A Note on Constructive Ordinals*

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In §11 of [6] G. Takeuti and the author presented a system of notations (in Putnam's style [4]) for extended constructive ordinals (cf. [2], [3]) as a candidate for a system of notations not expressible in both 2-function-quantifier forms.

In this note we shall define another system of notations for extended constructive ordinals, which contains the former one as a subsystem. We shall denote the former system by $\mathfrak{S}$ and the new one by $\hat{\mathfrak{S}}$. Let $\mathfrak{T}$ be a system of notations for extended constructive ordinals. Throughout this note we shall call an ordinal number constructively accessible in the sense of $\mathfrak{T}$, if it has a notation in $\mathfrak{T}$. Thus ' α is constructively accessible in the senses of $\mathfrak{S}$ ' means ' α is constructively accessible' in [6]. Therefore we may omit the phrase 'in the sense of $\mathfrak{S}$ '. By $A_{\mathfrak{T}}(\alpha)$ we shall understand the predicate ' α is a constructively accessible ordinal number in the sense of $\mathfrak{T}$ '. In the following we shall abbreviate $A_{\mathfrak{S}}(\alpha)$ and $A_{\hat{\mathfrak{S}}}(\alpha)$ by $A(\alpha)$ and $\hat{A}(\alpha)$, respectively.

Our results are: (i) $A(\alpha)$ is expressible in Π_2^1 (for notations like this cf. [1]) as a predicate of a function variable (this expression will be called the first form); (ii) $A(\alpha)$ is expressible in Π_2^1 as a predicate of only number variables (the second form); (iii) $\hat{A}(\alpha)$ is expressible in Π_1^1 as a predicate of a function variable (i.e. in $\Pi_1^{1,1}$ by Tugué's notation in [7]); (iv) $\hat{A}(\alpha)$ is expressible in $\Sigma_2^1 \cap \Pi_2^1$ as a predicate of only number variables (i.e. in $\Sigma_2^{1,0} \cap \Pi_2^{1,0}$). It should be noticed that the statement (iii) for $A_C(\alpha)$ is also true where C is the system of Kreider-Rogers [3] or Putnam [4]: We can express the predicate $A_C(\alpha)$ in Π_1^1 as a predicate of a function variable. The statement (iv) for $A_C(\alpha)$ has been proved in [3].

We shall use $\omega^\dagger$ and $\omega^\hat{\dagger}$ to denote the least ordinal numbers not constructively accessible in the sense of $\mathfrak{S}$ and $\hat{\mathfrak{S}}$, respectively. In the following we shall identify a (partial) recursive function with its Gödel number and use the notation $\varphi(\Phi: a_1, \dots, a_n)$, where Φ is a list of predicates or functions, to denote a function φ recursive in Φ .

The author wishes to express her heart-felt thanks to Prof. G. Takeuti for his valuable advice on this subject.

* This was presented to the Symposium of the Foundations of Mathematics held at Katada, Japan, 1962. Recently we were told that, in his paper "On hierarchies and system of notations", Putnam defined a far rich schema D for systems of notations and proved that the predicate ' α belongs to such a system' is $\Sigma_2^1 \cap \Pi_2^1$.

1. Definition of constructively accessible ordinal numbers.

First we shall restate the definition of constructively accessible ordinal numbers (with correction of some miswritings in [6]). An ordinal number α is called *constructively accessible* if the set N_α can be defined to satisfy the conditions given below. To state the definition of N_α , we shall write C_α for $\bigcup_{\beta < \alpha} N_\beta$, and also $n \leq_{C_\gamma} m$ to mean that for some ordinal numbers α and β less than γ we have $m \in N_\alpha$, $n \in N_\beta$ and that $\alpha \leq \beta$. A partial recursive function f is called an *order-preserving cofinal mapping* (abbreviated o.p.c.m.) from C_β into C_α , if and only if the following conditions are satisfied:

- (a) If $n \in C_\beta$, then $f(n) \in C_\alpha$.
- (b) Let $n \in C_\beta$ and $m \in C_\beta$. Then $n \leq_{C_\beta} m$ if and only if $f(n) \leq_{C_\alpha} f(m)$.
- (c) For each $n \in C_\alpha$, there exists an $m \in C_\beta$ such that $n \leq_{C_\alpha} f(m)$.

Definition of N_α :

- (1) There exists two functions d and e with resp. one and two arguments which are partial recursive uniformly in predicates $P(a)$ and $Q(a, b)$ and satisfy

(*) If A is a set and $\leq_A$ is a well-ordering on A , then $\lambda x d(A, \leq_A: x)$ and $\lambda x y e(A, \leq_A: x, y)$ are completely defined and $\lambda x y (e(A, \leq_A: x, y) = 0)$ is a well-ordering on the set $\{x | d(A, \leq_A: x) = 0\}$.

- (2) There exist two partial recursive functions p and q with two arguments.

