Notes on Measure and Category in Recursion Theory*)

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Introduction. Recently, several results obtained by associating Kleene's arithmetic or analytic sets with their measurability or Baire categoricity have been presented. (Hinman [1], Martin-Löf [5], Sacks [8], Tanaka [11, 12, 13], Thomason [15] and others.) In what follows, we shall note some results on this line.

(I) In [12] the author has noted that

$$\mu(E) = \mu(\bigcup \{ E_\nu | \nu < \omega_1 \})$$

where $E$ is a $\Pi^1_1$ set and $E_\nu$'s ($\nu < \omega$) are its constituents with respect to a recursive sieve. Hereafter $\omega_1$ denotes the first nonconstructive ordinal. (For notation and terminology, see below.) There we asked whether a similar result for the case of a $\Sigma^1_1$ set holds. In this note, the question will be solved negatively, thus:

Theorem. There is a $\Sigma^1_1$ set $A$ such that the measure of $A$ does not equal to that of the subset $\bigcup \{ A_\nu | \nu < \omega_1 \}$, where $A_\nu$'s ($\nu < \omega$) are the constituents of $A$ with respect to a recursive sieve.

(II) Martin-Löf [5] has defined the set $\mathcal{R}$ of all random sequences (consisting of 0's and 1's) by the intersection of all hyperarithmetic sets of measure one, and he has shown that $\mathcal{R}$ is a non-hyperarithmetic, $\Sigma^1_1$ set of measure one. There he wrote that it would be desirable to have a constructive proof of "$\mathcal{R}$ having measure one". In this note, however, it will be seen that this task is impossible in some sense. Namely, the complement of $\mathcal{R}$ is not hyperarithmetically null. Therefore, if we understand constructivity as "being hyperarithmetic", then we cannot have any constructive proof that $\mathcal{R}$ has measure one. But it can be proved that the complement $\mathcal{R}'$ is $A^*_1$-null. Further we shall give a generalized notion of randomness, thus:

$$\mathcal{R}_1 = \cap \{ E \subseteq \mathbb{N} | E \text{ is a } \Pi^1_1 \text{ set of measure one} \}.$$  

Then, unexpectedly, the following theorem is obtained:

Theorem. $\mathcal{R} = \mathcal{R}_1$.

(III) We can use the notion of Baire category instead of that of measure. Several results can be carried over to the Baire category case.

*) Results stated in this note were talked at Japan-U.S. Seminar on "Mathematical Logic, Model Theory" (Tokyo), October, 1969.
1. Preliminary. For notation and terminologies used here, we shall mostly follow Rogers [7]. We may not notice that one by one. Let \( a \) be a variable ranging over \( N^N (=\text{the set of all 1-place number-theoretic functions}) \) and let \( \xi \) be a variable over \( 2^N (=\text{the set of all sets of natural numbers}) \). We regard \( 2^N \) as a product space which has the Cantor-set topology. Further we define, as usual, a product measure \( \mu \) on \( 2^N \) by associating the equiprobable measure with \( 2=\{0, 1\} \). As a code of a sequence \( \langle a(0), a(1), \cdots, a(x-1) \rangle \) we take \( \hat{a}(x) \) defined as in Rogers [7; p.377]. As a notation of a paring function we take \( \pi \). We reserve \( \sigma \) to denote an order-type and \( \nu, \sigma \) to denote ordinal numbers. Let \( U \) be the set of all \( (\text{codes of}) \) finite sequences and let \( \{\delta_0, \delta_1, \cdots, \delta_n, \cdots\} \) be a fixed recursive enumeration of all basic open subsets of \( 2^N \) together with the empty set. We shall work on the space \( 2^N \), but almost all results in the paper remain true when we do on Baire's zerospace \( N^N \). Also, our previous results in [11-13] (which have been proved in \( N^N \)) used here remain true for the case of \( 2^N \).

