Quantum State Reduction and the Repeatability Hypothesis

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Abstract

The argument is re-examined that the program of deriving the rule for the state change caused by a measurement from the Schrödinger equation holding for the object-apparatus composite system falls into a vicious circle or an infinite regress called the von Neumann chain. It is shown that this argument suffers from a physical inconsistency concerning the causality between the process of reading of the outcome in the apparatus and the state change in the measured object caused by the measurement. A consistent argument which accomplishes the above program without falling into the circular argument is presented.

Introduction

Quantum mechanics includes a dualism concerning the principle of state changes. The Schrödinger equation, on the one hand, governs the state change caused by time evolution. The rule of quantum state reduction, on the other hand, governs the state change caused by measurement. The dualism is justified as long as the state change of only one system is concerned. The Schrödinger equation holds true only when the system is isolated, but the rule of quantum state reduction holds true only when the system is not isolated [1, p. 420].

Accepting that every measurement accompanies the interaction between the object and the apparatus, one can expect that the rule of quantum state reduction can be derived from the Schrödinger equation holding for the composite system of the object and the apparatus during the measurement. A negative view, however, prevails against this program. According to that view, the Schrödinger equation for the composite system transforms the problem of a measurement on the object to the problem of an observation on the apparatus, but in order to derive the rule of quantum state reduction holding for the object one still needs the rule of quantum state reduction applied to the composite system [2, p. 329]. This implies that the program of deriving the rule of quantum state reduction from the Schrödinger equation holding for the object-apparatus composite system falls into a vicious circle.

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sometimes called von Neumann's chain [3, Section 11.2].

The purpose of this paper is to show that the above argument, usually called the orthodox view of measurement theory, includes a serious physical inconsistency and then to present a consistent argument which derives the rule of quantum state reduction from the Schrödinger equation of the composite system without falling into the vicious circle.

In this paper, we are confined to the quantum state reduction caused by a measurement of an observable with nondegenerate purely discrete spectrum satisfying the repeatability hypothesis. Sections 2 re-examines the theory of state changes caused by measurements originated by von Neumann [1]. Here, we emphasize von Neumann's argument that the rule of quantum state reduction is equivalent to the repeatability hypothesis. Then, we derive a necessary condition for a unitary operator to represent the measuring interaction. The condition determines the form of the unitary operator representing the measuring interaction leading to quantum state reduction. The problem is then formulated as whether the unitary operator of this form is sufficient for deriving the rule of quantum state reduction. Section 3 reviews the orthodox view formulated by Wigner [2] that claims that the unitary operator does not lead to the rule of quantum state reduction without appealing to the rule of quantum state reduction to be applied to the measurement of the pointer position in the apparatus. Section 4 shows that the orthodox view suffers from a serious physical inconsistency concerning the causality between the reading of the outcome and the quantum state reduction. Section 5 presents a consistent argument which derives the rule of quantum state reduction from the Schrödinger equation holding for the object-apparatus composite system without appealing to the rule of quantum state reduction applied to the pointer-measurement. Concluding remarks are provided in Section 6.

2. State changes caused by measurements

2.1 Postulates for quantum mechanics

In this paper, all state vectors are supposed to be normalized, and mixed states are represented by density operators, i.e., positive operators with unit trace. Let $A$ be an observable with a nondegenerate purely discrete spectrum. Let $\phi_1, \phi_2, \ldots$ be a complete orthonormal sequence of eigenvectors of $A$ and $a_1, a_2, \ldots$ the corresponding eigenvalues; by assumption, all different from each other.

According to the standard formulation of quantum mechanics, on the result of a measurement of observable $A$ the following postulates are posed:

(A1) If the system is in a state $\psi$ at the time of measurement, the eigenvalue $a_n$ is obtained as the outcome of measurement with the probability $|\langle \phi_n | \psi \rangle|^2$.

(A2) If the outcome of measurement is the eigenvalue $a_n$ the system is left in
the corresponding eigenstate $\phi_n$ at the time just after measurement.

The postulate $(A1)$ is called the \textit{statistical formula}, and $(A2)$ the \textit{measurement axiom}. The state change $\psi \rightarrow \phi_n$ described by the measurement axiom is called the \textit{quantum state reduction}.

