Intuitionistic Predicate Calculus with ε -Symbol

Kokio SHIRAI

Tokyo University of Education

In 1970, Prof. S. Maehara has used a “weakened ε -Theorem”, as a lemma for proving a completeness theorem of intuitionistic predicate calculus as an application of his general theory of completeness proofs [5]. The intuitionistic predicate calculus with ε -symbol used in Maehara [5] contains incompletely the ε -terms in the usual sense. In this paper, we treat an intuitionistic predicate calculus with ε -symbol containing all of the usual ε -terms. Our main result (cf. §3) is an extension of Prof. Maehara’s weakened intuitionistic ε -Theorem and, in author’s opinion, it will be a natural translation of Hilbert-Bernays’s second ε -Theorem to the case of intuitionistic logic.

I would like to express my gratitude to Prof. S. Maehara who directed my attention to this problem and gave me valuable advice in the course of this work.

§ 1. Terms and formulas.

As the *primitive symbols* used in the formations of terms and formulas we shall admit

- 1) individual variables,
- 2) function symbols,
- 3) predicate symbols,
- 4) logical symbols: $\supset, \wedge, \vee, \supset, \forall, \exists, \varepsilon$,

and parentheses. If individual constants and proposition constants are needed, they can be regarded as special cases of function symbols or predicate symbols. In the following the word “variable” means “individual variable”, and we shall assume that as free and bound variables distinct letters are used.

1.1. Definition of “term” and “formula” (recursive definition).

1.11. A free variable is a *term*.

1.12. If $t_1, \dots, t_n$ are *terms* and f is an n -ary function symbol, then $f(t_1, \dots, t_n)$ is a *term*.

1.13. If $t_1, \dots, t_n$ are *terms*, and P is an n -ary predicate symbol, then $P(t_1, \dots, t_n)$ is a *formula*.

1.14. If A is a *formula*, then $\supset A$ is a *formula*. If A and B are *formulas*, then $A \wedge B$, $A \vee B$ and $A \supset B$ are *formulas*.

1.15. If $F(a)$ is a *formula*, a is a free variable, and $F(x)$ is the result of

substituting a bound variable x for a throughout $F(a)$, then $\forall xF(x)$ and $\exists xF(x)$ are formulas, and $\varepsilon xF(x)$ is a term.

1.16. The only terms and formulas are those given by 1.11–1.15.

1.2. A term or formula is said to be *open* or *closed* according as it contains at least one free variable or not.

A term of the form $\varepsilon xF(x)$ is called an ε -term.

1.3. Remark on substitution of term for bound variable.

When one of formulas $\forall xF(x)$ and $\exists xF(x)$ or a term $\varepsilon xF(x)$ is given to be considered, the substitution of a term t for x in $F(x)$ shall consist in substituting t for all occurrences of x in $F(x)$, which are contained in none of the scopes of the quantifiers $\forall x$, $\exists x$, and εx . The result of this substitution will be usually denoted by $F(t)$.

1.4. Subterm.

We shall define the concept 'subterm' of a given term or formula:

1.41. If t is a term, t is a subterm of t .

1.42. If $t_1, \dots, t_n$ are terms, and f is an n -ary function symbol, the subterms of t_i ($i=1, \dots, n$) are subterms of $f(t_1, \dots, t_n)$.

1.43. If $t_1, \dots, t_n$ are terms, and P is an n -ary predicate symbol, the subterms of t_i ($i=1, \dots, n$) are subterms of $P(t_1, \dots, t_n)$.

1.44. If A is a formula, the subterms of A are subterms of $\supset A$. If A and B are formulas, the subterms of A and the subterms of B are subterms of $A \wedge B$, $A \vee B$ and $A \supset B$.

1.45. If $F(a)$ is a formula, a is a free variable, and $F(x)$ is the result of substituting a bound variable x for a throughout $F(a)$, the subterms of $F(a)$ which do not contain a are subterms of $\forall xF(x)$, $\exists xF(x)$ and $\varepsilon xF(x)$.

1.46. A term or formula has only the subterms required by 1.41–1.45.

1.5. An ε -term which is a subterm of a term t is called an ε -subterm of t .

If a subterm of a term t is not identical with t itself, the subterm is called a *proper subterm* of t .

1.6. Subformula.

We define the concept 'subformula' of a given formula:

1.61. If A is a formula, A is a subformula of A .

1.62. If A is a formula, the subformulas of A are subformulas of $\supset A$. If A and B are formulas, the subformulas of A and the subformulas of B are subformulas of $A \wedge B$, $A \vee B$ and $A \supset B$.

1.63. If $F(a)$ is a formula, a is a free variable, and $F(x)$ is the result of substituting a bound variable x for a throughout $F(a)$, the subformulas of $F(a)$ which do not contain a are subformulas of $\forall xF(x)$ and $\exists xF(x)$.

1.64. A formula has only the subformulas required by 1.61–1.63.

1.7. Sequent.

By a 'sequent' we mean a formal expression of the form

$$A_1, \dots, A_n \rightarrow B$$

or

$$A_1, \dots, A_n \rightarrow ,$$

where $A_1, \dots, A_n$ and B are arbitrary formulas. The *antecedent* of the sequent is formed from the formulas $A_1, \dots, A_n$ and the *succedent* the only formula B or empty. The antecedent may be empty.

