On the Problem of Hidden Variables in Algebraic Quantum Field Theory

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Abstract

There have been many discussions about whether all observational propositions in nonrelativistic quantum mechanics can be taken as simultaneously definite without contradiction. Von Neumann, Jauch and Piron, and Bell have shown that not all observational propositions in nonrelativistic quantum mechanics can be interpreted in this way. In the present paper, I examine these theorems from an operator algebraic point of view in order to apply these theorems to algebraic quantum field theory. Then I point out that not all observational propositions in local algebras can be taken as definite, in addition to nonrelativistic quantum mechanics.

1. Introduction

In algebraic quantum field theory, some von Neumann algebra is associated with each bounded open set in Minkowski space-time. Any projection in that algebra is regarded as an observational proposition in the set in Minkowski space-time to which it is associated. Some physically proper conditions are imposed on that von Neumann algebra, which is called a local algebra. A state on a local algebra gives the probability of an observational proposition in the bounded open set in Minkowski space-time to which that local algebra is associated.

There have been many discussions about whether there exist hidden variables that determine truth-values of all observational propositions about a given nonrelativistic quantum mechanical system, and whether the probability given by a quantum mechanical state can be regarded as that probability measure on the set of all hidden variables. For example, von Neumann (1955), and Jauch and Piron (1963) mathematically defined such hidden variables, and showed that there is none in quantum mechanics. But they imposed a condition for incompatible observational propositions on hidden variables. Bell (2004) argued that it is not physically proper to impose that condition on hidden variables although it is proper to impose it on quantum mechanical states (pp. 4–6). Then Bell (2004) defined a hidden variable which was not bound by the condition on incompatible observational propositions, and showed that no such hidden variable in nonrelativistic quantum mechanics.

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mechanics exists (pp. 6-8). In sections 3, 4 and 5, we will see those arguments from an operator algebraic point of view and it will be seen that there is no hidden variable in either algebraic quantum field theory or nonrelativistic quantum mechanics.

On the other hand, for a local algebra in algebraic quantum field theory as well as for the set of all bounded operators on a Hilbert space in quantum mechanics, some subset of the set of all observational propositions does admit hidden variables. Halvorson and Clifton (1999) defined these sets from an operator algebraic point of view (see also Clifton, 1999), and named them 'beable algebras' following Bell's terminology (Bell, 2004, chapters 5, 7 and 19). In Section 6, we will look at Clifton's interpretation of algebraic quantum field theory in terms of beable algebras.

2. Mathematical preliminaries

2.1. Operator algebras

In the present paper, we use the following notation. If $S$ is a subset of the set of all bounded operators on a Hilbert space $H$ which is denoted by $B(H)$, let $S'$ denote $\{A \in B(H) \mid AB = BA \text{ for all } B \in S\}$. Where $\mathcal{M}$ is a von Neumann algebra, let $\mathcal{P}(\mathcal{M})$ denote the set of all projections in $\mathcal{M}$. Where $\mathcal{M}$ is a von Neumann algebra, $\mathcal{M} \cap \mathcal{M}'$ is called the center of $\mathcal{M}$ and let $\mathcal{Z}(\mathcal{M})$ denote the center of $\mathcal{M}$.

**Definition 2.1.** A linear functional $\rho$ on a unital $C^*$-algebra $\mathfrak{A}$ is called a state if $\rho$ satisfies following conditions:

1. $\rho(A^*A) \geq 0$ for any element $A \in \mathfrak{A}$;
2. $\rho(I) = 1$ where $I$ is the identity operator in $\mathfrak{A}$.

**Definition 2.2.** A state $\omega$ on a unital $C^*$-algebra $\mathfrak{A}$ is called a dispersion-free state if $\omega(A^2) = [\omega(A)]^2$ for any self-adjoint element $A \in \mathfrak{A}$.

**Definition 2.3.** A von Neumann algebra $\mathcal{M}$ is called a $\sigma$-finite von Neumann algebra if each orthogonal family of non-zero projections in $\mathcal{M}$ is countable.

**Definition 2.4.** Let $\mathcal{M}$ be a von Neumann algebra. Two projections $P$ and $Q$ in $\mathcal{P}(\mathcal{M})$ are called equivalent if there is a partial isometry $V$ in $\mathcal{M}$ such that $P = VV^*$ and $Q = V^*V$.

If two projections $P$ and $Q$ are equivalent, we shall write $P \sim Q$.

**Definition 2.5.** Let $\mathcal{M}$ be a von Neumann algebra. A projection $P$ in $\mathcal{M}$ is said to
be finite if $P \sim Q \leq P$ implies $P = Q$. Otherwise, it is said to be infinite. If $P \nabla P$ is an Abelian von Neumann algebra, then $P$ is said to be Abelian.

