$ -provable, then*

$$F^* = (\varphi^\tau, \psi^\sigma, \dots) = 1 \quad (\varphi^\tau \in \Phi_\tau, \psi^\sigma \in \Phi_\sigma, \dots).$$

PROOF (inductive proof).

- 1) Case where F is an identically true formula: The proposition evidently holds.

- 2) Case where F has the form

$$\forall x^\tau A(x^\tau, b^\sigma, \dots) \supset A(a^\tau, b^\sigma, \dots):$$

$$\begin{aligned} F^*(\varphi^\tau, \psi^\sigma, \dots) &= [\bigcap_{\xi^\tau \in \Phi_\tau} A^*(\xi^\tau, \psi^\sigma, \dots)]^c \vee A^*(\varphi^\tau, \psi^\sigma, \dots) \\ &\geq A^*(\varphi^\tau, \psi^\sigma, \dots)^c \vee A^*(\varphi^\tau, \psi^\sigma, \dots) = 1; \\ \therefore F^*(\varphi^\tau, \psi^\sigma, \dots) &= 1. \end{aligned}$$

- 3) Case where $F(b^\sigma, \dots)$ has the form

$$\exists x^{\tau+1} \forall y^\tau [(A(y^\tau, b^\sigma, \dots) \supset x^{\tau+1} y^\tau) \wedge (x^{\tau+1} y^\tau \supset A(y^\tau, b^\sigma, \dots))]:$$

$$F^*(\psi^\sigma, \dots) = \bigcup_{\xi^{\tau+1} \in \Phi_{\tau+1}} \bigcap_{\eta^\tau \in \Phi_\tau} [\xi^{\tau+1}(\eta^\tau) + A^*(\eta^\tau, \psi^\sigma, \dots)]^c,$$

where $+$ means the 'symmetric difference'. Hence,

$$F^*(\phi^\sigma, \dots) = \bigcap_{\eta^\tau \in \Phi_\tau} [A^*(\eta^\tau, \phi^\sigma, \dots) + A^*(\eta^\tau, \phi^\sigma, \dots)]^c = 1;$$

$$\therefore F^*(\phi^\sigma, \dots) = 1.$$

4) Case where F is $B(a^\tau, b^\sigma, \dots)$, and $A(a^\tau, b^\sigma, \dots)$ and $A(a^\tau, b^\sigma, \dots) \supset B(a^\tau, b^\sigma, \dots)$ are both **S**-provable:

By the hypothesis of induction,

$$A^*(\varphi^\tau, \phi^\sigma, \dots) = 1$$

and

$$A^*(\varphi^\tau, \phi^\sigma, \dots)^c \cup B^*(\varphi^\tau, \phi^\sigma, \dots) = 1;$$

$$\therefore F^* = B^*(\varphi^\tau, \phi^\sigma, \dots) = 1.$$

5) Case where F is $B(b^\sigma, \dots) \supset \forall x^\tau A(x^\tau, b^\sigma, \dots)$, $B(b^\sigma, \dots) \supset A(a^\tau, b^\sigma, \dots)$ is **S**-provable, and a^τ occurs in neither $A(x^\tau, b^\sigma, \dots)$ nor $B(b^\sigma, \dots)$:

By the hypothesis of induction,

$$B^*(\phi^\sigma, \dots)^c \cup A^*(\varphi^\tau, \phi^\sigma, \dots) = 1;$$

$$\therefore F^* = B^*(\phi^\sigma, \dots)^c \cup \left[\bigcap_{\varphi^\tau \in \Phi_\tau} A^*(\varphi^\tau, \phi^\sigma, \dots) \right] = 1.$$

6) Case where F is $\exists x^\tau A(x^\tau, b^\sigma, \dots) \supset B(b^\sigma, \dots)$, $A(a^\tau, b^\sigma, \dots) \supset B(b^\sigma, \dots)$ is **S**-provable, and a^τ occurs in neither $A(x^\tau, b^\sigma, \dots)$ nor $B(b^\sigma, \dots)$: The treatment is similar to 5), q. e. d.