2. Derived sieves. Let \( E \) be a \( \Pi^1_1 \) subset of \( 2^N \) and let \( A=\overline{E} \) (the complement of \( E \)). By Rogers [7; p. 426], there is a recursive predicate \( R(\xi, u) \) such that

\[
\xi \in E \iff (\forall a)(\exists x)R(\xi, \hat{a}(x)).
\]

We define a recursive sieve \( S\subset 2^N \times U \) as follows:

\[
\langle \xi, \hat{a}(x) \rangle \in S \iff (\forall y)_{\nu \leq \xi} \neg R(\xi, \hat{a}(y)).
\]

Let \( S^{<\xi>} = \{u | \langle \xi, u \rangle \in S\} \). It is well-known that

\[
\xi \in E \iff \{S^{<\xi>} \text{ is well-ordered by } <\}
\]

where \( < \) is the Kleene-Brouwer ordering on \( U \). Further, let \( I^{<\xi>} \) be the maximal initial segment of \( S^{<\xi>} \) which is well-ordered by \( < \). Now we define \( E_\nu \) and \( A_\nu \) for each \( \nu < \Omega (\Omega = \text{the first uncountable ordinal}) \) as follows:

\[
\xi \in E_\nu \iff \xi \in E \& \tau(I^{<\xi>}) = \nu (=\tau(S^{<\xi>})),
\]

and

\[
\xi \in A_\nu \iff \xi \in A \& \tau(I^{<\xi>}) = \nu. \quad 1)
\]

As stated in Introduction, it holds

\[
\mu(E) = \mu(\bigcup \{E_\nu | \nu < \omega_1\}).
\]

In the next section, we shall show that the formula corresponding to (1) does not, in general, hold when we take a \( \Sigma^1_1 \) set \( A \) instead of a \( \Pi^1_1 \) set \( E \). To prove this fact, we make a few lemmas ready.

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1) \( I^{<\xi>} \) is a \( \Pi^1_1 \)-in-\( \xi \) set and hence \( \tau(I^{<\xi>}) \) is a \( \Pi^1_1 \)-in-\( \xi \) ordinal. Therefore, it follows from [14; §5] that the ordinal is less than \( \omega_1^{\text{OF}} \).
Now we shall define derived sieves $S(a)$ for each $a \in \mathcal{O}$ in the following way:

1°) $a = 1$. $S(a) = S$.

2°) $a = 2^b + 1$. $S(a) = \{ \langle \xi, u \rangle : \langle \xi, u \rangle \in S(b) \land (\exists v) [v < u \land \langle \xi, v \rangle \in S(b)] \}$.

3°) $a = 3 \cdot 5^k$. $S(a) = \bigcap_{k=3}^{\infty} S(\varphi_k(k))$.

**Lemma 1.** For $a, b \in \mathcal{O}$,

$$|a| \leq |b| \Rightarrow S(a) \supseteq S(b)$$

$$|a| = |b| \Rightarrow S(a) = S(b).$$

Proof by induction on $b$.

**Lemma 2.** For each $a \in \mathcal{O}$, $S(a)$ is a hyperarithmetic subset of $2^\mathcal{N} \cup U$.

Proof. We shall show that there exist partial recursive functions $f$ and $g$ satisfying the following condition: If $a \in \mathcal{O}$, then both $f(a, u)$ and $g(a)$ are defined and

$$\langle \xi, u \rangle \in S(a) \Leftrightarrow f(a, u) \in H^k(g(a)) \text{ if } a = 2^{(\xi)*},$$

$$\langle \xi, u \rangle \in S(a) \Leftrightarrow f(a, u) \notin H^k(g(a)) \text{ if } a = 3 \cdot 5^{(\xi)*}.$$  

Further, if $b <_a a$ then $g(b) <_a g(a)$.

i) $a = 1$. There is a recursive function $\rho_0$ such that

$$S(1) = \{ \langle \xi, u \rangle : \rho_0(u) \in H^k(2) \}.$$  

Since $S(1)$ ($= S$) is recursive. Let

(2) \quad $\varphi_m(m, 1, u) = \rho_0(u)$ and $\varphi_m(m, 1) = 2$.

ii) $a = 2^b + 1$. Then we have

$$\langle \xi, u \rangle \in S(a) \Leftrightarrow (b = 2^{(\xi)*} \land \varphi_m(n, b, u) \in H^k(\varphi_m(n, b))) \land (\exists v) [v < u \land \varphi_m(n, b, v) \in H^k(\varphi_m(n, b))] \lor (b = 3 \cdot 5^{(\xi)*} \land \varphi_m(n, b, u) \in H^k(\varphi_m(n, b))) \land (\exists v) [v < u \land \varphi_m(n, b, v) \in H^k(\varphi_m(n, b))] \}.$$