**Definition 2.6.** Let $\mathcal{A}$ be a von Neumann algebra.

If for any non-zero projection $P$ in $\mathcal{A}$, there is a non-zero Abelian projection $Q$ such that $Q \leq P$, then $\mathcal{A}$ is said to be of type I. If $\mathcal{A}$ is of type I and the identity operator in $\mathcal{A}$ is the sum of $n$ mutually orthogonal and equivalent Abelian projections, then $\mathcal{A}$ is said to be of type I$_n$.

If for any non-zero projection $P$ in $\mathcal{A}$ there is a non-zero finite projection $Q$ such that $Q \leq P$, and if $\mathcal{A}$ has no non-zero Abelian projection, then $\mathcal{A}$ is said to be of type II. If $\mathcal{A}$ is of type II and the identity operator in $\mathcal{A}$ is finite, then $\mathcal{A}$ is said to be of type II$_1$. If $\mathcal{A}$ is of type II and the identity operator in $\mathcal{A}$ is infinite, then $\mathcal{A}$ is said to be of type II$\omega$.

If $\mathcal{A}$ has no non-zero finite projection, $\mathcal{A}$ is said to be of type III.

A von Neumann algebra of type I$_1$ is an Abelian von Neumann algebra. The set of all bounded operators on a Hilbert space is a von Neumann algebra of type I.

**Proposition 2.1 (Kadison and Ringrose, 1997, Theorem 6.5.2).** Let $\mathcal{A}$ be a von Neumann algebra. Then there are projections $P_I, P_{II_1}, P_{II\omega}$, and $P_{III}$ in the center of $\mathcal{A}$ and $\mathcal{A}$ can be expressed as

$$\mathcal{A} = \mathcal{A} P_I \oplus \mathcal{A} P_{II_1} \oplus \mathcal{A} P_{II\omega} \oplus \mathcal{A} P_{III}$$

where $\mathcal{A} P_I$ is of type I or $P_I = 0$, $\mathcal{A} P_{II_1}$ is of type II$_1$ or $P_{II_1} = 0$, $\mathcal{A} P_{II\omega}$ is of type II$\omega$ or $P_{II\omega} = 0$ and $\mathcal{A} P_{III}$ is of type III or $P_{III} = 0$.

**Definition 2.7.** A von Neumann algebra is called a properly infinite von Neumann algebra if for any non-zero projection $P$ in the center of $\mathcal{A}$, there is a projection $Q$ in $\mathcal{A}$ such that $P \sim Q \leq P$.

Any local algebra in algebraic quantum field theory is properly infinite (Baumgartel, 1995, Corollary 1.11.6).

**Definition 2.8.** A state $\rho$ on a von Neumann algebra $\mathcal{A}$ is called a normal state if there is a density operator $D$ such that $\rho(A) = \text{tr}(DA)$ for any operator $A \in \mathcal{A}$.

**Definition 2.9.** Let $\rho$ be a normal state on a von Neumann algebra $\mathcal{A}$. Define

$$P_{\rho} := \sup\{P \mid P \text{ is a projection of } \mathcal{A} \text{ such that } \rho(P) = 0\}$$

$I - P_{\rho}$ is called the support of $\rho$. 

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Definition 2.10. Let $\rho$ be a normal state on a von Neumann algebra $\mathcal{M}$. Define
$$
\mathcal{E}_{\rho,\mathcal{M}} := \{ A \in \mathcal{M} \mid \rho(AB - BA) = 0 \text{ for all } B \in \mathcal{M} \}.
$$
$\mathcal{E}_{\rho,\mathcal{M}}$ is called the centralizer of $\rho$ of $\mathcal{M}$.

A centralizer of any normal state on a von Neumann algebra $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{M}$ (Kitajima, 2004, Lemma 8).

2.2. Lattice theory

Definition 2.11. A set $\mathcal{Q}$ is called a partially ordered set if there is a binary relation $\leq$ on $\mathcal{Q}$ which satisfies the following conditions:

1. $\forall a \in \mathcal{Q} \ [a \leq a]$;
2. $\forall a, b \in \mathcal{Q} \quad [[a \leq b \text{ and } b \leq a] \Rightarrow a = b]$;
3. $\forall a, b, c \in \mathcal{Q} \quad [[a \leq b \text{ and } b \leq c] \Rightarrow a \leq c]$.

Definition 2.12. A partially ordered set $\mathcal{Q}$ is called a lattice if any elements $a$ and $b$ in $\mathcal{Q}$ have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. The least element and the greatest element, if they exist, are denoted by 0 and 1 respectively.

Definition 2.13. A non-empty subset $I$ of a lattice $\mathcal{Q}$ is called an ideal of $\mathcal{Q}$ if $I$ satisfies the following conditions:

1. $a \in I$ and $b \leq a$ imply $b \in I$.
2. $a \in I$ and $b \in I$ imply $a \vee b \in I$.