§ 5. By Propositions 3, 4 and 1, we have the following

THEOREM. *If an $\mathbf{S}_0$ -formula is **S**-provable, then the $\mathbf{S}_0$ -formula is $\mathbf{S}_0$ -provable.*

Chapter III. The Third Proof and an Extension of the Theorem (by T. NISHIMURA)

In this chapter we give a non-constructive consistency-proof of the theory of real numbers. The applied system is obtained by generalizing the system Z given by K. Schütte [3]. The system is denoted by GZ .

In § 1 we formulate GZ into G. Gentzen's style.

In § 2 we give a system equivalent to G. Takeuti's G^1LC and we

show that its super-arithmetical proof-figure is transformed to a proof-figure in GZ . Here a super-arithmetical proof-figure means a proof-figure which contains no predicates except $=$ and no functions except $'$, $+$ and $\cdot$.

In § 3 we give some important meta-theorems on GZ .

In § 4 we give the consistency-proof of the theory of real numbers and a theorem on G. Takeuti's fundamental conjecture for his G^1LC .

§1. The System GZ .

In this section we formulate the system GZ .

1. Symbols.

We apply the following *fundamental symbols*: Symbol 0 , function symbols $'$, $+$, $\cdot$, bound variables x, y, z etc., particular variety symbols with m arguments V_τ, V_ξ, V_η etc., (where τ, ξ, η etc. are arbitrary sets of m -tuples of natural numbers), bound V -variables with m arguments φ, ψ etc., symbol λ , logical symbols $\wedge, \supset, \vee$ and symbol $\rightarrow$.

The system GZ contains symbols of power of the continuum.

If necessary we apply several letters for abbreviation.

2. Terms are constructed as follows:

(1) the symbol 0 is a *term*; (2) if t is a *term*, so is t' , and if t_1 and t_2 are *terms*, so are t_1+t_2 and $t_1\cdot t_2$. Particularly figures of the form $0, 0', 0'', \dots$ are called *numerals*. For an term t we can decide its *numerical value* by the well-known way. When n is a natural number, $\overbrace{0''\dots}^n$ is called the *numeral* expressing n and denoted by n .

3. Varieties and formulas are constructed as follows.

(1) Particular varieties with m arguments are *varieties* with m arguments.

(2) If t_1 and t_2 are *terms*, then $t_1=t_2$ is a *formula*. If V is a *variety* with m arguments and $t_1, \dots, t_m$ are *terms*, then $[V](t_1, \dots, t_m)$ is a *formula*.

(3) If A is a *formula*, so is $\supset A$.

(4) If A and B are *formulas*, so is $A\wedge B$.

(5) If $F(t)$ is a *formula*, so is $\forall x F(x)$. If $F(t_1, \dots, t_m)$ is a *formula*, then $\lambda x_1 \dots x_m F(x_1, \dots, x_m)$ is a *variety* with m arguments.

(6) If $F(V_\tau)$ is a *formula* and V_τ is a particular variety with m arguments, then $\forall \varphi F(\varphi)$ is a *formula*, where φ is a bound V -variable with m arguments.

4. We call a figure of the following form a *sequent*,

$$A_1, \dots, A_\mu \rightarrow B_1, \dots, B_\nu$$

where $A_1, \dots, A_\mu, B_1, \dots, B_\nu$ are arbitrary formulas. And it may happen that $\mu=0$ or $\nu=0$. We say that $A_1, \dots, A_\mu$ are in the *antecedent* and $B_1, \dots, B_\nu$ are in the *succedent*.

5. Sequents of the following forms are called *beginning sequents*:

(1) a sequent of the form $\rightarrow P$, where P is a true formula of the form $t_1=t_2$,

(2) a sequent of the form $P \rightarrow$, where P is a false formula of the form $t_1=t_2$,

(3) a sequent of the form $\rightarrow [V_\tau] (t_1, \dots, t_m)$, where V_τ is a particular variety with m arguments and $t_1, \dots, t_m$ are terms numerical values of which are $n_1, \dots, n_m$ and the m -tuple of natural numbers $n_1, \dots, n_m$ belongs to τ ,

(4) a sequent of the form $[V_\tau] (t_1, \dots, t_m) \rightarrow$, where V_τ is a particular variety with m arguments and $t_1, \dots, t_m$ are terms numerical values of which are $n_1, \dots, n_m$ and the m -tuple of natural numbers $n_1, \dots, n_m$ does not belong to τ ,

(5) $D \rightarrow D$, where D is an arbitrary formula.