1.8. Formal proof.

By a 'formal proof' we mean a formal figure in tree-form consisting of one or more sequents with one lowermost sequent which is called the *endsequent*, and some uppermost sequents which must be *axioms*; the connection between the uppermost sequents and the endsequent is established by *inferences* admitted by some *rules of inference*. Suppose that an endsequent is given. This sequent is either already an uppermost sequent — in which case it alone constitutes at once the entire formal proof — or it is the lower sequent of an inference. Every upper sequent of this inference is in turn either an uppermost sequent of the formal proof or the lower sequent of another inference, etc.

When a set of the axioms and of the rules of inference are given, the concept of "formal proof" is completely determined.

The formal proof is called simply "proof" when no confusion results.

If a sequent is the endsequent of a proof, the proof is called a *proof of the sequent*.

A sequent is said to be *provable*, when there exists a proof of the sequent. A formula A is said to be *provable*, when a sequent

$$\rightarrow A$$

is provable.

§2. Auxiliary intuitionistic predicate calculus LDJ.

In order to clarify the structure of our proof of the main result in this paper, we shall introduce an auxiliary intuitionistic predicate calculus *LDJ*, which is equivalent in some sense to the ordinary intuitionistic predicate calculus (cf. 2.3).

In the auxiliary calculus *LDJ*, a unary predicate symbol D is fixed, and terms and formulas used in *LDJ* are only those which contain no ε -symbol.

2.1. Axioms.

Axioms of *LDJ* are sequents of the form

$$A \rightarrow A,$$

where A is an arbitrary *LDJ*-formula.

2.2. Rules of inference.

STIPULATIONS: A and B are formulas; $\forall xF(x)$ and $\exists xF(x)$ are formulas of such forms; a is a free variable; x is a bound variable; t is a term; $F(t)$ or $F(a)$ is a formula obtained from $F(x)$ by substituting a term t or a free variable a for x in $F(x)$ (cf. 1.3); Γ and Δ are finite sequences of zero or more formulas separated by formal commas and Θ is a sequence of at most one formula.

RESTRICTION ON VARIABLES: The free variable a in the introduction of $\forall$ in succedent or the introduction of $\exists$ in antecedent shall not occur in the lower sequent. We call such a free variable a the *eigenvariable* of the related inference.

2.21. Structural rules of inference:

$$\text{Thinning:} \quad \frac{\Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta} \quad \frac{\Gamma \rightarrow}{\Gamma \rightarrow A}$$

$$\text{Contraction:} \quad \frac{A, A, \Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta}$$

$$\text{Interchange:} \quad \frac{\Gamma, A, B, \Delta \rightarrow \Theta}{\Gamma, B, A, \Delta \rightarrow \Theta}$$

$$\text{Cut:} \quad \frac{\Gamma \rightarrow A \quad A, \Delta \rightarrow \Theta}{\Gamma, \Delta \rightarrow \Theta}$$

The two explicitly shown formulas A in the upper sequents are called the *cut formulas* of the cut.

2.22. Logical rules of inference:

Introduction of in antecedent.

in succedent.

$$\neg: \quad \frac{\Gamma \rightarrow A}{\neg A, \Gamma \rightarrow} \quad \frac{A, \Gamma \rightarrow}{\Gamma \rightarrow \neg A}$$

$$\wedge: \quad \frac{A, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta} \quad \frac{B, \Gamma \rightarrow \Theta}{A \wedge B, \Gamma \rightarrow \Theta} \quad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}$$

$$\vee: \quad \frac{A, \Gamma \rightarrow \Theta \quad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta} \quad \frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B} \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}$$

$$\supset: \quad \frac{\Gamma \rightarrow A \quad B, \Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta} \quad \frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}$$

$$\forall: \quad \frac{F(t), \Gamma \rightarrow \Theta}{D(t), \forall xF(x), \Gamma \rightarrow \Theta} \quad \frac{D(a), \Gamma \rightarrow F(a)}{\Gamma \rightarrow \forall xF(x)}$$

$$\exists: \quad \frac{D(a), F(a), \Gamma \rightarrow \Theta}{\exists xF(x), \Gamma \rightarrow \Theta} \quad \frac{\Gamma \rightarrow F(t)}{D(t), \Gamma \rightarrow \exists xF(x)}$$

The explicitly shown formula $\neg A$, $A \wedge B$, $A \vee B$, $A \supset B$, $\forall x F(x)$ or $\exists x F(x)$ is called the *principal formula* of the related inference; and the one or two explicitly shown formulas A , B , $F(a)$ or $F(t)$ in upper sequents the *side formulas* of the related inference.

2.3. Let A be a formula which contains neither D nor ε , and let $a_1, \dots, a_l$ and $f_1, \dots, f_m$ be all of the free variables and of the function symbols occurring in A . For each n -ary function symbol f , let $D[f]$ mean the formula

$$\forall x_1 \cdots \forall x_n \{ (D(x_1) \wedge \cdots \wedge D(x_n)) \supset D(f(x_1, \dots, x_n)) \}.$$

Then the sequent

$$\exists x D(x), D(a_1), \dots, D(a_l), D[f_1], \dots, D[f_m] \rightarrow A$$

is provable in *LDJ* when and only when A is provable in the ordinary intuitionistic predicate calculus.

§ 3. Intuitionistic ε -calculus LEJ. Main result.

We shall introduce an intuitionistic predicate calculus with ε -symbol, called *LEJ*. In this system the same unary predicate symbol D as in *LDJ* is fixed. All of the terms and formulas defined in 1.1 are used in *LEJ*.