2.2 Origin of measurement theory

The quantum state reduction can be thought to be caused by the following two factors: the dynamical change of the system and the change of the observer's knowledge. In order to separate these factors, let us suppose that we terminate the procedure of measurement of observable $A$ just before the observer's reading of the outcome; this procedure is called a \textit{nonselective measurement}. It follows from postulates $(A1)$ and $(A2)$ that the nonselective measurement leaves the system in the mixture of the states $|\phi_n\rangle\langle\phi_n|$ with the probability $|\langle\phi_n|\psi\rangle|^2$ and therefore yields the following state change:

$$|\psi\rangle\langle\psi| \rightarrow \sum_n |\langle\phi_n|\psi\rangle|^2 |\phi_n\rangle\langle\phi_n|.$$ (1)

Since a nonselective measurement does not change the observer's knowledge, this state change is considered to be caused entirely by the dynamical factor.

Even in classical mechanics, when the outcome of measurement is obtained, the information on the outcome changes the observer's knowledge and the probabilistic description of the state of the system changes according to the Bayes principle, which is formulated as follows: Let $X, Y$ be two (discrete) random variables. Suppose that we know the joint probability distribution $\Pr\{X=x, Y=y\}$. Then, the prior distribution of $X$ is defined as the marginal distribution of $X$, i.e.,

$$\Pr\{X=x\}=\sum_y \Pr\{X=x, Y=y\}.$$ (2)

If one measures $Y$, the \textit{information} "$Y=y$" changes the probability distribution of $X$ for any outcome $y$. The posterior distribution of $X$ is defined as the conditional probability distribution of $X$ given $Y=y$, i.e.,

$$\Pr\{X=x|Y=y\}=\frac{\Pr\{X=x, Y=y\}}{\sum_x \Pr\{X=x, Y=y\}}.$$ (3)

This principle of changing the probability distribution from the prior distribution to the posterior distribution is called the \textit{Bayes principle}.

Nonetheless, it is possible that a nonselective measurement, or a measurement without reading the result, in classical mechanics causes no change in the system. Therefore, it is a characteristic feature of quantum mechanics that a nonselective measurement changes the system dynamically, and this is the origin of von Neumann's measurement theory to explain this change.
2.3 Existence of measuring interaction

In quantum mechanics the state of an isolated system changes dynamically according to the Schrödinger equation, but this state change does not change the entropy of the system. On the other hand, the state change (1) caused by the nonselective measurement increases the entropy of the measured system, and hence this process of state change cannot be described by the Schrödinger equation of the measured system [1, p. 388]. It follows that this dynamical change of state must be caused by the interaction between the measured object and the measuring apparatus, a system external to the measured object including every system interacting with the measured object. Thus, from the basic postulates of quantum mechanics we conclude that there is an inevitable interaction between the measured object and the measuring apparatus. Since our discussion concerns only nonselective measurements, we do not need to mention the function of consciousness, although von Neumann's argument mentions the psycho-physical parallelism [1, pp. 418–420].

2.4 Repeatability hypothesis and quantum state reduction

In the measurement axiom (A2), quantum state reduction is described as a change of the state of the object. In order to consider quantum state reduction together with the interaction between the object and the apparatus, it is desirable to describe it independently of particular descriptions of states of systems. As one of such descriptions, von Neumann introduced the following repeatability hypothesis [1, pp. 213–218, 335].

(M) If a physical quantity is measured twice in succession in a system, then we get the same value each time.

Then, von Neumann showed that (A2) is equivalent to the repeatability hypothesis. In fact, according to (M) the state of the object just after the first measurement is the eigenstate corresponding to the outcome, and in the nondegenerate case it is determined uniquely so that we have (A2). It is obvious that (M) follows from (A2).

2.5 Projection postulate

In this paper, we are devoted to measurements of observables with nondegenerate discrete spectrum. In the conventional discussion explaining quantum state reduction, however, we need to consider a measurement on the object-apparatus composite system. Even if an observable in the object described by an operator $A$ is nondegenerate, the same observable is described by the operator $A \otimes I$ in the composite system and no longer nondegenerate. Thus, we need to extend the statistical formula and the measurement axiom to observables with degenerate spectrum. The statistical formula for a discrete observable $X = \sum_{n} \alpha_{n} E_{n}$ to be
measured in a state (density operator) $\rho$ is given as follows:

(B1) The eigenvalue $a_n$ is obtained as the outcome with the probability $\text{Tr}[E_n\rho]$.