6. RULES OF INFERENCE.

If $S_1, \dots, S_m$ and S are sequents, then a figure of the form

$$\frac{S_1 \dots S_m}{S}$$

is called a *rule of inference*. $S_1, \dots, S_m$ are called the *upper sequents* and S is called the *lower sequent* of the rule of inference. In our case GZ contains the following rules of inference.

In what follows, capital Greek letters Γ, Π etc. express finite sequences

of formulas.

(1) Structural rules of inference :

Thinning in antecedent

$$\frac{\Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$$

Thinning in succedent

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, D}$$

Interchange in antecedent

$$\frac{\Gamma, D, C, \Pi \rightarrow \Delta}{\Gamma, C, D, \Pi \rightarrow \Delta}$$

Interchange in succedent

$$\frac{\Gamma \rightarrow \Delta, D, C, A}{\Gamma \rightarrow \Delta, C, D, A}$$

Contraction in antecedent

$$\frac{D, D, \Gamma \rightarrow \Delta}{D, \Gamma \rightarrow \Delta}$$

Contraction in succedent

$$\frac{\Gamma \rightarrow \Delta, D, D}{\Gamma \rightarrow \Delta, D}$$

where C and D are arbitrary formulas called *principal formulas* of each rule of inference.

(2) Logical rules of inference :

$\wedge$ -in antecedent

$$\frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}, \quad \frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$$

$\wedge$ -in succedent

$$\frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}$$

$\neg$ -in antecedent

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}$$

$\neg$ -in succedent

$$\frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$$

where A and B are arbitrary formulas called *side-formulas* of each rule of inference. $A \wedge B$ or $\neg A$ is called *principal formula* of each rule of inference.

$\forall$ on individual-in antecedent

$$\frac{F(t), \Gamma \rightarrow \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}, \text{ where } t \text{ is an arbitrary term.}$$

$\forall$ on individual-in succedent

$$\frac{\Gamma \rightarrow \Delta, F(n) \text{ for every numeral } n}{\Gamma \rightarrow \Delta, \forall x F(x)}$$

This is called the *infinite induction*. $F(n)$ is called the *side-formula* and $\forall x F(x)$ is called the *principal formula* of each rule of inference.

$\forall$ on variety-in antecedent

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}, \text{ where } V \text{ is an arbitrary variety.}$$

$\forall$ on variety-in succedent

$$\frac{\Gamma \rightarrow \Delta, F(V_\tau) \text{ for every particular variety } V_\tau}{\Gamma \rightarrow \Delta, \forall \varphi F(\varphi)}$$

This is called the *infinite induction on variety*.

(3) Rules of inference with respect to the symbol λ :

in antecedent

in succedent

$$\frac{A(t_1, \dots, t_m), \Gamma \rightarrow \Delta}{[\lambda x_1 \dots x_m A(x_1, \dots, x_m)] (t_1, \dots, t_m), \Gamma \rightarrow \Delta},$$

$$\frac{\Gamma \rightarrow \Delta, A(t_1, \dots, t_m)}{\Gamma \rightarrow \Delta, [\lambda x_1 \dots x_m A(x_1, \dots, x_m)] (t_1, \dots, t_m)}$$

(4) Cut

$$\frac{\Gamma \rightarrow \Delta, D \quad D, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

where D is an arbitrary formula called the *cut-formula*.

7. The number of logical symbols and λ 's of a formula is called the *degree* of the formula. The *degree* of a cut is the degree of the cut-formula.

8. We introduce a concept "proof-figure" into the system. Under a proof-figure we understand a figure of finite or infinite sequents, built up in the following manner: *uppermost sequents* are always beginning sequent; every sequent is a lower sequent of at most one rule of inference; and every sequent, except just one, the *end-sequent*, is upper sequent of just one rule of inference. To every sequent of proof-figure corresponds an ordinal number of the third class as follows. (1) The ordinal number of a beginning sequent is zero. (2) The ordinal number of the lower sequent of a structural rule of inference is equal to that of the upper sequent. (3) The ordinal number of the lower sequent of a cut, a logical rule of inference or a rule of inference with respect to the symbol λ is greater than those of upper sequents.