3.1. Rank of ε -term and formula. Normal ε -term.

3.11. Definition of 'rank'.

3.111. If $t_1, \dots, t_n$ are terms and P is an n -ary predicate symbol, the *rank* of $P(t_1, \dots, t_n)$ is 0.

3.112. If A is a formula, the *rank* of $\neg A$ is identical with the *rank* of A . If A and B are formulas, the *ranks* of $A \wedge B$, $A \vee B$ and $A \supset B$ are the maximum of the *ranks* of A and B .

3.113. Let $\forall x F(x)$ and $\exists x F(x)$ be formulas, $\varepsilon x F(x)$ be an ε -term and a be an arbitrary free variable not contained in $F(x)$, and let $F(a)$ be the formula obtained from $F(x)$ by substitution of a for the bound variable x in $F(x)$ (cf. 1. 3). Then the *ranks* of $\forall x F(x)$ and $\exists x F(x)$ are the maximum of: i) the *rank* of $F(a)$; ii) the *ranks* of those ε -subterms of $F(a)$ which contain at least one occurrence of a . Then the *rank* of $\varepsilon x F(x)$ is greater by one than the *rank* of $\exists x F(x)$.

3.12. We have easily the following 3.121 and 3.122.

3.121. Let A be a formula and B be a subformula of A . Then the rank of B is not greater than that of A .

3.122. Let U be an ε -term or formula and V be the ε -term or formula obtained by substituting an arbitrary term t for a free variable a in U . Then the rank of V is equal to that of U .

3.13. Normal ε -term.

An ε -term is said to be *normal* if the rank of each closed ε -subterm is not

greater than the rank of the original ε -term, and if the rank of each proper open ε -subterm is less than the rank of the original ε -term.

For any ε -term, there exists a normal ε -term $\varepsilon xA(x, a_1, \dots, a_n)$, which contains only the distinct free variables $a_1, \dots, a_n$, and there exist terms $t_1, \dots, t_n$, such that the given ε -term is identical with the result $\varepsilon xA(x, t_1, \dots, t_n)$ of substituting $t_1, \dots, t_n$ for free variables $a_1, \dots, a_n$ throughout $\varepsilon xA(x, a_1, \dots, a_n)$ ($n \geq 0$). The choice of the normal ε -term $\varepsilon xA(x, a_1, \dots, a_n)$ and the terms $t_1, \dots, t_n$ is not uniquely determined for the given ε -term.

3.2. Axioms.

As axioms of *LEJ* we use the sequents of the forms

$$(1) \quad A \rightarrow A,$$

$$(2) \quad D(t_1), \dots, D(t_n), \exists y(\exists xA(x, t_1, \dots, t_n) \supset A(y, t_1, \dots, t_n)) \\ \rightarrow D(\varepsilon xA(x, t_1, \dots, t_n)),$$

and

$$(3) \quad D(t_1), \dots, D(t_n), \exists xA(x, t_1, \dots, t_n) \rightarrow A(\varepsilon xA(x, t_1, \dots, t_n), t_1, \dots, t_n)$$

under the following

STIPULATIONS: A is a formula; $t_1, \dots, t_n$ are terms; $\varepsilon xA(x, t_1, \dots, t_n)$ is the result of substituting $t_1, \dots, t_n$ for free variables $a_1, \dots, a_n$ throughout a normal ε -term $\varepsilon xA(x, a_1, \dots, a_n)$ which contains only the distinct free variables $a_1, \dots, a_n$; $A(\varepsilon xA(x, t_1, \dots, t_n), t_1, \dots, t_n)$ is the formula obtained from $A(x, t_1, \dots, t_n)$ by substitution of the ε -term $\varepsilon xA(x, t_1, \dots, t_n)$ for the bound variable x in $A(x, t_1, \dots, t_n)$ (cf. 1.3); n is an arbitrary non-negative integer.

3.3. Rules of inference.

As rules of inference of *LEJ* we use the same ones as those of *LDJ*, provided that terms and formulas may contain ε -symbol.

3.4. **MAIN RESULT.** *Let A be a formula which contains no ε -symbol. A is *LEJ*-provable if and only if A is *LDJ*-provable.*

REMARK. The main result implies a weakened ε -Theorem for intuitionistic logic, which is stated in Maehara [5]. The intuitionistic predicate calculus with ε -symbol, which was introduced in Maehara [5] to state and to prove that theorem, is the system obtained from *LJ* (cf. Gentzen [1]) by the following steps:

1) permit the ε -term $\varepsilon xF(x)$ to be used provided that it is obtained from any formula $F(a)$ having only one free variable a and permit the formulas containing the logical symbol ε to be used;

2) add the axiom schemata

$$\begin{aligned} \exists xA(x) &\rightarrow A(\varepsilon xA(x)) \\ \exists y(\exists xA(x) \supset A(y)), \quad \forall zF(z) &\rightarrow F(\varepsilon xA(x)) \\ \exists y(\exists xA(x) \supset A(y)), \quad F(\varepsilon xA(x)) &\rightarrow \exists zF(z); \end{aligned}$$

3) restrict the use of the following rules of inference within the case where t is a free variable:

$$\text{Introduction of } \forall \text{ in antecedent} \quad \frac{F(t), \Gamma \rightarrow \theta}{\forall x F(x), \Gamma \rightarrow \theta}$$

$$\text{Introduction of } \exists \text{ in succedent} \quad \frac{\Gamma \rightarrow F(t)}{\Gamma \rightarrow \exists x F(x)}.$$

For brevity we shall mean the above-mentioned system by “weakened LEJ ”.