In the degenerate case, the eigenstate corresponding to an eigenvalue is not determined uniquely, and hence (A2) is ambiguous. Moreover, the repeatability hypothesis (M) implies that the object is left in an eigenstate corresponding to the outcome, but (M) does not determine the eigenstate in which the object is left. For determining this eigenstate, von Neumann posed no special principle [1, p. 348], but later Lüders proposed a principle which has been widely accepted [4]. If the measurement of the discrete observable $X$ with spectral decomposition $X = \sum_n a_n E_n$ is carried out in the state (density operator) $\rho$, Lüders's principle is stated as follows.

(B2) If the outcome of measurement is the eigenvalue $a_n$ the object is left in the state $E_n\rho E_n/\text{Tr}[E_n\rho]$ at the time just after measurement.

This principle is called the projection postulate, because the eigenstate $\phi_n$ provoked by the measurement is chosen by the projection $E_n$ onto the corresponding eigenspace so that $\phi_n = E_n\phi/\|E_n\phi\|$ for the initial state $\phi$. It is clear that statements (B1) and (B2) imply (A1) and (A2) as special cases. It follows from (B1) and (B2) that the nonselective measurement of $X$ leads to the state change such as

$$\rho \rightarrow \sum_n E_n\rho E_n. \quad (4)$$

2.6 Interaction between the object and the apparatus

We shall turn to the discussion on the measurement of the discrete observable $A = \sum_n a_n |\phi_n\rangle\langle\phi_n|$ with nondegenerate eigenvalues. In subsection 2.3, we have concluded that the state change (1) in the nonselective measurement must be caused by the interaction between the object and the apparatus. Then, how is this interaction described?

Let us suppose that the object is in the eigenstate $\phi_n$ of observable $A$ pertaining to the eigenvalue $a_n$. By the statistical formula, the outcome is $a_n$ with probability one. Hence, the measurement changes the position of the pointer in the apparatus from the original position to the position $a_n$ on the scale.

Let $B$ be the observable corresponding to the position of the pointer in the apparatus, and $\xi$ the original state of the apparatus. Generally it is only required [1, p. 439] that the eigenvalues $\{b_n\}$ of $B$ are in one-to-one correspondence with the eigenvalues $\{a_n\}$ of $A$, but we assume for simplicity that observable $B$ has also the same eigenvalues, $a_1, a_2, \ldots$, as $A$. In the Hilbert space of the composite system of the object and the apparatus, observables $A$ and $B$ are represented by the self-adjoint operators $A \otimes 1$ and $1 \otimes B$ respectively.

The state change due to the interaction is represented by a unitary operator $U$ on the Hilbert space of the composite system:

$$U : \phi_n \otimes \xi \mapsto U(\phi_n \otimes \xi). \quad (5)$$
According to the statistical formula (A1), the state $U(\phi_n \otimes \xi)$ after the interaction must be the eigenstate of $1 \otimes B$ pertaining to the eigenvalue $a_n$. In fact, the position of the pointer takes the value $a_n$ with probability one after the interaction, and this implies that the state $U(\phi_n \otimes \xi)$ is the eigenvector of $1 \otimes B$ pertaining to the eigenvalue $a_n$.

On the other hand, according to the repeatability hypothesis (M) the state $U(\phi_n \otimes \xi)$ is the eigenvector of $A \otimes 1$ pertaining to the eigenvalue $a_n$. In fact, if the observer were to measure $A$ in this state again, then observable $A$ would be measured twice in succession and hence the outcome would be $a_n$ with probability one. This implies that the state $U(\phi_n \otimes \xi)$ is the eigenstate of $A \otimes 1$ pertaining to the eigenvalue $a_n$.