Let $K(X, m, n, b, u)$ be the formula obtained from the right hand side of the above equivalence relation by replacing $H^k(\varphi_m(m, b))$ by a free variable $X$ ranging over $2^\mathcal{N}$. Then for a suitable recursive function $\rho_1(m, n, b, u) \in K(X, m, n, b, u)$ holds, where $X'$ is the recursive-jump of $X$. Hence we have

$$\langle \xi, u \rangle \in S(a) \Leftrightarrow \rho_1(m, n, b, u) \in H^k(\psi_1(m, n, b)),$$

where $\psi_1(m, n, b) = (\varphi_{k}(m, b))^* (x^* = 2^*).$ Let

2) See Ljapunow et al. [3; p. 50]. For the notation on $\mathcal{O}$ here and $H^k(a)$ below, see Rogers [7; §§11 and 16]. Especially $\varphi^{(k)}$ denotes the $k$-place partial recursive function with Gödel number $z$. We shall omit superindex $k$. 

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(3) $\varphi_m(n, a, u) = \rho_1(m, n, (a)_0, u)$ and $\varphi_n(m, a) = \psi_1(m, n, (a)_0) \ (\Rightarrow \varphi_n(m, (a)_0))$.

   iii) $a = 3.5^b$. Then

   $$\langle \xi, n \rangle \in S(a) \iff (\forall k) [(\exists x) [\varphi_m(m, \varphi_0(k)) = 2^x] \rightarrow$$
   $$\varphi_m(n, \varphi_0(k), u) \in H^l(\varphi_m(m, \varphi_0(k)))$$

   & $(\exists x) [\varphi_n(m, \varphi_0(k)) = 3.5^x] \rightarrow \varphi_m(n, \varphi_0(k), u) \notin H^l(\varphi_m(m, \varphi_0(k)))].$

We can find a recursive function $\beta$ such that

   $$(\forall k) [\varphi_m(m, \varphi_0(k)) < 3.5^{(m, n, b)}],$$

Since we have $\varphi_m(m, \varphi_0(k)) < \varphi_m(m, \varphi_0(k + 1))$ by the induction hypothesis. By using $\beta$ we have

   $$(\exists x) [\varphi_m(m, \varphi_0(k)) = 2^x] \rightarrow (\exists y) [y = \varphi_m(n, \varphi_0(k), u)$$

   & $(\exists x) [\varphi_n(m, \varphi_0(k)) = 3.5^x] \rightarrow (\exists y) [y = \varphi_m(n, \varphi_0(k), u)$$

   & $(\exists x) [\varphi_m(n, \varphi_0(k)) = 3.5^x] \rightarrow H^l(\varphi_m(m, \varphi_0(k))).$$

By the same method as in (ii), we can find a recursive function $\rho_2(m, n, b, u)$ such that

   $$\langle \xi, u \rangle \in S(a) \iff \rho_2(m, n, b, u) \notin H^l(\varphi_2(m, n, b)),$$

where $\psi_2(m, n, b) = (3.5^{(m, n, b)})^{**}$. Let

   (4) $\varphi_m(n, a, u) = \rho_2(m, n, (a)_2, u)$ and $\varphi_n(m, a) = \psi_2(m, n, (a)_2) \ (\Rightarrow \varphi_n(m, \varphi_0(k))).$

By solving (2)-(4) with respect to $m$ and $n$ using Smullyan's double recursion theorem\(^3\), we can obtain desired functions $f$ and $g$:

   $$f(a, u) = \varphi_m(n, a, u) \quad \text{and} \quad g(a) = \varphi_n(m, a).$$

**Lemma 3.** For each constructive ordinal $\nu < \omega_1$, both $E_\nu$ and $A_\nu$ are hyperarithmetic subsets of $2^\nu$.

**Proof.** Let $a$ be a member of $O$ such that $|a| = \nu$. By a well-known classical device\(^4\) we can obtain the following relations:

   $$\xi \in E_\nu \iff (\forall b) [b <_0 a \Rightarrow (\exists u) (\langle \xi, u \rangle \in S(b))] \ & \ (\forall u) [(\xi, u) \notin S(a)],$$

   $$\xi \in A_\nu \iff (\exists u) [(\xi, u) \in S(a)] \ & \ (\forall b) [b <_0 a \Rightarrow$$

   $$(\exists u) (\langle \xi, u \rangle \in S(b) \ & \ (\forall v) [v < u \Rightarrow (\xi, v) \notin S(b)])]$$

   & $(\forall u) [(\xi, u) \in S(a) \Rightarrow (\exists v) (v < u \ & \ (\xi, v) \notin S(a))]].$

Hence, by uniformity of proof of Lemma 2, it is seen that both $E_\nu$ and $A_\nu$ are

\(^3\) See e.g., Rogers [7; p. 190].