Definition 2.14. A lattice $\mathcal{Q}$ with 0 and 1 is called orthocomplemented when there is a mapping $a \mapsto a^\perp$ from $\mathcal{Q}$ into itself satisfying the following conditions:

1. $a \vee a^\perp = 1$ and $a \wedge a^\perp = 0$;
2. $a \leq b$ implies $a^\perp \geq b^\perp$;
3. $(a^\perp)^\perp = a$ for any element $a \in \mathcal{Q}$.

When $a \leq b^\perp$, we write $a \perp b$.

Definition 2.15. An orthocomplemented lattice $\mathcal{Q}$ is called an orthomodular lattice when $b = a \vee (b \wedge a^\perp)$ for any elements $a, b \in \mathcal{Q}$ such that $a \leq b$.

When $\mathcal{M}$ is a von Neumann algebra, $\mathcal{P}(\mathcal{M})$ is an orthomodular lattice.

Definition 2.16. An orthocomplemented lattice $\mathcal{Q}$ is called a Boolean lattice when

\[
(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)
\]

\[
(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)
\]
for any elements $a$, $b$ and $c$ in $\mathcal{L}$.

As easily seen, a Boolean lattice is an orthomodular lattice.

3. von Neumann

In this section we see von Neumann’s ‘impossibility proof’ from an operator algebraic point of view (Proposition 3.1).

**Definition 3.1.** Let $\mathcal{M}$ be a von Neumann algebra. A mapping $\text{Exp}$ from the set of all self-adjoint operators in $\mathcal{M}$ to $\mathbb{R}$ is called a dispersion-free expectation value assignment on $\mathcal{M}$ if it satisfies following conditions.

1. $\text{Exp}(aR + bS) = a\text{Exp}(R) + b\text{Exp}(S)$ for any self-adjoint operators $R, S$ in $\mathcal{M}$ and for any $a, b \in \mathbb{R}$.
2. $\text{Exp}(R^2) = [\text{Exp}(R)]^2$ for any self-adjoint operator $R$ in $\mathcal{M}$.
3. $\text{Exp}(I) = 1$.

Von Neumann defined a dispersion-free expectation value assignment on the set $\mathcal{B}(\mathcal{H})$ of all bounded operators on an infinite dimensional Hilbert space $\mathcal{H}$ (von Neumann, 1955, pp. 311-312) and showed there is no dispersion-free expectation value assignment on $\mathcal{B}(\mathcal{H})$ (von Neumann, 1955, p. 321). In Proposition 3.1 we state this fact more generally.

**Lemma 3.1.** Let $\text{Exp}$ be a dispersion-free expectation value assignment on a von Neumann algebra $\mathcal{M}$. Then $\text{Exp}$ can be extended to a dispersion-free state on $\mathcal{M}$.

**Proof.** Define a mapping $\omega$ from $\mathcal{M}$ to $\mathbb{C}$ as

$$\omega(A) := \text{Exp}\left(\frac{A^* + A}{2}\right) + i\text{Exp}\left(\frac{i(A^* - A)}{2}\right) \quad (\forall A \in \mathcal{M}).$$

Let $A$ and $B$ be any operators in $\mathcal{M}$ and let $c$ be any complex number. $c$ can be expressed as $a + ib$ ($a, b \in \mathbb{R}$). Since

$$\omega(A^*A) = \text{Exp}(A^*A) = \text{Exp}(|A|^2) = [\text{Exp}(|A|)]^2 \geq 0,$$
\( \omega(cA) = \exp\left( \frac{(a-ib)A^* + (a+ib)A}{2} \right) + \frac{i \exp\left( \frac{i(a-ib)A^* - (a+ib)A}{2} \right)}{2} \)

\( = \exp\left( \frac{aA^* + A}{2} - \frac{ib(A^* - A)}{2} \right) + \frac{i \exp\left( \frac{iA^* - A}{2} \right)}{2} + \frac{a \exp\left( \frac{iA^* - A}{2} \right)}{2} + \frac{i \exp\left( \frac{iA^* - A}{2} \right)}{2} \)

\( = a \exp\left( \frac{A^* + A}{2} \right) - \frac{ib}{2} \exp\left( \frac{i(A^* - A)}{2} \right) + \frac{a}{2} \exp\left( \frac{i(A^* - A)}{2} \right) + \frac{ib}{2} \exp\left( \frac{i(A^* - A)}{2} \right) \)

and

\( \omega(A + B) = \exp\left( \frac{(A + B)^* + (A + B)}{2} \right) + \frac{i \exp\left( \frac{i((A + B)^* - (A + B))}{2} \right)}{2} \)

\( = \exp\left( \frac{A^* + A + B^* + B}{2} \right) + \frac{i \exp\left( \frac{i(A^* - A) + i(B^* - B)}{2} \right)}{2} \)

\( = \exp\left( \frac{A^* + A}{2} + \frac{iA^* - A}{2} \right) + \exp\left( \frac{B^* + B}{2} \right) \)

\( = \frac{i(B^* - B)}{2} \)

\( = \omega(A) + \omega(B), \)

\( \omega \) is a state on \( \mathcal{M} \). For any self-adjoint operator \( R \), \( \omega(R^2) = [\omega(R)]^2 \) because \( \omega(R) = \exp(R) \). Therefore \( \omega \) is a dispersion-free state on \( \mathcal{M} \).