The weakened ε -Theorem for intuitionistic logic is expressed as follows:

If a formula A contains no ε -symbol and is provable in the weakened LEJ , A is provable in LJ .

For any ε -term $\varepsilon x A(x)$ permitted in the weakened LEJ , the sequents

$$\exists x A(x) \rightarrow A(\varepsilon x A(x))$$

and

$$\exists y (\exists x A(x) \supset A(y)) \rightarrow D(\varepsilon x A(x))$$

are LEJ -provable, because such an ε -term $\varepsilon x A(x)$ is always closed and normal in our sense. On the other hand, by means of the sequents of those forms, the new added axioms of the weakened LEJ are LEJ -provable. Hence the sequents provable in the weakened LEJ are LEJ -provable. From this we can see the fact that our main result implies the above-mentioned weakened ε -Theorem.

§ 4. Proof of the Main Result.

4.1. We let LEJ' be the system obtained from LEJ by the following steps:

- 1) omit the all axioms of the forms (2) and (3);
- 2) add the sequents of the form

$$(4) \quad D(t_1), \dots, D(t_n), \exists x A(x, t_1, \dots, t_n) \rightarrow \\ D(\varepsilon x A(x, t_1, \dots, t_n)) \wedge A(\varepsilon x A(x, t_1, \dots, t_n), t_1, \dots, t_n)$$

as axioms instead, where $t_1, \dots, t_n$ are terms and $\varepsilon x A(x, a_1, \dots, a_n)$ is a normal ε -term which contains only the distinct free variables $a_1, \dots, a_n$.

PROPOSITION 1. *Given an LEJ -provable formula, the formula obtained from the given formula by replacing each part of the form $\varepsilon x F(x)$ by $\varepsilon y (\exists x F(x) \supset F(y))$ is LEJ' -provable.*

PROOF. For our purpose it is sufficient to prove the fact that the sequents of the forms

$$D(t_1), \dots, D(t_n), \exists y (\exists x A(x, t_1, \dots, t_n) \supset A(y, t_1, \dots, t_n)) \rightarrow \\ D(\varepsilon y (\exists x A(x, t_1, \dots, t_n) \supset A(y, t_1, \dots, t_n)))$$

and

$$D(t_1), \dots, D(t_n), \exists x A(x, t_1, \dots, t_n) \rightarrow \\ A(\varepsilon y(\exists x A(x, t_1, \dots, t_n) \supset A(y, t_1, \dots, t_n)), t_1, \dots, t_n),$$

which are obtained from the *LEJ*-axioms (2) and (3) by replacing the part $\varepsilon x A(x, t_1, \dots, t_n)$ by $\varepsilon y(\exists x A(x, t_1, \dots, t_n) \supset A(y, t_1, \dots, t_n))$, are *LEJ'*-provable. If the ε -term $\varepsilon x A(x, a_1, \dots, a_n)$ is normal, $\varepsilon y(\exists x A(x, a_1, \dots, a_n) \supset A(y, a_1, \dots, a_n))$ is normal. By using the axiom (4) for the normal ε -term $\varepsilon y(\exists x A(x, a_1, \dots, a_n) \supset A(y, a_1, \dots, a_n))$, we can see the *LEJ'*-provability of the above two sequents, q.e.d.

From Proposition 1, for the proof of the main result, it suffices to prove the fact that every *LEJ'*-provable formula containing no ε -symbol is *LDJ*-provable. In other words this fact runs as follows:

Every formal proof in LEJ' whose endsequent contains no ε -symbol can be transformed into a formal proof in LDJ with the same endsequent.

4.2. Instead of the axiom-schema (4), in the following, we shall use the rule of inference:

$$\frac{\Gamma \rightarrow \exists x A(x, t_1, \dots, t_n) \quad D(\varepsilon x A(x, t_1, \dots, t_n)), A(\varepsilon x A(x, t_1, \dots, t_n), t_1, \dots, t_n), \Gamma \rightarrow \Theta}{D(t_1), \dots, D(t_n), \Gamma \rightarrow \Theta},$$

where Γ is a finite sequence of zero or more formulas and Θ is a sequence of at most one formula. By ' ε -inference' we shall mean each inference obeying the above rule. The explicitly shown formulas $\exists x A(x, t_1, \dots, t_n)$ and $A(\varepsilon x A(x, t_1, \dots, t_n), t_1, \dots, t_n)$ are called the *principal formulas* of the related ε -inference; and the explicitly shown formula $D(\varepsilon x A(x, t_1, \dots, t_n))$ the *side formula* of the inference.

4.3. 'Rank of formal proof'.

To estimate a complexity of each *LEJ'*-proof, we define the rank of proof as follows.

The *rank* of an *LEJ'*-proof is the largest rank of ε -terms if the proof contains at least one ε -term, otherwise the rank of the proof is 0.

4.4. Descendant.

4.41. Each occurrence of formula in upper sequents of any *LEJ'*-inference has at most one *immediate descendant* in the lower sequent. The principal formula of a logical inference is the immediate descendant of the side formulas of the inference. When the two occurrences of one and the same formula in an inference, one in the upper sequents and the other in the lower sequent, correspond to one another, the latter is the immediate descendant of the former. The cut formula has no immediate descendant. The principal formulas and the side formula of an ε -inference have no immediate descendant.

4.42. Let an *LEJ'*-proof P be given. The set of *descendants* of an arbitrarily given occurrence of formula in P are recursively defined by using the concept

“immediate descendant” as the totality of all occurrences of formulas obtained by starting with the given occurrence of formula and collecting its immediate descendant, the immediate descendant of its immediate descendant, and so on.