\(^4\) Ljapunow et al. [3; p. 51].
hyperarithmetic.

**Corollary 4.** For each \( \nu < \omega_1 \), both sets \( \{ E_\sigma | \sigma < \nu \} \) and \( \cup \{ A_\sigma | \sigma < \nu \} \) are hyperarithmetic. Both sets \( \cup \{ E_\sigma | \sigma < \omega_1 \} \) and \( \cup \{ A_\sigma | \sigma < \omega_1 \} \) are \( \Sigma_1 \).

Proof by uniformity.

3. A counterexample.

**Theorem 5.** There is a \( \Sigma_1 \) set \( A \) such that
\[
\mu(A) \neq \mu(\cup \{ A_\nu | \nu < \omega_1 \}),
\]
where \( A_\nu 's (\nu < \omega) \) are the constituents of \( A \) with respect to a recursive sieve \( S \).

Proof. Let \( A \) be the set of all non-hyperarithmetic elements of \( 2^\omega \). Clearly, \( A \) is \( \Sigma_1 \) and it is co-countable, a fortiori \( \mu(A) = 1 \). Suppose it were \( \mu(A) = \mu(\cup \{ A_\nu | \nu < \omega_1 \}) \). Then we have \( \mu(\cup \{ A_\nu | \nu < \omega_1 \}) = 1 \). By Corollary 4, the set \( \cup \{ A_\nu | \nu < \omega_1 \} \) is \( \Sigma_1 \). Hence by Sacks' Theorem [8; Theorem 3.9] (also cf. [12]), there would be a hyperarithmetic element in it. This contradicts the definition of \( A \).

Then, for each \( \Sigma_1 \) set \( A \), what sort of \( \sigma \) satisfies the condition
\[
\mu(A) = \mu(\cup \{ A_\nu | \nu < \sigma \})?
\]
One can show that there is a \( \Delta_1 \) ordinal \( \sigma \) such that for every \( \Sigma_1 \) set \( A \) \( \mu(A) = \mu(\cup \{ A_\nu | \nu < \sigma \}) \) holds. Sacks has later pointed out that one can take \( \omega_1 + 2 \) as \( \sigma \).

4. Baire category case (I). Results obtained in the preceding section are carried over the Baire category case. By Thomason [15; Corollary 3 of Theorem...
13], the set $K$ is meager (first Baire category). Hence we have

**Lemma 7.** If $E$ is a $\Pi^1_1$ set, then the set $\bigcup \{E_v | v \geq \omega_1 \}$ is meager.

Now we have

**Theorem 8.** There is a $\sum^1_1$ set $A$ such that it is co-meager in $2^\omega$ (i.e., the complement of a meager set) but the set $\bigcup \{A_v | v < \omega_1 \}$ is meager.

**Proof.** We take $A$ defined in the proof of Theorem 5. That $A$ is a desired set can be proved by the same way as in the proof of Theorem 5, using a result of Hinman [1; Comment following Corollary 15] asserting that every non-meager, $\Pi^1_1$ set contains a hyperarithmetic element.

One can show that there is an $\mathcal{L}^1_{1/2}$ ordinal $\sigma$ such that for every $\sum^1_1$ set $A$, $\mu(A) = \mu(\bigcup A_v)$ holds. Sacks has later pointed out that one can take $\omega_1 + 2$ as $\sigma$.

5. **Martin-Löf's notion of randomness.** A subset $X$ of $2^\omega$ is said to be $\mathcal{L}^1_{1/2}$-null, if there is a $\mathcal{L}^1_{1/2}$ function $f$ such that

(1) $X \subseteq \bigcup \delta_{f(m,n)}$ for all $m$, and

(2) the sequence $(\mu(\bigcup \delta_{f(m,n)}))_{m=0}^{\infty}$ $\mathcal{L}^1_{1/2}$-converges to 0 as $m \to \infty$; that is, there exists a $\mathcal{L}^1_{1/2}$ function $g$ such that

(3) $m > g(p) \Rightarrow \mu(\bigcup \delta_{f(m,n)}) < 2^{-p}$.