We prove the following theorem, making reference to the proof of Lemma 19 of Doring (2005).

**Proposition 3.1**. There is no dispersion-free expectation value assignment on a von Neumann algebra without direct summand of type \( I_1 \).

**Proof.** 1. Let \( \mathcal{M} \) be a von Neumann algebra of type \( I_n \) where \( 2 \leq n < \infty \). Suppose that there is a dispersion-free expectation value assignment on \( \mathcal{M} \). There is a set \( \{ P_1, P_2, \ldots, P_n \} \) (\( 2 \leq n < \infty \)) of mutually orthogonal and equivalent projections in \( \mathcal{M} \) such that \( \sum_{i=1}^{n} P_i = I \). By Lemma 3.1 there is a dispersion-free state \( \omega \) on \( \mathcal{M} \). Since \( \sum_{i=1}^{n} \omega(P_i) = 1 \), only one of \( \omega(P_k) \)'s, say \( \omega(P_j) \), is unity, and \( \omega(P_k) = 0 \) (\( k \neq j \)). There are a partial isometry \( V \) in \( \mathcal{M} \) such that \( P_j = VV^* \) and \( P_k = V^*V \). By Lemma 2 of Misra (1967) \( \omega(P_j) = \omega(VV^*) = \omega(V) \omega(V^*) = \omega(V^*) \omega(V) = \omega(V^*V) = \omega(P_k) \). This is a contradiction. So there is no dispersion-free expectation value assignment on \( \mathcal{M} \).  

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2. Let \( \mathfrak{M} \) be either a properly infinite von Neumann algebra\(^1\) or a von Neumann algebra of type \( II_1 \). By Lemma 6.3.3 and Lemma 6.5.6 of Kadison and Ringrose (1997) there is a projection \( P \) in \( \mathfrak{M} \) such that \( P \sim P^\perp \). It follows that there is a partial isometry \( V \) in \( \mathfrak{M} \) such that \( P = VV^* \) and \( P^\perp = V^*V \). We can prove that there is no dispersion-free expectation value assignment on \( \mathfrak{M} \) in a similar manner to the case where \( \mathfrak{M} \) is of type \( I_n \) (\( 2 \leq n < 1 \)).

3. By Proposition 2.1 there is no dispersion-free expectation value assignment on a von Neumann algebra without type \( I_1 \) direct summand.

\[ \square \]

4. Jauch and Piron

Jauch and Piron (1963) evaluated von Neumann's argument (especially the condition 1 in Definition 3.1) as follows (see also Bell, 2004, pp. 4–5).

Concerning states, it is difficult to justify the additivity of the expectation values on non-compatible observables. (p. 828)

Then they defined the following truth-value assignment (p. 833).

**Definition 4.1 (Jauch-Piron truth-value assignment).** Let \( \mathfrak{Q} \) be an orthomodular lattice.\(^2\) We call a mapping \( \mu \) from \( \mathfrak{Q} \) to \( \{0, 1\} \) a Jauch-Piron truth-value assignment if \( \mu \) satisfies following conditions:

1. \( \mu(1) = 1 \);
2. \( \mu(a \lor b) = \mu(a) + \mu(b) \) for any elements \( a, b \in \mathfrak{Q} \) such that \( a \perp b \);
3. If \( \mu(a) = \mu(b) = 1 \), then \( \mu(a \land b) = 1 \).

Jauch and Piron proved that there is no Jauch-Piron truth-value assignment in quantum mechanics (Jauch and Piron, 1963, Theorem II). We prove Theorem II of Jauch and Piron (1963) in another way (Proposition 4.1).

**Lemma 4.1.** Let \( \mu \) be a Jauch-Piron truth-value assignment on an orthomodular lattice \( \mathfrak{Q} \). For any elements \( a, b \in \mathfrak{Q} \),

\[ \mu(1) = 1; \]
\[ \mu(a \lor b) = \mu(a) + \mu(b) \text{ for any elements } a, b \in \mathfrak{Q} \text{ such that } a \perp b; \]
\[ \text{If } \mu(a) = \mu(b) = 1, \text{ then } \mu(a \land b) = 1. \]

\(^1\) A type \( I_n \) von Neumann algebra, a type \( II_n \) von Neumann algebra and a type \( III \) von Neumann algebra are properly infinite.