Case 1. $N_0 = \{1\}$.

Case 2. $\alpha = \beta + 1$, where N_γ is already defined for all $\gamma < \alpha$. Then $N_\alpha = \{2^x | x \in N_\beta \text{ and } 2^x \notin C_\alpha\}$.

Case 3. α is a limit ordinal number such that N_γ is already defined for all $\gamma < \alpha$, and there exist an ordinal number $\beta < \alpha$ such that $a \in N_\beta$ for some number a and a partial recursive function f which gives an o.p.c.m. from C_β into C_α such that $p(a, f)$ is defined and $p(a, f) \notin C_\alpha$. Then

$N_\alpha = \{p(a, f) | a \in N_{\beta_0}, \text{ where } \beta_0 \text{ is the least ordinal number satisfying the above conditions, } a \text{ is a number required to exist for } \beta_0 \text{ and } f \text{ is a partial recursive function which gives an o.p.c.m. from } C_{\beta_0} \text{ into } C_\alpha \text{ such that } p(a, f) \text{ is defined and } p(a, f) \notin C_\alpha\}$.

Case 4. α is a limit ordinal number such that N_γ is already defined for all $\gamma < \alpha$, there exists no β as in Case 3, but there exists a number a satisfying the following conditions: $d(C_\alpha, \leq_{C_\alpha}: a) = 0$ and there exists a partial recursive function f which gives an o.p.c.m. from C_α into C_α such that $q(a, f)$ is defined and $q(a, f) \in C_\alpha$. Then,

$N_\alpha = \{q(a, f) | a \text{ is the least number satisfying the above conditions in the sense of } \lambda x y (e(C_\alpha, \leq_{C_\alpha}: x, y) = 0) \text{ and } f \text{ is a partial recursive function which gives an o.p.c.m. from } C_\alpha \text{ into } C_\alpha \text{ such that } q(a, f) \text{ is defined and } q(a, f) \in C_\alpha\}$.

2. Definition of the system $\hat{\mathcal{C}}$.

We shall modify the above system of notations. An ordinal number is called

constructively accessible in the sense of $\hat{\mathbb{C}}$, if the set $\hat{N}_\alpha$ can be defined to satisfy the conditions given below. To state the definition of $\hat{N}_\alpha$, we shall write $\hat{C}_\alpha$ for $\bigcup_{\beta < \alpha} \hat{N}_\beta$.

Definition of $\hat{N}_\alpha$.

- (1) There exist two functions d and e with resp. one and two arguments which are partial recursive uniformly in predicates $P(a)$ and $Q(a, b)$.
- (2) There exist two partial recursive functions p and q with two arguments.

We divide into four cases to define $\hat{N}_\alpha$ similarly as in the definition of N_α . In Cases 1–3, $\hat{N}_\alpha$ is defined quite similarly in the corresponding cases of N_α : We have only to replace the letters N and C by $\hat{N}$ and $\hat{C}$ in the definition of N_α . Let $\hat{E}_\alpha$ be the set of a such that $d(\hat{C}_\alpha, \leq_{\hat{C}_\alpha} : a)$ is defined, $d(\hat{C}_\alpha, \leq_{\hat{C}_\alpha} : a) = 0$ and there exists a partial recursive function f which gives an o.p.c.m. from C_α into C_α such that $q(a, f)$ is defined and $q(a, f) \in \hat{C}_\alpha$. By a first element of $\hat{E}_\alpha$ we shall understand such an element a of $\hat{E}_\alpha$ that $e(\hat{C}_\alpha, \leq_{\hat{C}_\alpha} : a, b) = 0$ for each $b \in \hat{E}_\alpha$, provided that $e(\hat{C}_\alpha, \leq_{\hat{C}_\alpha} : x, y)$ is defined for any pair of elements x, y of $\hat{E}_\alpha$.

Case 4. α is a limit ordinal number such that $\hat{N}_\gamma$ is already defined for all $\gamma < \alpha$, there exists no β as in Case 3, but $e(\hat{C}_\alpha, \leq_{\hat{C}_\alpha} : x, y)$ is defined for each pair of elements x, y of $\hat{E}_\alpha$ and there exists a first element of $\hat{E}_\alpha$. Then

$$\begin{aligned} \hat{N}_\alpha = \{ & q(a, f) \mid a \text{ is a first element of } \hat{E}_\alpha \text{ and } f \text{ is a partial recursive function} \\ & \text{which gives an o.p.c.m. from } \hat{C}_\alpha \text{ into } \hat{C}_\alpha \text{ such that } q(a, f) \text{ is} \\ & \text{defined and } q(a, f) \in \hat{C}_\alpha \}. \end{aligned}$$

PROPOSITION 1. $\omega^\dagger \leq \omega^\hat{\dagger}$.