4.5. For our purpose (cf. 4.2), it suffices to prove the following three theorems.

THEOREM 1. *Every LEJ' -proof can be transformed into another LEJ' -proof with the same endsequent, in which no cuts occur and whose rank is not greater than that of the original formal proof.*

THEOREM 2. *Let an LEJ' -proof be given and satisfy the following conditions:*

- 1) *the endsequent contains no ε -symbol;*
- 2) *the rank is positive;*
- 3) *it contains no cuts.*

Then we can find another LEJ' -proof with the same endsequent, whose rank is less than that of the given formal proof.

THEOREM 3. *Let an LEJ' -proof satisfy the following conditions:*

- 1) *the endsequent contains no ε -symbol;*
- 2) *the rank is 0;*
- 3) *it contains no cuts.*

Then it is an LDJ -proof.

4.6. PROOF OF THEOREM 1.

We only state the outline of the proof without going into details, because the proof is carried out in almost the same way as Gentzen's proof of 'Hauptsatz' for LJ -proofs (Gentzen [1]).

With the help of several thinnings and interchanges, we replace each cut in the original LEJ' -proof by an inference obeying the new rule of inference called 'Mischung', which constitutes a modified form of the cut. Then the new formal proof obtained by this replacement will have the same endsequent and the same rank as those of the original formal proof. In the proof of Theorem 1, as formal proofs we shall have only to consider ones, which contain no cuts but may contain some Mischungs instead. Hence it suffices to prove that every LEJ' -proof can be transformed into another LEJ' -proof with the same endsequent which contains no Mischungs and whose rank is not greater than that of the original formal proof. This will be easily proved by induction on the number of Mischungs in the original formal proof using the following

LEMMA 1. *Given an LEJ' -proof with a Mischung for the lowest inference and containing no other Mischung, another LEJ' -proof with the same endsequent can be found which contains no Mischungs and whose rank is not greater than that of the given formal proof.*

The lemma is proved by an induction on the grade of the 'Mischformel', where the grade of the Mischformel means the number of occurrences of $\neg, \wedge, \vee, \supset, \forall$ and $\exists$ contained in no subterms of the Mischformel. Within both the basis and the

induction step of this induction, an induction is used on the 'Rang' of the Mischung, which has the same meaning as that in Gentzen [1].

4.7. PROOF OF THEOREM 3.

Let P be an LEJ' -proof which satisfies the conditions 1), 2) and 3) of Theorem 3. To prove the fact that P is an LDJ -proof it is sufficient to show the fact that P contains no ε -symbol. Suppose that P contains the ε -symbol. Then there exists in P at least one formula which contains the ε -symbol. By the condition 2) of Theorem 3, no ε -terms occur in P . Hence each descendant of a formula which contains the ε -symbol contains the ε -symbol. By the conditions 2) and 3) of Theorem 3, P contains neither ε -inferences nor cuts. Thus the last descendant of such a formula must occur in the endsequent. Consequently the endsequent contains the ε -symbol. This contradicts to the condition 1).

4.8. PROOF OF THEOREM 2.

PROPOSITION 2.¹⁾ *Let an LEJ' -proof with positive rank ν be given and satisfy the following conditions:*

- a) *the endsequent contains no ε -symbol;*
- b) *it contains no cuts;*
- c) *it contains no open ε -terms with rank ν .*

Then we can find another LEJ' -proof which satisfies the following conditions:

- a') *the endsequent is the same as that of the given LEJ' -proof;*
- b') *the rank is not greater than ν ;*
- c') *it contains no open ε -terms with rank ν ;*
- d') *the number of the distinct closed ε -terms with rank ν is less than that in the given LEJ' -proof.*

PROPOSITION 3.¹⁾ *Let an LEJ' -proof with positive rank ν be given and satisfy the following conditions:*

- a) *the endsequent contains no ε -symbol;*
- b) *it contains no cuts;*
- c) *it contains at least one open ε -term with rank ν .*

Then we can find another LEJ' -proof which satisfies the following conditions:

- a') *the endsequent is the same as that of the given LEJ' -proof;*
- b') *the rank is not greater than ν ;*
- c') *it contains no open ε -terms with rank ν .*

PROPOSITION 4.²⁾ *Given an LEJ' -proof with positive rank ν which contains no open ε -terms with rank ν , an LEJ' -proof can be found which satisfies the following conditions:*

- a) *the endsequent is the same as that of the given LEJ' -proof;*
- b) *the rank is not greater than ν ;*

1) cf. §5.

2) Proposition 4 is proved in almost the same way as Theorem 1 (cf. 4.6).

- c) *it contains no cuts;*
- d) *all the ε -terms with rank ν occur in the given LEJ' -proof.*

Let P be an LEJ' -proof which satisfies the conditions 1), 2) and 3) of Theorem 2 and let $\nu (>0)$ be the rank of P . We prove the fact that there is an LEJ' -proof P' which has the same endsequent as P and has a rank smaller than ν .

To prove this, we distinguish two cases:

First case: P contains no open ε -terms with rank ν (4.81);

Second case: P contains at least one open ε -term with rank ν (4.82).

4.81. First case. P contains at least one closed ε -term with rank ν . Then we shall prove the existence of P' by an induction on the number of the distinct closed ε -terms with rank ν in P . By Proposition 2, there is an LEJ' -proof P_1 which satisfies the conditions:

- a') the endsequent is the same as that of P ;
- b') the rank is not greater than ν ;
- c') it contains no open ε -terms with rank ν ;
- d') the number of the distinct closed ε -terms with rank ν is less than that in P .