\(^2\) Jauch and Piron (1963) defined an orthocomplemented lattice \( \mathfrak{Q} \) that satisfies the following condition (p. 831).

\[ (P) \quad \forall a, b \in \mathfrak{Q}[a \leq b \rightarrow (a \land b^\perp) \lor b = (b \land a^\perp) \lor a] \]

As is easily seen, an orthocomplemented lattice that satisfies condition \( (P) \) is an orthomodular lattice.
1. \( \mu(a^+) = 1 - \mu(a) \);
2. \( a \leq b \Rightarrow \mu(a) \leq \mu(b) \);
3. \( \mu(a \wedge b) = \mu(a) \mu(b) \).

**Proof.**
1. Since \( 1 = \mu(a \vee a^+) = \mu(a) + \mu(a^+) = 1 - \mu(a) \), \( \mu(a^+) = 1 - \mu(a) \).
2. If \( a \leq b \), then \( a \leq (b^+)^+ \), that is, \( a \perp b^+ \). Since \( 1 \geq \mu(a \vee b^+) = \mu(a) + \mu(b^+) = 1 + \mu(a) - \mu(b) \), \( \mu(a) \leq \mu(b) \).
3. If either \( \mu(a) \) or \( \mu(b) \) is 0, \( \mu(a \wedge b) = 0 \) since \( a \wedge b \leq a \) and \( a \wedge b \leq b \). Therefore \( \mu(a \wedge b) = \mu(a) \mu(b) \).

**Proposition 4.1.** Let \( \Omega \) be an orthomodular lattice. The following conditions are equivalent.

1. For any non-zero element \( a \in \Omega \) there is a Jauch-Piron truth-value assignment \( \mu \) such that \( \mu(a) = 1 \).
2. \( \Omega \) is a Boolean lattice.

**Proof.**
1\( \Rightarrow \)2 Let \( a \) and \( b \) be any elements in \( \Omega \). Since \( a \wedge b \leq a \) and \( a \wedge b^+ \leq a \), \( (a \wedge b) \vee (a \wedge b^+) \leq a \). Because \( \Omega \) is an orthomodular lattice,
\[
a = (a \wedge b) \vee (a \wedge b^+) \vee (a \wedge ((a \wedge b) \vee (a \wedge b^+))^+) .
\]
Suppose that \( a \wedge ((a \wedge b) \vee (a \wedge b^+))^+ \neq 0 \). It follows that there is a Jauch-Piron truth-value assignment \( \mu \) such that \( \mu(a \wedge ((a \wedge b) \vee (a \wedge b^+))^+) = 1 \). By Lemma 4.1 \( \mu(a) = 1 \) since \( a \wedge ((a \wedge b) \vee (a \wedge b^+))^+ \leq a \). Further,
\[
1 = \mu(a \wedge ((a \wedge b) \vee (a \wedge b^+))^+) = \mu(a \wedge ((a \wedge b)^+ \wedge (a \wedge b^+)^+)) = \mu(a)(1 - \mu(a)\mu(b))(1 - \mu(a)(1 - \mu(b))) \quad (\because \text{Lemma 4.1})
\]
\[
= (1 - \mu(b))\mu(b) \quad (\because \mu(a) = 1)
\]
\[
= 0 \quad (\because \mu(b) = 0 \text{ or } 1).
\]
This is a contradiction. Therefore \( a \wedge ((a \wedge b) \vee (a \wedge b^+))^+ = 0 \), that is, \( a = (a \wedge b) \vee (a \wedge b^+) \). By Theorem 36.7 of Maeda (1970), \( \Omega \) is a Boolean lattice.

2\( \Rightarrow \)1 This we prove, making reference to Lemma 11.3 of Maeda (1980).

Let \( a \) be any non-zero element in \( \Omega \) and let \( \mathcal{J} \) be the set of all proper ideals of \( \Omega \). Define \( \mathcal{J}_a \) as
\[
\mathcal{J}_a := \{ J \in \mathcal{J} \mid a \in \mathcal{J} \}.
\]
\( \mathcal{J}_a \) is partially ordered by set inclusion. Every linearly ordered subset \( \mathcal{J}_a' \) of \( \mathcal{J}_a \) has an upper bound in \( \mathcal{J}_a \) because \( \bigcup \mathcal{J}_a' \) is a proper ideal such that \( a \in \bigcup \mathcal{J}_a' \). \( \mathcal{J}_a \) is not empty because \( \{0\} \in \mathcal{J}_a \). Zorn's lemma implies that \( \mathcal{J}_a \) has a maximal element \( \mathcal{J}_a' \).