3. Expression of the predicate $A(\alpha)$.

To express the predicate $A(\alpha)$ in the analytic hierarchy, we shall begin with some comments on notations. In the following d, e, p and q will work in the same way as the above; ψ gives a well-ordering of natural numbers whose order-type corresponds to a constructively accessible ordinal number, say α , for the first form, while it gives a well-ordering of natural numbers whose order type corresponds to an ordinal number α greater than any constructively accessible ordinal numbers determined in depending on given d, e, p, q for the second form; a, b, c correspond to ordinal numbers $\tilde{a}, \tilde{b}, \tilde{c}$ not greater than α ; r, s, t, u, v, w express the notations; $\varphi(a, r) = 0$ means that a expresses an ordinal number in the given well-ordering of natural numbers and r is its notation i.e. $r \in N_{\tilde{a}}$; f, g stand for the Gödel numbers of partial recursive functions which give o.p.c.m.'s.

We shall use the following abbreviations:

$$\begin{aligned} D(\psi, a, b) & \quad \text{for} & \psi(a, b) = 0 \vee \psi(b, a) = 0; \\ D(\psi, a) & \quad \text{for} \end{aligned}$$

$$\exists x D(\psi, a, x),$$

which means 'a is in the domain of ψ ';

$$\begin{aligned} W'(\psi) \quad & \text{for} \\ & \forall x \forall y (D(\psi, x) \wedge D(\psi, y) \vdash D(\psi, x, y)) \wedge \exists x D(\psi, x) \\ & \wedge \forall x \forall y \forall z (\psi(x, y) = 0 \wedge \psi(y, z) = 0 \vdash \psi(x, z) = 0) \\ & \wedge \forall \xi \exists x (\psi(\xi(x+1), \xi(x)) \neq 0 \vee \psi(\xi(x), \xi(x+1)) = 0), \end{aligned}$$

which means ' ψ gives a well-ordering of natural numbers' and which is expressible in $\Pi_1^{1,1}$;

$$\begin{aligned} S(\psi, a, b) \quad & \text{for} \\ & \psi(b, a) = 0 \wedge \psi(a, b) \neq 0 \wedge \forall x (\psi(b, x) = 0 \wedge \psi(x, b) \neq 0 \vdash \psi(a, x) = 0), \end{aligned}$$

which means ' a is a successor of b in ψ ';

$$\begin{aligned} Z(\psi, a) \quad & \text{for} \\ & \forall x (D(\psi, x) \vdash \psi(a, x) = 0), \end{aligned}$$

which means ' a is a first element in ψ ';

$$\begin{aligned} L(\psi, a) \quad & \text{for} \\ & \neg \exists x S(\psi, a, x) \wedge \neg Z(\psi, a), \end{aligned}$$

which means ' a is a limit ordinal in ψ ';

$$\begin{aligned} R(d, e) \quad & \text{for} \\ & \forall \xi \forall \eta (\forall x (\xi(x) = 0 \vdash D(\eta, x)) \wedge W'(\eta) \\ & \vdash \forall x \exists y T_1(\xi, \eta: d, x, y) \wedge \forall x \forall y \exists z T_2(\xi, \eta: e, x, y, z) \\ & \wedge \forall x (d(\xi, \eta: x) = 0 \vdash D(\lambda u v e (\xi, \eta: u, v), x)) \wedge W'(\lambda u v e (\xi, \eta: u, v))), \end{aligned}$$

which corresponds to the condition (*); ($R(d, e)$ is expressible in $\Pi_2^{1,1}$ -form, while we suspect that it is not expressible in the dual form.)

$$\begin{aligned} C(\psi, \varphi; a; b; r) \quad & \text{for} \\ & \psi(b, a) = 0 \wedge \psi(a, b) \neq 0 \wedge \varphi(b, r) = 0; \end{aligned}$$

$$\begin{aligned} C(\psi, \varphi; a; r) \quad & \text{for} \\ & \exists x C(\psi, \varphi; a, x, r), \end{aligned}$$

which means $r \in C_{\bar{a}}$;

$$\begin{aligned} \leq(\psi, \varphi; a; r, s) \quad & \text{for} \\ & \exists x \exists y (\psi(x, y) = 0 \wedge \psi(y, a) = 0 \wedge \psi(a, y) \neq 0 \wedge \varphi(x, r) = 0 \wedge \varphi(y, s) = 0), \end{aligned}$$

which means $r \leq_{C_{\bar{a}}} s$;

$$\begin{aligned} M(\psi, \varphi; b, a; f) \quad & \text{for} \\ & \forall u (C(\psi, \varphi; b; u) \vdash \exists x T_1(f, u, x) \wedge C(\psi, \varphi; a; f(u))) \\ & \wedge \forall u \forall v (C(\psi, \varphi; b; u) \wedge C(\psi, \varphi; b; v) \vdash (\leq(\psi, \varphi; b; u, v) \vdash \leq(\psi, \varphi; a; f(u), f(v)))) \\ & \wedge \forall u \exists v (C(\psi, \varphi; a; u) \vdash C(\psi, \varphi; b; v) \wedge \leq(\psi, \varphi; a; u, f(v))), \end{aligned}$$