"$\mathcal{L}^1_{1/2}$-null" is also said to be hyperarithmetically null.

**Lemma 9.** Let $X$ be a hyperarithmetic set. If $\mu(X) = 0$, then $X$ is hyperarithmetically null.

**Proof.** By [12; Theorem 1], for each hyperarithmetic set $Y$ there exist hyperarithmetically closed (uniformly in $p$) sets $Y(p)$ contained in $Y$ satisfying the condition $\mu(Y - Y(p)) < 2^{-p}$. Take $Y = CX$. The sets $CY(p)$ can be expressed in the form $\bigcup \delta_{f(p,m)}$ where $f$ is a hyperarithmetic function. Clearly $f$ satisfies (1)-(3) with $g(p) = p$.

Again we state here Martin-Löf's definition of the set $\mathcal{M}$ of all random binary sequences:

$$\mathcal{M} = \bigcap \{H \subseteq 2^\omega | H \in \mathcal{L}^1_1 \& \mu(H) = 1\}.$$ 

Then we have

**Proposition 10.** $\xi \in \mathcal{M}$ if and only if the singleton $\{\xi\}$ is a hyperarithmetically null set.

**Proof.** Assume $\xi \in \mathcal{M}$. Then there is a hyperarithmetic set $X$ such that $\mu(X) = 0$ but $\xi \in X$. By Lemma 9, $X$ is hyperarithmetically null and hence so is $\{\xi\}$. Conversely, if $\{\xi\}$ is hyperarithmetically null, then by the definition, $CX(\xi)$ contains a hyperarithmetic subset $X$ which is of measure one. Hence $\xi \notin \mathcal{M}$.

But in contrast to this proposition we have

6) Cf. Martin-Löf [4; p. 611]. Also, in [11; §7.1] we have defined a similar notion.
Theorem 11. The complement of M cannot be hyperarithmetically null.
Proof. By Martin-Löf [5], M contains no hyperarithmetic elements. Now suppose C_M were hyperarithmetically null. Then it would have a hyperarithmetic masslgeichen Hülle. Hence M would contain a hyperarithmetic element which is a contradiction.

However, in what follows, it is seen that C_M is \( \Delta^1_2 \)-null (Theorem 14).

By a method stated in [11: §5.4], we can obtain

Proposition 12. If A is a \( \sum^1_1 \) set of measure 0, then A is \( \Delta^1_2 \)-null.

Let A be a \( \sum^1_1 \) set. Then there exists a monotone recursive Souslin system \( \{A_{\{a_0, a_1, \ldots, a_k\}}\} \) consisting of closed-open subsets of \( 2^N \) such that

\[
A = \bigcup_{a, k \geq 0} A_{a(k)},
\]

where we abbreviate \( A_{\{a(0), \ldots, a(k-1)\}} \) by \( A_{a(k)} \).

Let us define, as usual in classical descriptive set theory, \( P(a, k) \) in the following way:

\[
P(a, k) \leftrightarrow \bigcup_{m_0 = 0}^{a(0)} \cdots \bigcup_{m_{k-1} = 0}^{a(k-1)} A_{\{m_0, \ldots, m_{k-1}\}},
\]

where \( A_{\{m_0, \ldots, m_{k-1}\}} = \bigcup_{a, j > 0} A_{\{m_0, \ldots, m_{k-1}, a(0), \ldots, a(j-1)\}} \).

By the proof of [11; Theorem 8], we have

\[
(\forall p) (\exists a) [\mu(A - \bigcap_{k \geq 0} P(a, k)) < 2^{-p}].
\]