Let \( b \) and \( c \) be any elements in \( \Omega \) such that \( b \wedge c \in J_a \) and \( b \in J_a \). Suppose that \( c \in J_a \). Define \( J' \) as
$J' := \{ y \in \mathfrak{B} \mid \exists x \in J_a \ [ y \leq b \lor x] \}.$

$J'$ is an ideal. $J_a \subseteq J'$ since $b \in J'$. Because $J'$ is a maximal element in $\mathfrak{B}$, $a \subseteq J'$. Then there is an element $x_1$ in $J_a$ such that $a \leq b \lor x_1$. Similarly there is an element $x_2$ in $J_a$ such that $a \leq c \lor x_2$. Because $\mathfrak{B}$ is a Boolean lattice,

$$a \leq (b \lor x_1 \lor x_2) \land (c \lor x_1 \lor x_2) = (b \land c) \lor (x_1 \lor x_2) \in J_a.$$  

This contradicts that $a \notin J_a$. Therefore $c \in J_a$. Therefore any elements $b, c \in \mathfrak{B},$

$$b \land c \in J_a \implies b \in J_a \text{ or } c \in J_a.$$  

If $b \notin J_a$, then $b^+ \notin J_a$ since $b \in J_a$ or $b^+ \in J_a$. Therefore for any elements $d, e \in \mathfrak{B}$ such that $d \perp e$ and $e \in J_a$, $d \in J_a$ because $e^+ \in J_a$ and $d \leq e^+$. Define a mapping $\mu_a$ from $\mathfrak{B}$ to $\{0, 1\}$ as

$$\mu_a(x) = \begin{cases} 1 & (x \in J_a) \\ 0 & (x \notin J_a). \end{cases}$$

Then $\mu_a$ is a Jauch–Piron truth-value assignment, and $\mu_a(a) = 1$.

In quantum theory, the set of all observational propositions is not a Boolean lattice. Therefore there is no Jauch–Piron truth-value assignment in quantum theory.

5. Gleason

Bell (2004) stated that the condition 3 of Definition 4.1 is a 'quite peculiar property' of quantum mechanical states (p. 6). If this condition cannot be justified, the argument of Jauch and Piron does not hold. Then we define a truth-value assignment which is not bounded the condition on incompatible observational propositions as follows.

Definition 5.1 (A finitely additive truth-value assignment). A mapping $\mu$ from the set of all projections in a von Neumann algebra $\mathfrak{N}$ to $\{0, 1\}$ is called a finitely additive truth-value assignment on $\mathfrak{N}$ if $\mu$ satisfies following conditions:

1. $\mu(I) = 1$;
2. For any mutually orthogonal projections $P, Q \in \mathfrak{N}$, $\mu(P \lor Q) = \mu(P) + \mu(Q)$.

Hamhalter (1993) proved the following lemma using a generalised Gleason’s theorem (see, e.g. Maeda, 1990, Theorem 12.1 or Hamhalter, 2003, Chapter 5).

Lemma 5.1 (Hamhalter, 1993, Lemma 5.1). Let $\mathfrak{N}$ be a von Neumann algebra
without direct summand of type $I_2$ and let $\mu$ be a finitely additive truth-value
assignment on $\mathcal{A}$. Then $\mu$ can be extended to a dispersion-free state on $\mathcal{A}$.

The following lemma was proved in Corollary 5.2 of Hamhalter (1993). We prove it in another way.

Lemma 5.2. Let $\mathcal{A}$ be a von Neumann algebra without direct summand of type $I_2$,
let $\mu$ be a finitely additive truth-value assignment on $\mathcal{A}$ and let $P$, $Q$ be any
projections in $\mathcal{A}$. If $\mu(P) = \mu(Q) = 1$, then $\mu(P \land Q) = 1$.

Proof. Let $\mu$ be a finitely additive truth-value assignment on $\mathcal{A}$. By Lemma 5.1,
$\mu$ can be extended to a dispersion-free state on $\mathcal{A}$. Let $P$, $Q$ be any projections in
$\mathcal{A}$. By Theorem 6.1.7 of Kadison and Ringrose (1997), $(P \lor Q - Q) \sim (P - P \land Q)$.
It follows that there is a partial isometry $V$ in $\mathcal{A}$ such that $P \lor Q - Q = VV^*$, $P - P \land Q = V^*V$.
Because
\[
\omega(VV^*) = \omega(V)\omega(V^*) = \omega(V^*)\omega(V) = \omega(V^*V),
\]
by Lemma 2 of Misra (1967),
\[
\omega(P \land Q) = \omega(P - V^*V) = \omega(P) - \omega(V^*V)
= \omega(P) - \omega(VV^*) = \omega(P) - \omega(P \lor Q - Q)
= \omega(P) + \omega(Q) - \omega(P \lor Q).
\]
Therefore if $\omega(P) = \omega(Q) = 1$, then $\omega(P \land Q) = 1$ (cf. Hamhalter, 2003, Example 10.1.1).

We can prove the following theorem using Proposition 4.1.

Theorem 5.1. Let $\mathcal{A}$ be a von Neumann algebra without direct summand of type $I_2$.
Then the following conditions are equivalent.

1. For any non-zero projection $P$ in $\mathcal{A}$ there is a finitely additive truth-value
   assignment $\mu$ such that $\mu(P) = 1$.
2. $\mathcal{A}$ is an Abelian von Neumann algebra.