which means ' f is an o.p.c.m. from $C_{\bar{b}}$ into $C_{\bar{a}}$ ';

$$\begin{aligned} G(p; \psi, \varphi; b, a; r; f) \quad & \text{for} \\ & M(\psi, \varphi; b, a; f) \wedge \exists x T_2(p, r, f, x) \wedge \neg C(\psi, \varphi; a; p(r, f)) \wedge C(\psi, \varphi; a, b; r), \end{aligned}$$

which corresponds to a part of the condition in Case 3;

$$H(d, q; \psi, \varphi; a; r; f) \quad \text{for}$$

$$M(\psi, \varphi; a, a; f) \wedge \exists x T_2(q, r, f, x) \wedge \neg C(\psi, \varphi; a; q(r, f)) \\ \wedge d(\lambda w C(\psi, \varphi; a; w), \lambda v w \leq (\psi, \varphi; a; v, w) : r) = 0,$$

which corresponds to a part of the condition in Case 4;

$$P_1(\psi, \varphi) \quad \text{for} \quad \forall x (Z(\psi, x) \vdash \forall u (\varphi(x, u) = 0 \vdash u = 1)),$$

which corresponds to Case 1;

$$P_2(\psi, \varphi; a) \quad \text{for} \quad \forall x \forall y (\psi(x, a) = 0 \wedge \psi(a, x) \neq 0 \wedge S(\psi, y, x) \\ \vdash \forall u (\psi(x, u) = 0 \wedge \neg C(\psi, \varphi; y, 2^u) \vdash \varphi(y, 2^u) = 0) \\ \wedge \forall u \forall v (\varphi(y, u) = 0 \vdash u = 2^v \wedge \varphi(x, v) = 0 \wedge \neg C(\psi, \varphi; y; u))),$$

which corresponds to Case 2;

$$P_3(p; \psi, \varphi; a) \quad \text{for} \quad \forall x (\psi(x, a) = 0 \wedge L(\psi, x) \wedge \exists y \exists u \exists f G(p; \psi, \varphi; y, x; u; f) \\ \vdash \forall y \forall u \forall f (G(p; \psi, \varphi; y, x, u, f) \\ \wedge \forall z \forall v \forall g (G(p; \psi, \varphi; z, x; v, g) \vdash \psi(y, z) = 0) \vdash \varphi(x, p(u, f)) = 0) \\ \wedge \forall u (\varphi(x, u) = 0 \\ \vdash \exists y \exists v \exists f (u = p(v, f) \wedge G(p; \psi, \varphi; y, x; v; f) \\ \wedge \forall z \forall w \forall g (G(p; \psi, \varphi; z, x; w; g) \vdash \psi(y, z) = 0))),$$

which corresponds to Case 3;

$$P_4(d, e, q; \psi, \varphi; a) \quad \text{for} \quad \forall x (\psi(x, a) = 0 \wedge L(\psi, x) \wedge \neg \exists y \exists u \exists f G(p; \psi, \varphi; y, x; u; f) \\ \vdash \forall u \forall f (H(d, q; \psi, \varphi; x; u; f) \wedge \forall v \forall g (H(d, q; \psi, \varphi; x; v; g) \\ \vdash e(\lambda w C(\psi, \varphi; w; x), \lambda w w' \leq (\psi, \varphi; x; w, w') : u, v) = 0) \\ \vdash \varphi(x, q(u, f)) = 0) \\ \wedge \forall u (\varphi(x, u) = 0 \vdash \exists v \exists f (u = q(v, f) \wedge H(d, q; \psi, \varphi; x; v; f) \\ \wedge \forall w \forall g (H(d, q; \psi, \varphi; x; w; g) \\ \vdash e(\lambda w' C(\psi, \varphi; w'; x), \lambda v' w' \leq (\psi, \varphi; x; v', w') : v, w) = 0))))),$$

which corresponds to Case 4;

$$P(d, e, p, q; \psi, \varphi; a) \quad \text{for} \quad P_1(\psi, \varphi) \wedge P_2(\psi, \varphi; a) \wedge P_3(p; \psi, \varphi; a) \wedge P_4(d, e, q; \psi, \varphi; a),$$

which corresponds to the definition on transfinite induction up to a depending on d, e, p, q ;

$$P(d, e, p, q; \psi) \quad \text{for} \quad \forall x \forall \varphi \exists u (D(\psi, x) \wedge P(d, e, p, q; \psi, \varphi; x) \vdash \varphi(x, u) = 0).$$

We denote

$$W'(\psi) \wedge \exists d \exists e \exists p \exists q (R(d, e) \wedge P(d, e, p, q; \psi)),$$

the *first form* expressing $A(\alpha)$, by $N_1(\psi)$. $N_1(\psi)$ is expressible in Π_1^1 and can be regarded as ' ψ gives a well-ordering of natural numbers whose order-type is constructively accessible'.