Since \( \bigcap_{k \geq 0} P(a, k) \) is arithmetic in a (see (5) below) and it is contained in A, A – \( P(a, k) \) is \( \sum^1_1 \) in a. Hence the bracketed relation of (4) is \( \Pi^1_1 \). Therefore by the Kondo-Addison Theorem\(^7\), there is a \( \Delta^1_2 \) function f such that

\[
(\forall p) [\mu(A - \bigcap_{k \geq 0} P(\lambda t E(p, t), k)) < 2^{-p}],
\]

where \( E \) is a pairing function. The set \( \bigcap_{k \geq 0} P(\lambda t E(p, t), k) \) is \( \Delta^1_2 \)-closed uniformly in p. For, by [11; (6) in p. 33], it holds

\[
C \bigcap_{k \geq 0} P(a, k) = \bigcup_{k \geq 0} \bigcup_{m_0 = 0}^{a(0)} \cdots \bigcup_{m_{k-1} = 0}^{a(k-1)} A_{\{m_0, \ldots, m_{k-1}\}}.
\]

Since the Souslin system is recursive, there exist arithmetic functions g and h such that for any \( \beta \in N^N \)

\[
A_{\beta(k)} = \bigcup \{g_1(\beta(k), i) \mid i < h(\beta(k))\}
\]

(a finite union of basic open subsets of \( 2^N \)). Therefore we can find arithmetic functions \( g' \) and \( h' \) such that for any \( \alpha \in N^N \)

\[
C \bigcup_{m_0 = 0}^{a(0)} \cdots \bigcup_{m_{k-1} = 0}^{a(k-1)} A_{\{m_0, \ldots, m_{k-1}\}} = \bigcup \{g_1'(\alpha(k), i) \mid i < h'(\alpha(k))\}.
\]

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\(^7\) See Rogers [7; p. 426 and p. 430].
By this together with the fact that \( f \) is \( \mathcal{A}_k \), we can find a \( \mathcal{A}_k \) function \( f' \) such that

\[
C \cap P(\lambda f(\pi(p, t)), k) = \bigcup_k \delta_{f'(\pi(p, k))}.
\]

This means that \( \bigcap_k P(\lambda f(\pi(p, t)), k) \) is \( \mathcal{A}_k \)-closed uniformly in \( p \). Thus we obtain

**Proposition 13.** If \( E \) is a \( \Pi^1_1 \) set and if \( \mu(E) = 0 \), then \( E \) is \( \mathcal{A}_k \)-null.

And hence we have

**Theorem 14.** \( \mathcal{M} \) is \( \mathcal{A}_k \)-null.

**Remark.** As stated above by Martin-Löf, none of hyperarithemetic elements is random. It is well-known that there is a non-empty arithmetic subset \( Z \) of \( 2^N \) containing no hyperarithemetic element. By Sacks' Theorem quoted above (or by [8; Corollary 2.3], [11;Theorem 16]), we have \( \mu(Z) = 0 \). Therefore, \( Z \subseteq \mathcal{M} \). By Feferman-Harrison-Kripke's Theorem (see e.g. Mathias [6; T3200]), \( Z \) must contain a perfect subset. Further \( Z \) has also a (non-\( \mathcal{A}_k \) \( \mathcal{A}_k \)) element and hence so does \( \mathcal{M} \).

Of course \( \mathcal{M} \) also contains a \( \mathcal{A}_k \) element.

6. **A generalization.** Let \( k \) be a natural number. If \( \xi \) is a \( \Pi^1_k \) element of \( 2^N \), then \( C(\xi) = 2^N - \{\xi\} \) is a \( \Pi^1_k \) set of measure one. Hence the family

\[
\{ E \subseteq 2^N \mid E \in \Pi^1_k (\Sigma^1_k) \land \mu(E) = 1 \},
\]

is not empty. So we may define as follows:

\[
\mathcal{M}_k = \bigcap \{ E \subseteq 2^N \mid E \in \Pi^1_k \land \mu(E) = 1 \},
\]

\[
\mathcal{M}_k' = \bigcap \{ A \subseteq 2^N \mid A \in \Sigma^1_k \land \mu(A) = 1 \}.
\]

Let \( k \geq 0 \) and let \( Q^*_k(n = 0, 1, 2, \cdots) \) be an enumeration of all \( \Pi^1_k \) subsets of \( 2^N \) such that \( \lambda \xi \in Q^*_k \) itself is \( \Pi^1_k \). Then we have

\[
\begin{align*}
(1) & \quad \xi \in \mathcal{M}_k \iff (\forall n)[\mu(Q^*_n) = 1 \implies \xi \in Q^*_n], \\
(2) & \quad \xi \in \mathcal{M}_k' \iff (\forall n)[\mu(Q^*_n) = 0 \implies \xi \notin Q^*_n].
\end{align*}
\]

We shall distinguish into two cases: \( k = 1 \) and \( k \geq 2 \).