Suppose here that the eigenvalues of $B$ are also nondegenerate. Then the state that satisfies the above two conditions is represented by the state vector $\phi_n \otimes \xi_n$, where $\xi_n$ is an arbitrary eigenvector of $B$ with unit length pertaining to the eigenvalue $a_n$. In order to represent the measurement, the unitary $U$ must satisfy the following relation

$$U(\phi_n \otimes \xi) = \phi_n \otimes \xi_n. \quad (6)$$

If the unitary operator $U$ satisfies the above condition, then for the arbitrarily given original state $\phi = \sum_n c_n \phi_n$ we have by linearity

$$U(\phi \otimes \xi) = \sum_n c_n \phi_n \otimes \xi_n. \quad (7)$$

Thus the problem is whether equation (7) is a sufficient condition for the unitary operator $U$ to represent the measuring interaction or whether, in other words, equation (7) implies (A1) and (A2) even when $\phi$ is a superposition of eigenstates $\phi_n$. If equation (7) were not a sufficient condition, further interaction — though ineffective in the case where the object is initially in the eigenstate — might be necessary for the explanation leading to the quantum state reduction.

3 The Orthodox view

The conventional approach adopted by most of the text books on measurement theory, the so-called orthodox view, is negative about the above problem. The orthodox view maintains that (7) does not imply the measurement axiom (A2). The argument runs as follows.

The state transformation by the unitary $U$,

$$\left( \sum_n c_n \phi_n \right) \otimes \xi \rightarrow \sum_n c_n \phi_n \otimes \xi_n, \quad (8)$$

makes a one-to-one correspondence between the state of the object and the state of the apparatus. The state of the object is mirrored by the state of the apparatus, and
the problem of a measurement on the object is transformed into the problem of an observation on the apparatus [2]. If the observer observes the pointer position of the apparatus to obtain the outcome of measurement, the state in the right-hand side of (8) changes as follows:

\[ | \sum_n c_n \phi_n \otimes \xi_n \rangle \langle \sum_n c_n \phi_n \otimes \xi_n | \rightarrow \sum_n | c_n |^2 | \phi_n \otimes \xi_n \rangle \langle \phi_n \otimes \xi_n |. \] (9)

The state change in (9) is derived by the projection postulate (B2) applied to the state change caused by the measurement of the observable \( A \) of the composite system. According to the projection postulate (B2), the new state is the mixture of the states \( \phi_n \otimes \xi_n \), and hence when the outcome \( a_n \) is obtained, a system in the state \( \phi_n \otimes \xi_n \) is selected from the ensemble described by the right-hand side of (9):

\[ \sum_n | c_n |^2 | \phi_n \otimes \xi_n \rangle \langle \phi_n \otimes \xi_n | \rightarrow | \phi_n \otimes \xi_n \rangle \langle \phi_n \otimes \xi_n |. \] (10)

Finally, the composite system is in the state \( \phi_n \otimes \xi_n \), and this implies that the object is led to the state \( \phi_n \).

Nevertheless, if we further describe the measuring process that leads to the state change (9) in terms of the coupling with the second apparatus, having an orthonormal system \( \{ \xi'_n \} \) and being prepared in a state \( \xi' \), measuring the pointer position in the first apparatus, then instead of (9) we have the following state change:

\[ (\sum_n c_n \phi_n \otimes \xi_n) \otimes \xi' \rightarrow \sum_n c_n \phi_n \otimes \xi_n \otimes \xi'_n. \] (11)

From this state change, we cannot conclude that the measurement leads the object with the outcome \( a_n \) to the state \( \phi_n \). The original problem of explaining the quantum state reduction caused by the first apparatus is not solved but only transferred to the the problem of explaining the quantum state reduction caused by the second apparatus [3, Section 11.2]. This vicious circle is often called von Neumann’s chain.

### 4 Inconsistency of the orthodox view

A difficulty in the orthodox view is to apply the projection postulate to the object-apparatus composite system in order to show that the state of the object that leads to the outcome \( a_n \) is in the state \( \phi_n \) at the time just after measurement. This argument suffers from the circular argument that assumes the rule of quantum state reduction for the composite system in order to explain the rule of quantum state reduction for the object. The conventional studies of measurement theory, how-
ever, have not detected any physical inconsistency or empirical inadequacy of the
above argument in the orthodox view and have aimed at circumventing the above
circular argument, an epistemological difficulty, by adding, for instance, the element
of macroscopic nature of the measuring apparatus [5, 6].