To define the second form expressing $A(\alpha)$, we shall give further abbreviations:

$$\begin{aligned}
& A(d, e, p, q; \psi, \varphi, a) \quad \text{for} \\
& \quad Z(\psi, a) \vee \exists x(S(\psi; a, x) \wedge \exists u(\varphi(x, u) = 0 \wedge \neg C(\psi, \varphi; a; 2^u))) \\
& \quad \vee (L(\psi, a) \wedge (\exists y \exists u \exists f G(p; \psi, \varphi; y, a; u; f) \\
& \quad \vee \exists u \exists f H(d, q; \psi, \varphi; a; u; f))) ;
\end{aligned}$$

$$\begin{aligned}
& B(d, e, p, q; \psi, \varphi; a) \quad \text{for} \\
& \quad D(\psi, a) \wedge P(d, e, p, q; \psi, \varphi; a) \wedge \neg A(d, e, p, q; \psi, \varphi; a) \\
& \quad \wedge \forall x(\varphi(x, a) = 0 \wedge \varphi(a, x) \neq 0 \vdash A(d, e; p, q; \psi, \varphi; x)),
\end{aligned}$$

which means ' $\tilde{\alpha}$ is the least ordinal number which has not a notation depending on d, e, p, q '.

Let $N_2(d, e, p, q; r)$ be

$$R(d, e) \wedge \forall \varphi \forall \psi \forall x (W'(\psi) \wedge B(d, e, p, q; \psi, \varphi; x) \vdash C(\psi, \varphi; x; r)).$$

$N_2(d, e, p, q; r)$ is the *second form* expressing the predicate $A(\alpha)$. This is expressible in $\Pi_2^{1,0}$, while we do not know if it is expressible in the dual form. The ordering $\preceq$ of the set $\{ \langle d, e, p, q, r \rangle \mid N_2(d, e, p, q; r) \}$ of the five-tuple of natural numbers is defined as follows:

$$\langle d, e, p, q, r \rangle \preceq \langle d', e', p', q', r' \rangle$$

if and only if

$$R(d, e) \wedge R(d', e') \wedge \forall \psi \forall \varphi \forall x (W'(\psi) \wedge B(d', e', p', q'; \psi, \varphi; x) \vdash \leq (\psi, \varphi; x; r, r')).$$

$\langle d, e, p, q, r \rangle \preceq \langle d', e', p', q', r' \rangle$ is also expressible in $\Pi_2^{1,0}$.

PROPOSITION 2. *The predicate $A(\alpha)$ is expressible as a Π_2^1 -predicate of a function variable and also as a Π_2^1 -predicate of only number variables in the analytic hierarchy.*

4. Expression of the predicate $\hat{A}(\alpha)$.

We shall express $\hat{A}(\alpha)$ in the analytic hierarchy in the same way as the above. We shall define further abbreviations:

$$\begin{aligned}
& F(d; \psi, \varphi; a; r) \quad \text{for} \\
& \quad \exists x T_1(\lambda w C(\psi, \varphi; a, w), \lambda v w \leq (\psi, \varphi; a; v, w): d, r, x); \\
& E(d, e, q; \psi, \varphi; a) \quad \text{for} \\
& \quad \forall u \forall v \forall f \forall g (F(d, \psi, \varphi; a; u) \wedge F(d; \psi, \varphi; a; v) \\
& \quad \wedge H(d, q; \psi, \varphi; a, u, f) \wedge H(d, q; \psi, \varphi; a, v, g) \\
& \quad \vdash \exists x T_2(\lambda w C(\psi, \varphi; a; w), \lambda w w' \leq (\psi, \varphi; a; w, w'): e, u, v, x)); \\
& \hat{H}(d, e, q; \psi, \varphi; a; r; f) \quad \text{for} \\
& \quad E(d, e, q; \psi, \varphi; a) \wedge F(d; \psi, \varphi; a, r) \wedge H(d, q; \psi, \varphi; a; r; f) \\
& \quad \wedge \forall v \forall g (F(d; \psi, \varphi; a; v) \wedge H(d, q; \psi, \varphi; a; v; g) \\
& \quad \vdash e(\lambda w' C(\psi, \varphi; x; w'), \lambda v' w' \leq (\psi, \varphi; x; v', w'): r, v) = 0),
\end{aligned}$$

which corresponds to a part of the definition in Case 4;

$$\begin{aligned}
& \hat{P}_4(d, e, p; \psi; a) \quad \text{for} \\
& \quad \forall x(\varphi(x, a) = 0 \wedge L(\psi, x) \wedge \neg \exists y \exists u \exists f G(p; \psi, \varphi; y, x; u; f) \\
& \quad \vdash \forall u \forall f (H(d, e, q; \psi, \varphi; x; u; f) \vdash \varphi(x, q(u, f)) = 0) \\
& \quad \wedge \forall u(\varphi(x, u) = 0 \vdash \exists v \exists f (\hat{H}(d, e, q; \psi, \varphi; x; v; f) \wedge u = q(v, f))))),
\end{aligned}$$

which corresponds to Case 4;