Proof. 1$\Rightarrow$2 By Lemma 5.2 any finitely additive truth-value assignment on $\mathcal{A}$ is a
Jauch–Piron truth-value assignment on $\mathcal{P} (\mathcal{A})$. Since $\mathcal{P} (\mathcal{A})$ is a Boolean
lattice by Proposition 4.1, $P = (P \land Q) \lor (P \land Q^\perp)$ for any projections $P$, $Q \in \mathcal{P}$
(\mathcal{A}). Then $PQ = QP$. Therefore $\mathcal{A}$ is an Abelian von Neumann algebra.

2$\Rightarrow$1 Let $P$, $Q$ be any projections in $\mathcal{A}$. Since $PQ = QP$, $P = (P \land Q) \lor (P \land Q^\perp)$ by
Proposition 2.5.3 of Kadison and Ringrose (1997). Then $\mathcal{P} (\mathcal{A})$ is a Boolean
lattice by Theorem 36.7 of Maeda (1970). Therefore for any non-zero projection $P$ in $\mathcal{A}$
there is a finitely additive truth-value assignment $\mu$ such that $\mu(P) = 1$ by Proposition 4.1.

We can prove the following theorem using Proposition 3.1 (cf. Hamhalter,
Theorem 5.2. Let $\mathcal{A}$ be a von Neumann algebra that has neither direct summand of type $I_1$ nor direct summand of type $I_2$. Then there is no finitely additive truth-value assignment on $\mathcal{A}$.

Proof. Suppose that there is a finitely additive truth-value assignment on a von Neumann algebra $\mathcal{A}$ which has neither direct summand of type $I_1$ nor direct summand of type $I_2$. It follows that there is a dispersion-free state on $\mathcal{A}$ by Lemma 5.1. This contradicts Proposition 3.1, since a dispersion-free state on $\mathcal{A}$ is a dispersion-free expectation value assignment on $\mathcal{A}$. Therefore there is no finitely additive truth-value assignment on $\mathcal{A}$. □

By Theorem 5.2 no finitely additive truth-value can be assigned simultaneously to all projections on the Hilbert space whose dimension is greater than 2. Moreover, no finitely additive truth-value can be assigned to all projections which belong to any local algebra in algebraic quantum field theory because any local algebra is a properly infinite von Neumann algebra (Baumgartel, 1995, Corollary 1.11.6).

6. Clifton's interpretation of algebraic quantum field theory

Theorem 5.2 shows that there is no finitely additive truth-value assignment on any local algebra. But some subalgebra of a local algebra can be assigned a finitely additive truth-value and a state on that subalgebra can be regarded as a mixture of finitely additive truth-value assignments. Halvorson and Clifton (1999) called such a subalgebra a beable algebra, following Bell's terminology (Bell, 2004, chapters 5, 7 and 19). For example, Bell (2004) said the following:

We will exclude the notion of 'observable' in favour of that of 'beable'. The beables of the theory are those elements which might correspond to elements of reality, to things which exist. Their existence does not depend on 'observation'. (p. 174)

A beable algebra is defined the following.

Definition 6.1 (Halvorson and Clifton, 1999, p. 2447). Let $\mathcal{A}$ be a unital $C^*$-algebra, let $\mathcal{B}$ be a unital $C^*$-subalgebra of $\mathcal{A}$ and let $\rho$ be a state on $\mathcal{A}$. $\mathcal{B}$ is a beable algebra for $\rho$ if and only if $\rho \mid \mathcal{B}$ is a mixture of dispersion-free states, that is, if and only if there is a probability measure $I$ on the space $\mathcal{S}$ of dispersion-free states on $\mathcal{B}$ such that
\[ \rho(A) = \int_{\mathcal{B}} \omega_s(A) d\mu(s) \quad (\forall A \in \mathcal{B}). \]

Clifton (2000) determined the maximal beable algebra for each faithful normal state in a von Neumann algebra under the condition that the beable algebra is determined solely in terms of the faithful normal state and the algebraic structure of the von Neumann algebra (Clifton, 2000, Proposition 1). The following theorem is a generalized Clifton’s theorem.

**Theorem 6.1 (Kitajima, 2004, Theorem 11).** Let \( \mathfrak{N} \) be a \( \sigma \)-finite von Neumann algebra\(^3\) on a Hilbert space, let \( \rho \) be a normal state on \( \mathfrak{N} \) and let \( S \) be the support of \( \rho \). Let \( \mathfrak{B} \) be a C*-subalgebra of \( \mathfrak{N} \) and let \( \mathfrak{B} \) satisfy the following conditions.

1. \( \mathfrak{B} \) is a beable algebra for \( \rho \).
2. \( \sigma(\mathfrak{B}) = \mathfrak{B} \) for any automorphism \( \sigma \) on \( \mathfrak{N} \) such that \( \rho \circ \sigma = \rho \).
3. \( \mathfrak{B} \) is maximal with respect to Conditions 1 and 2.