A proof-figure in GZ is called a *GZ-proof-figure*. If we have a GZ-

proof-figure to a sequent $\Gamma \rightarrow \Delta$, then we say that $\Gamma \rightarrow \Delta$ is *GZ-provable*.

9. Logical system *PGZ* and *FPGZ*.

The system *PGZ* is obtained by modifying *GZ* as follows: in logical rules of inference of *GZ*, $\forall$ on variety-in antecedent, we restrict V to particular variety. A proof-figure in *PGZ* is called a *P-proof-figure*. A sequent to be provable in *PGZ* is said to be *P-provable*.

A proof-figure of the system *FPGZ* is a proof-figure of *PGZ* the order of which is finite and which has the maximum degree of cuts. A proof-figure of *FPGZ* is called a *F-proof-figure*. A sequent to be provable in *FPGZ* is said to be *F-provable*.

§ 2. G. Takeuti's G^1LC .

1. By the following modification of *GZ* we obtain the system equivalent to G. Takeuti's G^1LC .

(1) As fundamental symbols, we add *free variables* a, b, c etc. and *free V-variables* with m arguments α, β, γ etc., $m=1, 2, 3, \dots$.

Moreover we add *function symbols* with m arguments f, h , etc. ($m=1, 2, \dots$) and *predicate symbols* with m arguments p, q etc. ($m=1, 2, \dots$).

(2) To the construction rules of terms we add 'if f is a function symbol with m arguments and if $t_1, \dots, t_m$ are *terms*, then $f(t_1, \dots, t_m)$ is a *term*.' And to the construction rules of formulas we add 'if p is a predicate symbol with m arguments and if $t_1, \dots, t_m$ are *terms*, then $p(t_1, \dots, t_m)$ is a *formula*.'

(3) The logical rule of inference $\forall$ on individual-in succedent of *GZ* is omitted, and we add $\forall$ on individual-in succedent of G^1LC of the form

$$\frac{\Gamma \rightarrow \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}$$

where a is a free variable to be not contained in the lower sequent.

(4) The logical rules of inference $\forall$ on variety-in succedent are omitted, and we add $\forall$ on variety of G^1LC of the form

$$\frac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow \Delta, \forall \varphi F(\varphi)}$$

where α is a free V -variable to be not contained in the lower sequent.

(5) Moreover we add a rule of inference (inference of *complete induction*) to the system, which is of the form

$$CI \quad \frac{F(a), \Gamma \rightarrow \Delta, F(a')}{F(0), \Gamma \rightarrow \Delta, F(t)}$$

where a is a free variable to be not contained in the lower sequent and t is an arbitrary term.

(6) Beginning sequents of GZ are omitted except those of the form $D \rightarrow D$.

In what follows, we call our system the system G^1LC with complete induction or briefly G^1LC , which contains the rule of inference CI .

2. We call a formula a *super-arithmetical formula*, which contains no predicates except $=$ and no functions except $', +, \cdot$. A sequent $A_1, \dots, A_\mu \rightarrow B_1, \dots, B_\nu$ is a *super-arithmetical sequent*, if $A_1, \dots, A_\mu, B_1, \dots, B_\nu$ are super-arithmetical. A proof-figure is called a *super-arithmetical proof-figure*, if it contains only super-arithmetical sequent.

A sequent $A_1, \dots, A_\mu \rightarrow B_1, \dots, B_\nu$ is called a *simple sequent*, if A_i ($i=1, \dots, \mu$) are super-arithmetical and of the forms

$$\forall x_1 \dots \forall x_l F(x_1, \dots, x_l)$$

where $F(a_1, \dots, a_l)$ contains no quantifiers, and if B_j ($j=1, \dots, \nu$) are super-arithmetical and of the forms

$$\neg \forall y_1 \dots \forall y_m H(y_1, \dots, y_m)$$

where $H(b_1, \dots, b_m)$ contains no quantifiers.