In fact, the above argument leading to the quantum state reduction has been
generalized to the following argument for any measurements to determine the state
change caused by measurement conditional upon the outcome: Let us given the
initial state \( \rho \) of the object, the initial state of the apparatus \( \sigma \), and the unitary
evolution operator \( U \) of the object-plus-apparatus. Then, compute the state of the
composite system just after the interaction as \( U(\rho \otimes \sigma)U^* \) and apply the projection
postulate to the measurement of the pointer observable \( B \) in the apparatus, and the
state \( \rho_{an} \) just after the measurement conditional upon the outcome \( a_n \) is given by

\[
\rho_{an} = \frac{\text{Tr}_A[(I \otimes E^\sigma(a_n))U(\rho \otimes \sigma)U^*(I \otimes E^\sigma(a_n))]}{\text{Tr}[(I \otimes E^\sigma(a_n))U(\rho \otimes \sigma)U^*(I \otimes E^\sigma(a_n))]}
\]  

(12)

where \( E^\sigma(a_n) \) is the projection operator onto the eigenspace of \( B \) corresponding to
the eigenvalue \( a_n \) and \( \text{Tr}_A \) stands for the partial trace over the Hilbert space of the
apparatus. This unitary-evolution-plus-projection-postulate argument has been a
standard argument for determining the general quantum state reduction, see for
example [7, 8, 9, 10, 11].

The purpose of this section is, despite the conventional arguments, to show that
the orthodox view suffers from a serious physical inconsistency concerning the
causality between the reading of the outcome and the quantum state reduction.

In order to explain the rule of quantum state reduction in terms of the time
evolution of the object-plus-apparatus, it is necessary to clarify the meanings of the
words the “time of measurement” and the “time just after measurement” in the
context as to what happens in the object and the apparatus. Let us suppose that
one measures an observable \( A = \sum_n a_n |\phi_n><\phi_n| \) of the object in the state \( \phi \) at the
time \( t \). The measurement, carried out by an interaction with the apparatus, takes finite
time \( \Delta t > 0 \). Thus, the object interacts with the apparatus from the time \( t \) to \( t + \Delta t \)
and is free from the apparatus after the time \( t + \Delta t \). It follows that the time of
measurement is the time \( t \), the time just after measurement is \( t + \Delta t \), and that the
object is in the state \( \phi \) at the time of measurement. The statistical formula (A1)
means, in this case, that the observer obtains the outcome \( a_n \) with probability
\( |<\phi_n|\phi>|^2 \). The measurement axiom (A2) means that the object that leads to the
outcome \( a_n \) is in the state \( \phi_n \) at the time \( t + \Delta t \). Moreover, the repeatability
hypothesis (M) means that if observable \( A \) is measured at the time \( t_1 = t + \Delta t \) again
in the same object then the outcome coincides with the one obtained by the
measurement of \( A \) at the time \( t \).

In the orthodox view, the state changes given by (8) and (9) represent dynami-
cal changes of the system, and the state change (10) represents a change of the knowledge of the observer. The state change (8) represents the interaction between the object and the apparatus. The state change (9) represents the interaction between the “apparatus” and the “apparatus measuring the apparatus”. It follows that the state change (8) shows that the object-plus-apparatus is in the state \( \sum_n c_n \phi_n \otimes \xi_n \) at the time \( t \) and in the state \( \sum_n c_n \phi_n \otimes \xi_n \) at the time \( t + \Delta t \).

Suppose that the state change (9) takes time \( \tau \). Then, it is at the time \( t + \Delta t + \tau \) that the object-plus-apparatus turns to be in the state described by the right-hand side of (9). Since the state change (10) represents the change of the observer’s knowledge, it does not accompany the change of time so that at the time \( t + \Delta t + \tau \) the object turns to be in the state \( \phi_n \) as the result of the quantum state reduction. In other words, the orthodox view leads to the conclusion that the quantum state reduction occurs at the time \( t + \Delta t + \tau \) which is later in time \( \tau \) than the time \( t + \Delta t \) just after measurement. Thus, if \( \tau \) is not negligible in the time scale of the time evolution of the object then this contradicts the measurement axiom that the quantum state reduction leaves the system in the state \( \phi_n \) at the time \( t + \Delta t \).