4.811. The case where P_1 contains no closed ε -terms with rank ν . P_1 has a rank smaller than ν . As P' we may take P_1 .

4.812. The case where P_1 contains at least one closed ε -term with rank ν . By applying Proposition 4 to P_1 we can see the fact that there is an LEJ' -proof P_2 which satisfies the conditions:

- a) the endsequent is the same as that of P_1 ;
- b) the rank is not greater than ν ;
- c) it contains no cuts;
- d) all the ε -terms with rank ν occur in P_1 .

4.812.1. The case where P_2 contains no closed ε -terms with rank ν . P_2 has the same endsequent as P and has a rank smaller than ν . As P' we may take P_2 .

4.812.2. The case where P_2 contains at least one closed ε -term with rank ν . P_2 is an LEJ' -proof with positive rank ν which satisfies the conditions:

- a) the endsequent is the same as that of P ;
 - b) it contains no cuts;
 - c) it contains no open ε -terms with rank ν ;
 - d) the number of the distinct closed ε -terms with rank ν is less than that in P .
- By regarding P_2 as P and applying the induction hypothesis to P_2 , we can see the existence of P' .

4.82. Second case. By Proposition 3 there is an LEJ' -proof P_1 which satisfies the conditions:

- a') the endsequent is the same as that of P ;
- b') the rank is not greater than ν ;

c') it contains no open ε -terms with rank ν .

4.821. The case where P_1 contains no closed ε -terms with rank ν . P_1 has a rank smaller than ν . As P' we may take P_1 .

4.822. The case where P_1 contains at least one closed ε -term with rank ν . By applying Proposition 4 to P_1 we can see the fact that there is an LEJ' -proof P_2 which satisfies the conditions:

- a) the endsequent is the same as that of P_1 ;
- b) the rank is not greater than ν ;
- c) it contains no cuts;
- d) all the ε -terms with rank ν occur in P_1 .

4.822.1. The case where P_2 contains no closed ε -terms with rank ν . P_2 has the same endsequent as P and has a rank smaller than ν .

As P' we may take P_2 .

4.822.2. The case where P_2 contains at least one closed ε -term with rank ν . P_2 is an LEJ' -proof with positive rank ν which satisfies the conditions:

- a) the endsequent is the same as that of P ;
- b) it contains no cuts;
- c) it contains no open ε -terms with rank ν . By regarding P_2 as P and using 4.81, we can see the existence of P' .

§ 5. Proofs of Propositions 2 and 3.

5.1. Proof of Proposition 3.

Let P be the given LEJ' -proof with positive rank ν which satisfies the conditions:

- a) the endsequent contains no ε -symbol;
- b) it contains no cuts;
- c) it contains at least one open ε -term with rank ν .

We prove the fact that there is an LEJ' -proof P' which satisfies the conditions:

- a') the endsequent is the same as that of P ;
- b') the rank is not greater than ν ;
- c') it contains no open ε -terms with rank ν .

5.11. P contains no formulas with rank ν .

PROOF. If P contained a formula F with rank ν , the formula F should contain at least one ε -symbol occurring in none of the subterms, and therefore every descendant of F (cf. 4.4) should contain an ε -symbol. On the other hand, by the condition a), the endsequent of P contains no ε -symbol, accordingly the endsequent should not contain a descendant of F . By the condition b), the last descendant of F should be a principal formula of an ε -inference, and it should have the form $\exists yA(y, t_1, \dots, t_n)$ or $A(\varepsilon yA(y, t_1, \dots, t_n), t_1, \dots, t_n)$. Then the rank of $\varepsilon yA(y, t_1, \dots, t_n)$ should be greater than that of F , because the rank of F should not be

greater than that of the last descendant of F (cf. 3.121). Hence P should contain the ε -term $\varepsilon y A(y, t_1, \dots, t_n)$ with the rank greater than ν , and it should contradict to the hypothesis that P is of rank ν , q.e.d.

5.12. Let $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$ be all of the distinct open ε -terms with rank ν in P , and $\varepsilon x_i E_i(x_i)$ be a subterm of none of $\varepsilon x_{i+1} E_{i+1}(x_{i+1}), \dots, \varepsilon x_m E_m(x_m)$ ($i=1, \dots, m-1$).

Let $c_1, \dots, c_m$ be m distinct free variables not contained in P .

5.13. The formal expression obtained from a formal expression U by the following replacements will be denoted by U^* : first we replace all of $\varepsilon x_1 E_1(x_1)$ by c_1 in U ; secondly all of $\varepsilon x_2 E_2(x_2)$ by c_2 in the first result; thirdly all of $\varepsilon x_3 E_3(x_3)$ by c_3 in the second result; and so on, till none of the ε -terms $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$ occur in the result.

When $\forall x F(x)$, $\exists x F(x)$ or $\varepsilon x F(x)$ is a formula or term of such a form and t is an arbitrary term, $F^*(x)$ means $(F(x))^*$ and $F^*(t)$ means the result of substituting t for x in $F^*(x)$ (cf. 1.3).

When Γ stands for a finite sequence of formulas $A_1, \dots, A_l$, Γ^* means the sequence $A_1^*, \dots, A_l^*$.

5.131. If U is $\varepsilon x_i E_i(x_i)$, U^* is c_i ($i=1, \dots, m$).

5.132. If U is an ε -term other than $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$, U^* is also an ε -term and the rank of U^* is equal to that of U (cf. 3.122).