$\hat{P}(d, e, p, q; \psi, \varphi; a)$ for

$$P_1(\psi, \varphi) \wedge P_2(\psi, \varphi; a) \wedge P_3(p; \psi, \varphi; a) \wedge \hat{P}_4(d, e, q; \psi, \varphi; a),$$

which corresponds to the definition on transfinite induction up to $\tilde{a}$ depending on d, e, p, q ;

$\hat{P}(d, e, p, q; \psi)$ for

$$\forall x \forall \varphi \mathcal{E}u(D(\psi, x) \wedge \hat{P}(d, e, p, q; \psi, \varphi; x) \vdash \varphi(x, u) = 0).$$

We denote

$$W'(\psi) \wedge \mathcal{E}d\mathcal{E}e\mathcal{E}p\mathcal{E}q\hat{P}(d, e, p, q; \psi),$$

the *first form* expressing $\hat{A}(\alpha)$, by $\hat{N}_1(\psi)$. $\hat{N}_1(\psi)$ is expressible in Π_1^1 and can be regarded as ' ψ gives a well-ordering of natural numbers whose order-type is constructively accessible in the sense of $\hat{\mathcal{C}}$ '.

To define the second form we shall give further abbreviations:

$\hat{A}(d, e, p, q; \psi, \varphi; a)$ for

$$\begin{aligned} & Z(\psi, a) \vee (\mathcal{E}xS(\psi; a, x) \wedge \mathcal{E}u(\varphi(x, u) = 0 \wedge \neg C(\psi, \varphi; a, 2^u))) \\ & \vee (L(\psi, a) \wedge (\mathcal{E}y\mathcal{E}u\mathcal{E}fG(p; \psi, \varphi; y, a; u; f) \\ & \vee \mathcal{E}u\mathcal{E}f\hat{H}(d, e, q; \psi, \varphi; a; u; f))); \end{aligned}$$

$\hat{B}(d, e, p, q; \psi, \varphi; a)$ for

$$\begin{aligned} & D(\psi, a) \wedge \hat{P}(d, e, p, q; \psi, \varphi; a) \wedge \neg \hat{A}(d, e, p, q; \psi, \varphi; a) \\ & \wedge \forall x(\hat{\psi}(x, a) = 0 \wedge \hat{\psi}(a, x) \neq 0 \vdash \hat{A}(d, e, p, q; \psi, \varphi; x)), \end{aligned}$$

where $\hat{A}$ and $\hat{B}$ are the counterparts of A and B , respectively.

Let $\hat{N}_2(d, e, p, q; r)$ be

$$\mathcal{E}\psi\mathcal{E}\varphi\mathcal{E}x(W'(\psi) \wedge \hat{B}(d, e, p, q; \psi, \varphi; x) \wedge C(\psi, \varphi; x; r))$$

or equivalently

$$\forall \psi \forall \varphi \forall x(W'(\psi) \wedge \hat{B}(d, e, p, q; \psi, \varphi; x) \vdash C(\psi, \varphi; x; r)).$$

This is the *second form* expressing $\hat{A}(\alpha)$ and as is seen from the above, it is expressible in $\Sigma_2^{1,0} \cap \Pi_2^{1,0}$. The ordering $\hat{\preceq}$ of the set $\{ \langle d, e, p, q, r \rangle \mid \hat{N}_2(d, e, p, q, r) \}$ of the five-tuple of natural numbers is defined as follows:

$$\langle d, e, p, q, r \rangle \hat{\preceq} \langle d', e', p', q', r' \rangle$$

if and only if

$$\mathcal{E}\psi\mathcal{E}\varphi\mathcal{E}x(W'(\psi) \wedge \hat{B}(d', e', p', q'; \psi, \varphi; x) \wedge \leq(\psi, \varphi; x; r, r'))$$

(or equivalently

$$\forall \psi \forall \varphi \forall x(W'(\psi) \wedge \hat{B}(d', e', p', q'; \psi, \varphi; x) \vdash \leq(\psi, \varphi; x; r, r')).$$

This predicate is also expressible in $\Sigma_2^{1,0} \cap \Pi_2^{1,0}$.

We can express $r \in \hat{\mathcal{C}}$ by $\mathcal{E}d\mathcal{E}e\mathcal{E}p\mathcal{E}qN_2(d, e, p, q; r)$ and the ordering on $\hat{\mathcal{C}}$ by

$$\mathcal{E}d\mathcal{E}e\mathcal{E}p\mathcal{E}q\mathcal{E}d'\mathcal{E}e'\mathcal{E}p'\mathcal{E}q'(\langle d, e, p, q, r \rangle \hat{\preceq} \langle d', e', p', q', r' \rangle).$$

PROPOSITION 3. The predicate $\hat{A}(\alpha)$ is expressible by Π_1^1 -predicate of a function variable, while it is expressible by a $\Sigma_2^1 \cap \Pi_2^1$ -predicate of only number variables.