Since the object is free from the apparatus after the time \( t_1 = t + \Delta t \), one can make the object interact with the second apparatus at the time \( t_1 \). If this apparatus also measures \( A \), according to the repeatability hypothesis it is predicted, and will

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FIG. 1: Repeated position measurements by photon scattering from an atom.
be confirmed by experiments, that the outcome from the first apparatus and the outcome from the second are always the same. But, this fact cannot be explained by the orthodox view which concludes that the quantum state reduction occurs at the time \( t_1 + \tau \).

Is \( \tau \) negligible in the time scale of the time evolution of the object? In general, the process of the state change (9) is regarded as a process in which a macroscopic instrument operates or a directly-sensible variable feature is produced — otherwise, the state change in the apparatus measurement might not necessarily satisfy the repeatability hypothesis — and hence the duration \( \tau \) of this process cannot be negligible in the time scale of the time evolution of the microscopic measured object.

Consider, for instance, the experiment in which the light is scattered by an atom in a low intensity atom beam. Regarding the paths before the collision as known, the measurement of the path of the photon after the collision suffices to determine the point of scattering. In order to measure the position of the atom (at the time of collision) twice in succession in this method, suppose to use two nearly placed light beams I and II; see FIG. 1. Suppose that the atom interacts with the beam I from the time \( t \) to \( t' \) and with the beam II from \( t_1 \) to \( t'_1 \) and that \( t_1 - t' \) is so small that the time evolution of the atom in this period can be neglected — hence, we can put \( t_1 = t' \). Suppose that the photon scattered from the beam I is detected by a photoelectric detector at the time \( t'' \), and the one from the beam II is detected by another photoelectric detector at the time \( t''_1 \). In this experiment, the collision is accomplished in quite a short time and the photoelectric detectors are necessary to place sufficiently far from the light beams, so that it is taken for granted in scattering theory [12, p. 375] that

\[
\Delta t = t'' - t' .
\]  

(13)

It is taken for granted from the Compton-Simons experiment that there is the uncertainty of the position at which the beam I is scattered depending on the initial state of the atom but that the position of the scattering from the beam II is always near the position of the scattering from the beam I. It follows that this experiment can be considered as the position measurement of the atom satisfying the repeatability hypothesis [1, pp. 212-214]. In this example, the state change (8) corresponds to the interaction between the atom and the light beam, and hence we have \( \Delta t = t' - t \). On the other hand, the apparatus corresponds to the scattered photon, the state change (9) corresponds to the process including the free propagation of the photon after scattering and the interaction between the photon and the photoelectric detector, and hence we have \( \tau = t'' - t' \). Thus, from (13) we have

\[
\Delta t \ll \tau ,
\]  

(14)

and consequently we cannot neglect \( \tau \). This means that the orthodox view claims
that the quantum state reduction of the atom occurs after the photon is detected despite the fact that the quantum state reduction of the atom occurs just after the scattering of the light.

The inconsistency of the orthodox view is in the claim of causality between the reading of the outcome and the quantum state reduction such that the state change (9) of the composite system causes the quantum state reduction of the object system. It is obvious, however, that such causality does not exist, since the result, the quantum state reduction, occurs before the cause, the reading of the outcome or the manifestation of the directly-sensible variable feature. It is not the case that the observer's knowing or reading of the outcome at the time \( t'' = t + \Delta t + \tau \) causes the quantum state reduction of the object at the time \( t' = t + \Delta t \). But, it is the case that by knowing or reading of the outcome at the time \( t'' \) the observer obtains the information to determine the state of the object at the time \( t' \). The orthodox view confuses the time at which the outcome of measurement is obtained and the time at which the object is left in the state determined by the outcome. Or, in other words, it confuses the time just after the reading of the outcome and the time just after the interaction between the object and the apparatus. There is no causality relation between the outcome and the state just after measurement but there is coincidence between them yielded by the measuring interaction.