It is well known that the theory of real numbers is derived from axioms Γ_0 in the system G^1LC with complete induction restricted to super-arithmetical proof-figures, where Γ_0 are following axioms:

$$\forall x \forall y (x' = y' \supset x = y)$$

$$\forall x \neg (x' = 0)$$

$$\forall x \forall y \forall z (x = y \supset (y = z \supset x = z))$$

$$\forall x \forall y (x = y \supset x' = y')$$

$$\begin{aligned}
& \forall x(x+0=x) \\
& \forall x \forall y(x+y'=(x+y)') \\
& \forall x(x \cdot 0=0) \\
& \forall x \forall y(x \cdot y'=x \cdot y+x)
\end{aligned}$$

Here $A \supset B$ is the abbreviation of $\neg(A \wedge \neg B)$.

3. We transform any super-arithmetical proof-figure of the system G^1LC with complete induction to a GZ -proof-figure. In what follows, if Γ is of the form $A_1(\alpha_1, \dots, \alpha_l, a_1, \dots, a_m), \dots, A_\mu(\alpha_1, \dots, \alpha_l, a_1, \dots, a_m)$, then $A_1(V_1, \dots, V_l, t_1, \dots, t_m), \dots, A_\mu(V_1, \dots, V_l, t_1, \dots, t_m)$ is expressed by $\Gamma(V_1, \dots, V_l, t_1, \dots, t_m)$.

3. 1 THEOREM. *Let a sequent $\Gamma(\alpha_1, \dots, \alpha_l, a_1, \dots, a_m) \rightarrow \Delta(\alpha_1, \dots, \alpha_l, a_1, \dots, a_m)$ be super-arithmetical and G^1LC -provable. Then for every l -tuple of particular varieties $V_1, \dots, V_l$ and every m -tuple of numerals $n_1, \dots, n_m$, the sequent*

$$\Gamma(V_1, \dots, V_l, n_1, \dots, n_m) \rightarrow \Delta(V_1, \dots, V_l, n_1, \dots, n_m)$$

is GZ -provable by a smaller order than ω^2 .]

PROOF is carried out in similar way as author's proof of Theorem 3 of § 3 in [7].

§ 3. Preliminary Theorems.

In this section we discuss about the relations between systems GZ , PGZ and $FPGZ$. First we easily see that an F -proof-figure is a P -proof-figure and a P -proof-figure is a GZ -proof-figure.

By the Theorems 2 and 6 in this section we see that these three systems are mutually equivalent.

1. THEOREM. (Elimination theorem of cut for $FPGZ$). *If a sequent $\Gamma \rightarrow \Delta$ is F -provable, then we have an F -proof-figure without cut to $\Gamma \rightarrow \Delta$.*

PROOF is carried out in the same way as K. Schütte's proof in his cut-elimination theorem for Z -proof-figure with finite order.

2. THEOREM (Elimination theorem of cut for PGZ). *If a sequent $\Gamma \rightarrow \Delta$ is P -provable, then we have an F -proof-figure without cut to $\Gamma \rightarrow \Delta$*

PROOF is carried out in the same way as author's proof in Theorem

7 of § 2 in his paper [7].

3. **THEOREM** (Decidability theorem). *If A is an arbitrary formula, then either A or $\neg A$ is F -provable without cut.*

PROOF is carried out by the same way or author's proof in Theorem 8 of § 2 in his paper [7].

4. **LEMMA**. *If V is an arbitrary variety with m arguments, then we have a particular variety V_τ with m arguments such that the sequent*

$$\rightarrow \forall x_1 \cdots \forall x_m \{ [V](x_1, \dots, x_m) \sim [V_\tau](x_1, \dots, x_m) \}$$

is F -provable without cut, where $A \sim B$ is the abbreviation of

$$(A \supset B) \wedge (B \supset A).$$

PROOF. If V is a particular variety, then it is clear. Let V be of the form $\lambda y_1 \cdots y_m A(y_1, \dots, y_m)$. Then for every m -tuple of natural numbers $n_1, \dots, n_m$ either the formula $A(n_1, \dots, n_m)$ or $\neg A(n_1, \dots, n_m)$ is F -provable without cut. Let V_τ be the set of m -tuples of natural number $n_1, \dots, n_m$ such that $A(n_1, \dots, n_m)$ is F -provable. Then the sequent

$$\rightarrow \forall x_1 \cdots \forall x_m \{ [V](x_1, \dots, x_m) \sim [V_\tau](x_1, \dots, x_m) \}$$

is F -provable without cut, q. e. d.