An ordinal number is called *recursively expressible*, if it is expressible by using recursive functions of ordinal numbers, 0 and ω (cf. [6]). ω^* stands for the least ordinal number which is not recursively expressible.

PROPOSITION 4. $\omega^\dagger$ is recursively expressible (i.e. $\omega^\dagger < \omega^*$).

PROOF. By the above argument every ordinal number a less than $\omega^\dagger$ is represented by a five-tuple $\langle d, e, p, q, r \rangle$ of natural numbers such that $\hat{N}_2(d, e, p, q, r)$ and the predicate $\langle d, e, p, q, r \rangle \hat{=} \langle d', e', p', q', r' \rangle$ (which is the ordering over the set $\{\langle d, e, p, q, r \rangle \mid \hat{N}_2(d, e, p, q, r)\}$) is expressible in $\Sigma_2^1 \cap \Pi_2^1$. By the result of Shoenfield ([5]) noticed in 'Added in proof' of [6], this is expressible as a predicate of natural numbers general recursive in the sense of [6]. Thus we can find a general recursive function $\zeta(a, b)$ of ordinal numbers such that $\zeta(a, b) = 0$ means ' a and b express five-tuples of natural numbers, say, $a = j_5(a_1, a_2, a_3, a_4, a_5)$ and $b = j_5(b_1, b_2, b_3, b_4, b_5)$, respectively, and

$$\langle a_1, a_2, a_3, a_4, a_5 \rangle \hat{=} \langle b_1, b_2, b_3, b_4, b_5 \rangle \wedge \neg \langle b_1, b_2, b_3, b_4, b_5 \rangle \hat{=} \langle a_1, a_2, a_3, a_4, a_5 \rangle.'$$

Now we define a general recursive function $A(c, b)$ as follows:

$$\begin{aligned} A(c, 0) &= \mu z_{z < \omega} (\zeta(z, c) = 0 \wedge \forall y (y < \omega \wedge z \neq y \wedge \zeta(y, c) = 0 \vdash \zeta(z, y) = 0)) \\ \wedge \forall x (0 < x \vdash A(c, x) &= \mu z_{z < \omega} (\zeta(z, c) = 0 \wedge \forall y (y < x \vdash z \neq A(c, y)) \\ &\quad \wedge \forall u (u < \omega \wedge z \neq u \wedge \zeta(u, c) = 0 \wedge \forall y (y < x \vdash u \neq A(c, y)) \\ &\quad \vdash \zeta(z, u) = 0))) . \end{aligned}$$

Let $P(c, b)$ be

$$A(c, b) = 0 \wedge \forall x (x < b \vdash \exists z (0 < z \wedge z < \omega \wedge A(c, x) = z))$$

and $A(c)$ be $\mu z P(c, z)$. Then $\omega^\dagger$ is expressible by the general recursive function $\mu x \forall y (y < \omega \vdash A(y) < x)$.

5. Remark on expression of C .

In the same way as the above we shall show that the predicate ' ϕ gives a well-ordering of natural numbers whose order-type is constructively accessible in the sense of C (Kreider-Rogers and Putnam's system) is in $\Pi_2^{1,1}$ '. We shall follow the definition of C given by Putnam [4]. Let i be the Gödel number of the identity function. We shall define further abbreviations:

$I(\psi, \varphi; a; n)$ for

$$\mathcal{I}u C(\psi, \varphi; a; 3^n \cdot 5^u),$$

which means ' n is an index in C_a ';

$G^0(\psi, \varphi; b, a; m; f)$ for

$$M(\psi, \varphi; b, a; f) \wedge C(\psi, \varphi; a, b; 3^m \cdot 5^f),$$

which corresponds a part of the condition in Case 3;

$$\begin{aligned}
H^0(\psi, \varphi; a, b; m; f) & \quad \text{for} \\
& M(\psi, \varphi; a, a; f) \wedge \neg I(\psi, \varphi; a; m) \wedge C(\psi, \varphi; a, b; m); \\
H^0(\psi, \varphi; a; m; f) & \quad \text{for} \\
& \exists x H^0(\psi, \varphi; a, x; m; f),
\end{aligned}$$

which corresponds to a part of the condition in Case 4;

$$\begin{aligned}
P_2^0(\psi, \varphi; a) & \quad \text{for} \\
& \forall x \forall y (\psi(x, a) = 0 \wedge \psi(a, x) \neq 0 \wedge S(\psi, y, x) \vdash \forall u (\varphi(x, u) = 0 \vdash \varphi(y, 2^u) = 0) \\
& \quad \wedge \forall u \exists v (\varphi(y, u) = 0 \vdash u = 2^v \wedge \varphi(x, v) = 0)),
\end{aligned}$$