5 New interpretation

Our new interpretation presented in the following does not include the inconsistency of the orthodox view discussed in the previous section. Moreover, the quantum state reduction can be explained only from (8) without assuming the process (9) so that the circular argument of the von Neumann chain is circumvented.

As in the preceding section, suppose that the observer measures the observable \( A = \sum n a_n |\phi_n\rangle \langle \phi_n| \) of the object in the state \( \phi \) at the time \( t \). The object interacts with the apparatus from the time \( t \) to \( t + \Delta t \) and is free from the apparatus after the time \( t + \Delta t \). The repeatability hypothesis (M) means that if the observer measures \( A \) at the time \( t_1 = t + \Delta t \) again then the outcome coincides with the outcome of the measurement at the time \( t \). As shown previously, the measurement axiom (A2) is equivalent to the repeatability hypothesis (M). Hence, if the statistical formula (A1) and the repeatability hypothesis (M) is derived from (7), it is demonstrated that the quantum state reduction is derived from (7). Let \( \Pr\{B(t+\Delta t) = a_n\} \) be the probability of obtaining the outcome \( a_n \) when the pointer position \( B = \sum n a_n |\xi_n\rangle \langle \xi_n| \) is measured at the time \( t + \Delta t \). By (7) and the statistical formula (B1) for the degenerate observable \( 1 \otimes B \) we have

\[
\Pr\{B(t+\Delta t) = a_n\}
\]
Let $\Pr\{A(t) = a_n\}$ be the probability that the measurement of $A$ at the time $t$ leads to the outcome $a_n$. Since this outcome is obtained as the outcome of the measurement of $B$ at the time $t + \Delta t$, we have

$$\Pr\{A(t) = a_n\} = \Pr\{B(t + \Delta t) = a_n\} = |c_n|^2.$$  

Thus if we regard this process as the measurement of $A$ — namely, if we interpret the outcome of the measurement of $B$ at the time $t + \Delta t$ as the outcome of the measurement of $A$ at the time $t$ — then it is shown that this measurement satisfies the statistical formula (A1).

We shall show that this measurement satisfies the measurement axiom (A2) in the following. Since the measurement axiom (A2) is equivalent to the repeatability hypothesis (M), we need only to show that this measurement satisfies the repeatability hypothesis (M).

In order to show the last statement, it suffices to show that if the observer measures $A$ again at the time $t_1 = t + \Delta t$, immediately after the first measurement, then the outcome of the first measurement at the time $t$ and that of the second at the time $t_1$ are always the same. Let $\Pr\{A(t) = a_n, A(t_1) = a_m\}$ be the joint probability that the first outcome is $a_n$ and the second outcome is $a_m$.

The outcome of the first measurement of $A$ at the time $t$ is the same as the outcome of the measurement of the pointer position $B$ at the time $t_1 = t + \Delta t$. Since the measurements of $B$ and $A$ at the time $t_1$ does not interfere each other, the joint probability distribution of the outcomes of these two measurements satisfies the statistical formula for the simultaneous measurements:

$$\Pr\{B(t_1) = a_m, A(t_1) = a_n\} = |\langle m|\xi_n|\phi_k\otimes\xi_k\rangle|^2 = \delta_{m,n}|c_n|^2.$$  

Thus if $m \neq n$ then we have

$$\Pr\{A(t) = a_n, A(t_1) = a_m\} = \Pr\{B(t_1) = a_m, A(t_1) = a_n\} = 0.$$  

It follows that the outcome of the first measurement and that of the second are always the same. Therefore, this process satisfies the repeatability hypothesis (M) and hence satisfies the measurement axiom (A2).

We have thus demonstrated that the unitary operator $U$ satisfying (7) represents the interaction between the object and the apparatus and leads to the quantum state reduction in the object at the time just after measurement.
6 Concluding remarks

In this paper, we define quantum state reduction as the state change in the measured system caused by a measurement postulated by the measurement axiom as formulated by (A2). This definition is also consistent with those given in standard textbooks, e.g., Ref. [12].

The logical relationship between axiom (A1) and axiom (A2) are independent. According to the argument given in Section 2, in the measuring interaction with the apparatus prepared in a state $\xi$ having the pointer-position $B$, the unitary operator $U$ characterizes the measurement as follows:

1. The measurement satisfies (A1) for observable $A$ if and only if $U(\phi_n \otimes \xi)$ is an eigenstate of $1 \otimes B$ pertaining to the eigenvalue $a_n$.