(i) \( k = 1 \). Firstly we shall show that \( \mathcal{M}_1 \) coincides with \( \mathcal{M} \).

**Lemma 15.** If \( E \) is a \( \Pi^1_1 \) set and if \( \mu(E) = 1 \), then there exists a hyperarithemetic subset \( H \) of \( E \) such that \( \mu(H) = 1 \).

**Proof.** As in \( \S 2 \), let \( E = \bigcup \{ E_\nu \mid \nu < \omega \} \) be the decomposition into constituents with respect to a recursive sieve. Then, since \( \bigcup \{ E_\nu \mid \nu < \omega_1 \} \) has the same measure as \( E \) (see Introduction), we have

\[
\mu \left( \bigcup \{ E_\nu \mid \nu < \omega_1 \} \right) = 1.
\]

Hence it holds

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8) As a matter of course, we stipulate that the condition \( \mu(Q) = 1 \) means "\( Q \) is measurable and has measure one".
(3) \((\forall p) (\exists a) [a \in O & \mu(\bigcup (E_v | v < |a|)) > 1 - 2^{-p}]\).  

As the bracketed predicate of (3) is \(\Pi_1\) (by [11; Corollary 4 with parameters]), we can apply Kreisel’s Lemma [2; Lemma 1] to (3), thus:

\[(\forall p) [f(p) \in O & \mu(\bigcup (E_v | v < |f(p)|)) > 1 - 2^{-p}]\]

holds for some hyperarithmetic function \(f\). By a result known as a consequence of Spector [10; Theorem 6], we can find an \(a \in O\) such that

\[(\forall p) [|f(p)| < |a|]\]

holds true. For this \(a\)

\[(\forall p) [\mu(\bigcup (E_v | v < |a|)) > 1 - 2^{-p}]\]

and hence we have

\[\mu(\bigcup (E_v | v < |a|)) = 1.\]

Take \(H = \bigcup (E_v | v < |a|)\). By Corollary 4, \(H\) is a hyperarithmetic subset of \(E\).

This Lemma can be generalized in the following form:

**Lemma 15'.** If \(E\) is a \(\Pi_1\) set and if it has the measure which is a hyperarithmetical real number, then \(E\) contains a hyperarithmetic subset \(H\) which has the same measure as \(E\).

**Theorem 16.** \(\mathcal{M}_1 = \mathcal{M}\).

Proof. For each \(\Pi_1\) set \(E\) of measure one, a hyperarithmetic subset of the same measure whose existence is guaranteed by Lemma 15 we denote by \(H_E\). Then

\[\mathcal{M}_1 \supset \bigcap \{H_E | E \in \Pi_1 \& \mu(E) = 1\}\]

\[\supset \bigcap \{H | H \in \mathcal{A}_1 \& \mu(H) = 1\} = \mathcal{M}.\]

Since \(\mathcal{M}_1 \subseteq \mathcal{M}\), we obtain: \(\mathcal{M}_1 = \mathcal{M}\).

**Problem.** Does it hold \(\mathcal{M}_1' = \mathcal{M}\) ?

We conjecture that this answer is no.\(^9\)

By the definition, \(\mathcal{M}_1'\) is a \(\mathcal{A}_1\) set of measure one \(\subseteq \mathcal{M}\); in fact it is a \(C(\Sigma_1^0)\) set. Hence

(A) \(\mathcal{M}_1'\) contains a \(\mathcal{A}_1\) element.

Similarly as in Martin-Löf [5], it is seen that

(B) \(\mathcal{M}_1'\) cannot have any element \(\xi\) such that the singleton \(\{\xi\}\) is \(\Pi_1\), and hence \(\mathcal{M}_1'\) is not a \(\Pi_1\) set. Further, it contains no non-empty \(\Pi_1\) subset.\(^10\)

(ii) \(k \geq 2\). Let \(G(a) = \bigcup_{i} \delta_{a(i)}\) (a universal set of open subsets of \(2^N\) which is itself \(\Sigma_1^0\)). Then

\[\mu(Q_k^a) = 1 \Leftrightarrow (\forall p) (\exists a) [C Q_k^a \subseteq G(a) \& \mu(G(a)) < 2^{-p}]\]

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9) Sacks has shown \(\mathcal{M}_1' = \mathcal{M}\). (Private communication.)