5.133. If c_j is contained in $E_i^*(c_i)$, and c_j is not c_i , then $\varepsilon x_j E_j(x_j)$ is a proper subterm of $\varepsilon x_i E_i(x_i)$ (cf. 5.12).

5.134. If a is a free variable other than $c_1, \dots, c_m$, and is contained in U^* , then it is contained in U .

5.135. $(E_i(\varepsilon x_i E_i(x_i)))^*$ is $E_i^*(c_i)$ ($i=1, \dots, m$).

PROOF. To prove the above assertion, it suffices to show the fact that the formula $E_i(\varepsilon x_i E_i(x_i))$ contains no open ε -subterms with rank ν which contain the ε -term $\varepsilon x_i E_i(x_i)$ as a proper subterm.

If $E_i(\varepsilon x_i E_i(x_i))$ contained such an open ε -subterm with rank ν , for a free variable a not occurring in $\varepsilon x_i E_i(x_i)$ $E_i(a)$ should contain an open ε -subterm with rank ν which contains the free variable a . Hence the rank of $\varepsilon x_i E_i(x_i)$ should be greater than ν , and it should contradict to the hypothesis that the rank of $\varepsilon x_i E_i(x_i)$ is of rank ν .

5.14. Let S be a sequent

$$\Gamma \rightarrow \Theta$$

in P , and $\varepsilon x_{i_1} E_{i_1}(x_{i_1}), \dots, \varepsilon x_{i_k} E_{i_k}(x_{i_k})$ be the all of the distinct open ε -terms with rank ν contained in S . Then S^* means the sequent

$$\Gamma^*, \Psi \rightarrow \Theta^*,$$

where Ψ is a finite sequence of formulas $D(c_{i_1}), \dots, D(c_{i_k}), E_{i_1}^*(c_{i_1}), \dots, E_{i_k}^*(c_{i_k})$.

5.141. If S is a sequent in P , S^* contains no open ε -terms with rank ν .

PROOF. If S^* contained an open ε -term with rank ν , it should be identical with $(\varepsilon xF(x))^*$ for some ε -term $\varepsilon xF(x)$ with rank ν in P (cf. 5.132, 5.135) and $\varepsilon xF(x)$ should be distinct from $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$ (cf. 5.131).

5.141.1. The case where $(\varepsilon xF(x))^*$ contains none of $c_1, \dots, c_m$. The ε -term $(\varepsilon xF(x))^*$ should be identical with $\varepsilon xF(x)$, and accordingly $\varepsilon xF(x)$ should be an open ε -term with rank ν in P . On the other hand, $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$ are the only open ε -terms with rank ν in P . Hence $\varepsilon xF(x)$ should be identical with one of $\varepsilon x_1 E_1(x_1), \dots, \varepsilon x_m E_m(x_m)$. This is a contradiction.

5.141.2. The case where $(\varepsilon xF(x))^*$ contains at least one of $c_1, \dots, c_m$. Some $\varepsilon x_i E_i(x_i)$ should be a subterm of $\varepsilon xF(x)$. By the fact that $\varepsilon x_i E_i(x_i)$ is open, $\varepsilon xF(x)$ should be open. Hence $\varepsilon xF(x)$ should be an open ε -term with rank ν in P . By the same reason in 5.141.1, it implies the contradiction.

5.142. If S is a sequent in P , S^* contains no ε -terms with greater ranks than ν (cf. 5.131, 5.132).

5.15. A sequent S is said to be (ν) -provable when there exists an LEJ' -proof which satisfies the following conditions:

- a) the endsequent is S ;
- β) the rank is not greater than ν ;
- γ) it contains no open ε -terms with rank ν .

To prove the existence of an LEJ' -proof P' which satisfies the previously stated conditions a'), b') and c'), it is sufficient to prove the following

LEMMA 2. If S is a sequent in P , then P^* is (ν) -provable.

Because, if S is the endsequent of P , then S^* is identical with S itself and therefore S is (ν) -provable.

PROOF OF LEMMA 2.

5.151. The case where S is an axiom

$$A \rightarrow A.$$

S^* has the form

$$A^*, \psi \rightarrow A^*.$$

It is clearly (ν) -provable (cf. 5.141, 5.142).

5.152. The case where S is the lower sequent of an inference I in P . We shall show that S^* is (ν) -provable under the hypothesis that for each $\bar{S}$ of the one or two upper sequents of I $\bar{S}^*$ is (ν) -provable.

5.152.1. When I is an introduction of $\forall$ in succedent:

Let S be

$$\Gamma \rightarrow \forall x F(x).$$

The upper sequent S_1 of I has the form

$$D(a), \Gamma \rightarrow F(a),$$

where a is a free variable not contained in S . S^* and S_1^* have the forms

$$\Gamma^*, \Psi \rightarrow \forall x F^*(x)$$

and

$$D(a), \Gamma^*, \Psi_1 \rightarrow (F(a))^*,$$

respectively. By 5.11, P contains no open ε -terms with rank ν which are subterms of $F(a)$ and contain the eigenvariable a of I . Hence $(F(a))^*$ is identical with $F^*(a)$ and Ψ_1 is identical with Ψ . The free variable a does not occur in S^* (cf. 5.134). From these facts, by using an inference of introduction of $\forall$ in succedent, we can obtain the LEJ' -provability of S^* from that of S_1^* . Moreover S^* contains neither open ε -terms with rank ν nor ε -terms with ranks greater than ν (cf. 5.141, 5.142). Hence S^* is (ν) -provable when S_1^* is (ν) -provable. By the hypothesis of the induction S_1^* is (ν) -provable, and therefore S^* is (ν) -provable.