2. In the above case, the measurement satisfies (A2) for observable $A$ if and only if $U(\phi_n \otimes \xi)$ is an eigenstate of $A \otimes 1$ pertaining to the eigenvalue $a_n$.

Thus, by choosing an appropriate $U$ we can construct a measurement model in which (A1) is satisfied but (A2) is not. One of such models describes a measurement of the photon number in a small cavity by an ideal photon counter. In this case, all photons in the cavity are counted and absorbed, so that we have

$$U(|n> \otimes |0>_A) = |0> \otimes |n>_A.$$ (16)

where $|n>$ is the photon number eigenstate with $n$ photons and $|n>_A$ is the eigenstate corresponding to the pointer of the counter pertaining to the outcome $n$. For position measurement, a model is also known that satisfies (A1) surely but does not satisfy (A2) even approximately [14], in an appropriate generalization of the formulation to measurements of continuous observables. We can also construct a measurement model in which (A2) is satisfied but (A1) is not. Thus, (A1) and (A2) are logically independent.

By the way, we can logically classify epistemological problems concerning quantum measurements into the class of problems to explain (A1) and the class of problems to explain (A2). The problem treated in this paper obviously belongs to the second class. However, the problem of explaining the process in which the measurement produces the unique and objective result in each instance of measurement, the objectification problem, belongs to the first class. It is confusing that the latter process is also often called the "state reduction." As argued above, this process explains the state change of the pointer described by (A1) which is independent of (A2) describing the state change of the measured object.

It follows from the basic postulate requiring the quantum state reduction that even in the case where the observer obtains no information from the measurement, namely the case of nonselective measurement, the state of the system changes. This change is not accompanied with the change of knowledge so that it is purely...
dynamical, but it is irreversible so that it cannot be described by the Schrödinger equation of the object. The only way to explain this by the rules of quantum mechanics is to derive this change from the Schrödinger equation of a larger system than the object, which describes the interaction between the object and the apparatus. This interaction is turned on during a finite time interval, from the time \( t \) of measurement to the time \( t + \Delta t \) just after measurement. After the object-apparatus interaction, the object turns to be free from the apparatus again. Thus, the quantum state reduction describes the state change from the time \( t \) to the time \( t + \Delta t \) conditional upon the outcome of measurement.

The state change in the nonselective measurement is derived without any difficulties from the interaction between the object and the apparatus. In fact, the state change (1) is explained as the open system dynamics of the object yielded by the unitary evolution of the object-apparatus composite system described by the unitary \( U \) in (7), i.e.,

\[
\sum_n |\phi_n> |\phi_n><\phi_n| = \text{Tr}_A [U|\psi \otimes \xi><\psi \otimes \xi| U^*],
\]  

where \( \text{Tr}_A \) is the partial trace over the Hilbert space of the apparatus [13, p. 136]. The problem is to explain the change of state dependent on the outcome, namely the quantum state reduction. The answer of the orthodox view to this problem is that the quantum state reduction of the object is resulted from the quantum state reduction of the object-plus-apparatus caused by the measurement of the pointer position carried out after the object-apparatus interaction. Applying the above argument that quantum state reduction needs the time for the nonselective measurement to the measurement of the pointer position, it is concluded that the quantum state reduction, explained by the orthodox view, occurs apparently later than the time \( t + \Delta t \) at which the quantum state reduction should occur. This time difference leads to the detectable difference as to whether the outcomes obtained by measuring the same object twice in succession satisfy the repeatability hypothesis, and hence we can conclude that the inconsistency of the orthodox view can be tested by an experiment.

The photon scattering experiment from the atom beam in an atom interferometer has been realized already by Chapman et al. [15]. The double scattering gedanken experiment suggested in Section 4 will be realized in future along with a similar experimental setting with the additional second laser beam for the repeated measurement of the point of scattering of a single atom, if a conceivable difficulty can be circumvented in distinguishing the case where two detected photons from the two beams have been scattered by a single atom from the other cases.
References