10) See [11; Theorem 3].
holds. Hence the predicate \( \mu(Q^*_k)=1 \) is \( \Sigma^k_1 \). Therefore

\[(C) \quad \mathcal{M}_k \text{ is a } \Pi^k_1 \text{ set of measure one.} \]

For \( \mathcal{M}_k' \), similarly to \( \mathcal{M}_{k+1} \). Evidently, both \( \mathcal{M}_k \) and \( \mathcal{M}_k' \) contain \( \mathcal{M}_{k+1} \cup \mathcal{M}_{k+1}' \).

Now the \( d^k_1 \) elements constitute a countable \( \Sigma^k_1 \) set.\(^{11}\) Hence we have

\[(D) \quad \mathcal{M}_2 \text{ contains no } d^k_1 \text{ element, and hence, by (A), } \mathcal{M}_2 \text{ is a proper subset of } \mathcal{M}_1'. \]

Obviously both \( \mathcal{M}_{k+1} \) and \( \mathcal{M}_{k+1}' \) contain no \( d^k_1 \) element.

7. Baire category case (II). As in §6, let \( k \) be a natural number and let \( \xi \) be a \( d^k_1 \) element. Then \( C(\xi) \) is a co-meager, \( d^k_1 \) set. Hence we may define as follows:

\[ \mathcal{N}_k = \bigcap \{ E \subseteq 2^N \mid E \in \Pi^k_1 \& \text{ (E is co-meager)} \}. \]

Similarly for \( \mathcal{E}_k' \) with \( \Sigma^k_1 \). Unfortunately \( \mathcal{M} \) and \( \mathcal{N}_0 \) (\( =\mathcal{N}_0' \)) are disjoint (and hence so are \( \mathcal{M}_k \) and \( \mathcal{N}_k \)), because there exists a meager, arithmetic set of measure one.\(^{13}\)

Now let \( F(a) = C \cup \delta_{\alpha(n)} \) for \( a \in N^N \). The set \( \{ \langle \xi, a \rangle \mid \xi \in F(a) \} \) is universal for closed subsets of \( 2^N \) and it is itself \( \Pi^k_1 \). Let \( N(a) \) be the predicate \( \{ F(a) \text{ is nowhere dense} \} \). Since

\[ N(a) \iff \left( \forall m \right) \left( \exists n \right) \left[ \delta_m \supset \delta_n \& \ F(a) \cap \delta_n = \emptyset \right], \]

it is \( \Pi^k_1 \); in fact, \( N(a) \) is arithmetic because basic open subsets of \( 2^N \) are compact.

That a set \( P \) is meager (denoted by \( M(P) \)) can be expressed in the following form:

\[ M(P) \iff \left( \exists a \right) \left( P \subseteq \bigcup_n F(\lambda i \alpha \langle n, i \rangle) \right) \& \left( \forall n \right) N(\lambda i \alpha \langle n, i \rangle). \]

Therefore we obtain

**Lemma 17.** If \( P \) is a \( \Sigma^k_1 \) set, then the predicate \( M(P) \) is \( \Sigma^k_1 \).

Now we have

\[ \xi \in \mathcal{N}_k \iff \left( \forall m \right) \left( Q^*_k \text{ is co-meager} \right) \Rightarrow \xi \in Q^*_k. \]

Hence by Lemma 17, \( \mathcal{N}_k \) is a \( \Pi^k_1 \) set which is also co-meager.

By a result of Hinman [1] quoted in §4 (instead of Sacks' Theorem), we can see that \( \mathcal{N}_1 \) is not \( \Pi^1_1 \). (B) in §6 holds for \( \mathcal{N}_1' \). The first-half of (D) holds for \( \mathcal{M}_2 \).

We do not know whether or not \( \mathcal{N}_1=\mathcal{N}_1' \) and whether \( \mathcal{M}_2 \) is a proper subset of \( \mathcal{M}_1 \).\(^{14}\)

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11) See Rogers [7; p. 417 or Suzuki's Theorem].
12) Solovay [9] has defined a similar notion which he calls a generic real.
13) See, e.g. [13; Corollary 4].
14) For category case, Sacks also has obtained a corresponding result to one in §6. See Footnote 9).
References