5. **LEMMA**. *If $A(n_1, \dots, n_m)$ and $B(n_1, \dots, n_m)$ are arbitrary formulas and*

$$\rightarrow \forall x_1 \cdots \forall x_m \{ A(x_1, \dots, x_m) \sim B(x_1, \dots, x_m) \}$$

is F -provable without cut, then the sequents $F(A) \rightarrow F(B)$ and $F(B) \rightarrow F(A)$ are F -provable without cut.

PROOF. We can easily prove by induction on the number of logical symbols in $F(V_\tau)$.

6. **THEOREM**. *If a sequent $\Gamma \rightarrow \Delta$ is GZ -provable, then it is F -provable without cut.*

PROOF. We prove by transfinite induction on the order of the proof-figure to $\Gamma \rightarrow \Delta$.

If the order is zero, then it is clear.

If the order is not zero and the last rule of inference of the proof-figure is a structural rule of inference, then it is clear. If the last rule of inference is a logical rule of inference or a cut, then we have nine cases: $\wedge$ -in antecedent, $\wedge$ -in succedent, $\neg$ -in antecedent, $\neg$ -in succedent,

$\forall$ on individual-in antecedent, $\forall$ on individual-in succedent, $\forall$ on variety-in antecedent, $\forall$ on variety-in succedent and cut. In cases that the last rule of inference is neither $\forall$ on variety in antecedent nor cut, we add the rule of inference to the F -proof-figure without cut to the upper sequent where is such F -proof-figure by the assumption of the transfinite induction. In case that the last rule of inference is a cut, we add the cut to the F -proof-figures without cut to the upper sequents, where are such F -proof-figures by the assumption of the transfinite induction, and we eliminate the new cut by Theorem 1 in this section. Let the last rule of inference be $\forall$ on variety-in antecedent and of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$$

where V is a variety. By Lemmas 4 and 5 we have a particular variety V_τ such that the sequent

$$F(V_\tau) \rightarrow F(V)$$

is F -provable. By the assumption of the transfinite induction we have an F -proof-figure to $F(V), \Gamma \rightarrow \Delta$. Therefore we have an F -proof-figure to $F(V_\tau), \Gamma \rightarrow \Delta$ by the cut with $F(V_\tau) \rightarrow F(V)$. And so we have an F -proof-figure without cut to $\forall \varphi F(\varphi), \Gamma \rightarrow \Delta$ by Theorem 1 in this section.

§ 4. Main Theorems.

In this section we prove two main theorems.

1. MAIN THEOREM. *The system GZ is consistent.*

PROOF. If the sequent ' $\rightarrow$ ' is GZ -provable, then it is F -provable by Theorem 6 in § 3. The sequent ' $\rightarrow$ ', however, is not F -provable by Theorem 1 in § 3, q. e. d.

In section 2 we showed that G^1LC with complete induction is included in GZ . Therefore we can see that GZ contains the theory of real numbers. Hence from Main theorem 1 we can see that the theory of real numbers is consistent.

2. MAIN THEOREM. *If a super-arithmetical simple sequent $\Gamma_0, \Gamma \rightarrow \Delta$*

without free variable and free V -variable is G^1LC -provable, then we have a LK -proof figure without cut to $\Gamma_0, \Gamma \rightarrow \Delta$.

PROOF. If we have a proof-figure in G^1LC with complete induction to $\Gamma_0, \Gamma \rightarrow \Delta$, then we have a GZ -proof-figure to $\Gamma_0, \Gamma \rightarrow \Delta$ by Theorem 2 in § 2. Then we have an F -proof-figure without cut to $\Gamma_0, \Gamma \rightarrow \Delta$ from Theorem 6 in § 3. Because $\Gamma_0, \Gamma \rightarrow \Delta$ is simple, the proof-figure contains neither infinite induction nor $\forall$ on variety. Moreover it contains neither beginning sequents of the form 5, (3), (4) and (5) in § 1. However beginning sequents of the form 5 (1) and 5 (2) are LK -provable from Γ_0 . Therefore we obtain the theorem, q. e. d.

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