which corresponds to Case 2;

$$\begin{aligned}
P_3^0(\psi, \varphi; a) & \quad \text{for} \\
& \forall x (\psi(x, a) = 0 \wedge L(\psi, x) \wedge \exists y \exists u \exists f G^0(\psi, \varphi; y, x; u, f) \\
& \quad \vdash \forall y \forall u \forall f (G^0(\psi, \varphi; y, x; u; f) \vdash \varphi(x, 3^u \cdot 5^f) = 0) \\
& \quad \wedge \forall u (\varphi(x, u) = 0 \vdash \exists y \exists v \exists f (G^0(\psi, \varphi; y, x; v; f) \wedge u = 3^v \cdot 5^f))),
\end{aligned}$$

which corresponds to Case 3;

$$\begin{aligned}
P_4^0(\psi, \varphi; a) & \quad \text{for} \\
& \forall x (\psi(x, a) = 0 \wedge L(\psi, x) \wedge \neg \exists y \exists u \exists f G^0(\psi, \varphi; y, x; u, f) \\
& \quad \vdash \forall y \forall u \forall f (H^0(\psi, \varphi; x, y; u; f) \\
& \quad \quad \wedge \forall z \forall v \forall g (H^0(\psi, \varphi; x, z; v; g) \vdash \psi(y, z) = 0) \vdash \varphi(x, 3^u \cdot 5^f) = 0) \\
& \quad \wedge \forall u (\varphi(x, u) = 0 \vdash \exists y \exists v \exists f (u = 3^v \cdot 5^f \wedge H^0(\psi, \varphi; x, y; v; f) \\
& \quad \quad \wedge \forall z \forall w \forall g (H^0(\psi, \varphi; x, z; w; g) \vdash \varphi(y, z) = 0))),
\end{aligned}$$

which corresponds to Case 4;

$$\begin{aligned}
P^0(\psi, \varphi; a) & \quad \text{for} \\
& P_1(\psi, \varphi) \wedge P_2^0(\psi, \varphi; a) \wedge P_3^0(\psi, \varphi; a) \wedge P_4^0(\psi, \varphi; a); \\
P^0(\psi) & \quad \text{for} \\
& \forall x \forall \varphi \exists u (D(\psi, x) \wedge P^0(\psi, \varphi; x) \vdash \varphi(x, u) = 0); \\
N_1^0(\psi) & \quad \text{for} \\
& W'(\psi) \wedge P^0(\psi),
\end{aligned}$$

which is expressible in $\Pi_1^{1,1}$.

We can regard $N_1^0(\psi)$ as a predicate which means ‘ ψ gives a well-ordering of natural numbers whose order-type is constructively accessible in the sense of C ’.

We can show easily that the predicate ‘ a is a notation in C ’ is expressible in $\Sigma_2^1 \cap \Pi_2^1$ in the same way as in §3 (cf. §6 of [3]). Let $A^0(\psi, \varphi; a)$ be

$$\begin{aligned}
Z(\psi, a) \vee \exists x S(\psi, a, x) \vee (L(\psi, a) \wedge (\exists y \exists u \exists f G^0(\psi, \varphi; y, x; u; f) \\
\vee \exists u \exists f H^0(\psi, \varphi; x; u; f)));
\end{aligned}$$

$B^0(\psi, \varphi; a)$ be

$$D(\psi, a) \wedge \neg A^0(\psi, \varphi; a) \wedge \forall x (\psi(x, a) = 0 \wedge \psi(a, x) \neq 0 \vdash P^0(\psi, \varphi; x) \wedge A^0(\psi, \varphi; x)).$$

Let $N_2^0(m)$ be

$$\exists \psi \exists \varphi \exists x (W'(\psi) \wedge B^0(\psi, \varphi; x) \wedge C(\psi, \varphi; x; m))$$

or equivalently

$$\forall \psi \forall \varphi \forall x (W'(\psi) \wedge B^0(\psi, \varphi; x) \vdash C(\psi, \varphi; x; m)).$$

$N_2^0(m)$ corresponds to $m \in C$ and is expressible in $\Sigma_2^1 \cap \Pi_2^1$ -form. Let $m \equiv n$ be

$$\exists \psi \exists \varphi \exists x (W'(\psi) \wedge B^0(\psi, \varphi; x) \wedge \leq(\psi, \varphi; x; m, n))$$

or equivalently

$$\forall \psi \forall \varphi \forall x (W'(\psi) \wedge B^0(\psi, \varphi; x) \vdash \leq(\psi, \varphi; x; m, n)).$$

$m \equiv n$ corresponds to $m \leq_c n$ and is expressible in $\Sigma_2^1 \cap \Pi_2^1$ -form.

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