Then \( \mathfrak{B} \) can be uniquely expressed as \( S^+ \mathfrak{N} S^+ \oplus \mathcal{Z}(\mathfrak{E}_{\rho,\mathfrak{N}})S \), where \( \mathcal{Z}(\mathfrak{E}_{\rho,\mathfrak{N}}) \) is the center of the centralizer \( \mathfrak{E}_{\rho,\mathfrak{N}} \) of \( \rho \) of \( \mathfrak{N} \).

The set of all projections in \( S^+ \mathfrak{N} S^+ \oplus \mathcal{Z}(\mathfrak{E}_{\rho,\mathfrak{N}})S \) can be regarded as the set of definite propositions in a local algebra which are determined solely in terms of the normal state \( \rho \) on that local algebra. When \( \mathfrak{N} \) is the set of all bounded operators on a separable Hilbert space, the set of all projections in \( S^+ \mathfrak{N} S^+ \oplus \mathcal{Z}(\mathfrak{E}_{\rho,\mathfrak{N}})S \) coincides with the set \( \text{Def}_{\rho,\mathfrak{N}}(W) \) in Clifton (1995)) of all projections which have simultaneously definite values under the Kochen–Dieks modal interpretation (Kitajima, 2004, Corollary 12). Therefore Clifton’s interpretation of algebraic quantum field theory is an extension of the Kochen–Dieks modal interpretation of nonrelativistic quantum mechanics to algebraic quantum field theory.

There are two problems in this interpretation. One problem concerns faithful normal states of which centralizers are trivial. When a von Neumann algebra \( \mathfrak{N} \) is a hyperfinite type III\(_1\) factor,\(^5\) there are many faithful normal states of which centralizers are trivial (Clifton, 2000, Proposition 3). The maximal beable algebra for such a faithful normal state which is determined solely in terms of that state have no nontrivial definite propositions.

Another problem concerns an invalid truth-value assignment. If we regard dispersion-free states on \( S^+ \mathfrak{N} S^+ \oplus \mathcal{Z}(\mathfrak{E}_{\rho,\mathfrak{N}})S \) as truth-value assignments, it is natural to

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\(^3\) Since local algebras have cyclic and separating vectors by the Reeh–Schlieder theorem (Baumgartel, 1995, Corollary 1.3.3), these algebras are \( \sigma \)-finite von Neumann algebras (Bratteli and Robinson, 1987, Proposition 2.5.6).

\(^4\) To represent that \( \mathfrak{B} \) is determined solely in terms of \( \rho \) and the algebraic structure of \( \mathfrak{N} \), Clifton assumed this condition in Proposition 1 of Clifton (2000).

\(^5\) In physically reasonable models, any local algebra is a hyperfinite type III\(_1\) factor (Haag, 1996, section V.6).
think that for any dispersion-free state $\omega$, $\bigvee_{i \in \mathbb{N}} P_i$ is false ($\omega(\bigvee_{i \in \mathbb{N}} P_i) = 0$) whenever all projections in a set $\{P_i \mid i \in \mathbb{N}\}$ of mutually orthogonal projections in $S^+ \otimes S^+ \oplus \mathcal{B}(\mathcal{L}_\rho, \mathcal{H})$ are false ($\omega(P_i) = 0$ for any $i \in \mathbb{N}$). If $\omega$ is a normal state, this holds. But when $\mathcal{B}(\mathcal{L}_\rho, \mathcal{H})$ contains a set $\{P_i \mid i \in \mathbb{N}\}$ of mutually orthogonal countably infinite non-zero projections, there is a dispersion-free state $\omega'$ on $S^+ \otimes S^+ \oplus \mathcal{B}(\mathcal{L}_\rho, \mathcal{H})$ such that $\omega'(\bigvee_{i \in \mathbb{N}} P_i) = 1$ and $\omega'(P_i) = 0$ for any $i \in \mathbb{N}$ (Kitajima, 2005, Proposition 3.1). Then $\bigvee_{i \in \mathbb{N}} P_i$ is true and $P_i$ is false for any $i \in \mathbb{N}$. Therefore this state cannot be regarded as a valid truth-value assignment.

7. Concluding Remarks

In section 5 we pointed out that there is no hidden variable in algebraic quantum field theory as well as nonrelativistic quantum mechanics. Therefore there are interpretation problems in algebraic quantum field theory as well as nonrelativistic quantum mechanics.

In section 6 we saw Clifton’s interpretation of algebraic quantum field theory which is an extension of the Kochen–Dieks modal interpretation of nonrelativistic quantum mechanics to algebraic quantum field theory, and saw problems with this interpretation. In nonrelativistic quantum mechanics there are some interpretations other than the Kochen–Dieks modal interpretation. It therefore needs to be examined whether algebraic quantum field theory can be interpreted in another way.

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