5.152.2. When I is an introduction of $\forall$ in antecedent:

Let S be

$$D(t), \forall x F(x), \Gamma \rightarrow \Theta.$$

The upper sequent S_1 of I has the form

$$F(t), \Gamma \rightarrow \Theta.$$

S^* and S_1^* have the forms

$$D(t^*), \forall x F^*(x), \Gamma^*, \Psi \rightarrow \Theta^*$$

and

$$(F(t))^*, \Gamma^*, \Psi_1 \rightarrow \Theta^*,$$

respectively. If a is a free variable not occurring in $F(t)$, P contains no open ε -terms with rank ν which are subterms of $F(a)$ and contain the free variable a (cf. 5.11). Hence $(F(t))^*$ is identical with $F^*(t^*)$, and Ψ_1 is identical with Ψ . From these facts, by using an inference of introduction of $\forall$ in antecedent, we can prove that S^* is LEJ' -provable if S_1^* is LEJ' -provable. Moreover S^* contains neither open ε -terms with rank ν nor ε -terms with ranks greater than ν (cf. 5.141, 5.142). Hence S^* is (ν) -provable when S_1^* is (ν) -provable. By the hypothesis of the induction S_1^* is (ν) -provable, and therefore S^* is (ν) -provable.

5.152.3. When I is an ε -inference:

Let S be

$$D(t_1), \dots, D(t_n), \Gamma \rightarrow \Theta.$$

The upper sequents S_1 and S_2 of I have the forms

$$\Gamma \rightarrow \exists x A(x, t_1, \dots, t_n)$$

and

$$D(\varepsilon x A(x, t_1, \dots, t_n)), A(\varepsilon x A(x, t_1, \dots, t_n), t_1, \dots, t_n), \Gamma \rightarrow \Theta,$$

respectively. As stipulated in 4.1 $t_1, \dots, t_n$ are terms and $\varepsilon x A(x, a_1, \dots, a_n)$ is a

normal ε -term in which $a_1, \dots, a_n$ are the only distinct free variables. S^* has the form

$$D(t_1^*), \dots, D(t_n^*), \Gamma^*, \Psi \rightarrow \Theta^*.$$

5.152.31. The case where $\varepsilon xA(x, t_1, \dots, t_n)$ is not an open ε -term with rank ν :

Since $\varepsilon xA(x, a_1, \dots, a_n)$ is normal and is of rank not greater than ν , it contains no proper open ε -subterms with rank ν (cf. 3.13). Hence the open ε -subterms with rank ν of $\varepsilon xA(x, t_1, \dots, t_n)$ are the only those of $t_1, \dots, t_{n-1}$ or t_n . Consequently S_1^* and S_2^* have the forms

$$\Gamma^*, \Psi_1 \rightarrow \exists xA(x, t_1^*, \dots, t_n^*)$$

and

$$D(\varepsilon xA(x, t_1^*, \dots, t_n^*)), A(\varepsilon xA(x, t_1^*, \dots, t_n^*), t_1^*, \dots, t_n^*), \Gamma^*, \Psi_2 \rightarrow \Theta^*,$$

respectively.

All formulas in Ψ_1 are contained in Ψ , and Ψ contains neither open ε -terms with rank ν nor ε -terms with rank greater than ν (cf. 5.141, 5.142). By the hypothesis of the induction S_1^* is (ν) -provable. Hence the sequent

$$\Gamma^*, \Psi \rightarrow \exists xA(x, t_1^*, \dots, t_n^*)$$

is (ν) -provable. On the other hand, the formulas in Ψ_2 are the same formulas that are in Ψ , and therefore S_2^* is

$$D(\varepsilon xA(x, t_1^*, \dots, t_n^*)), A(\varepsilon xA(x, t_1^*, \dots, t_n^*), t_1^*, \dots, t_n^*), \Gamma^*, \Psi \rightarrow \Theta^*,$$

which is (ν) -provable by the hypothesis of the induction.

S^* contains neither open ε -terms with rank ν nor ε -terms with rank greater than ν (cf. 5.141, 5.142). Hence, using an ε -inference which has the above two sequents as the upper sequents, S^* is (ν) -provable.

5.152.32. The case where $\varepsilon xA(x, t_1, \dots, t_n)$ is an open ε -term with rank ν :

Since $\varepsilon xA(x, a_1, \dots, a_n)$ is normal and is of rank ν , it contains no open ε -subterms with rank ν (cf. 3.13). Hence the proper open ε -subterms with rank ν of $\varepsilon xA(x, t_1, \dots, t_n)$ are the only subterms of $t_1, \dots, t_{n-1}$ or t_n . By this fact, S_1^* has the form

$$\Gamma^*, \Psi_1 \rightarrow \exists xA(x, t_1^*, \dots, t_n^*).$$

Let c be the free variable by which $\varepsilon xA(x, t_1, \dots, t_n)$ is replaced in the definition of U^* (cf. 5.13). Then $(D(\varepsilon xA(x, t_1, \dots, t_n)))^*$ is identical with $D(c)$ (cf. 5.131), and by above fact $(A(\varepsilon xA(x, t_1, \dots, t_n), t_1, \dots, t_n))^*$ is identical with $A(c, t_1^*, \dots, t_n^*)$ (cf. 5.135). Hence S_2^* has the form

$$D(c), A(c, t_1^*, \dots, t_n^*), \Gamma^*, \Psi_2 \rightarrow \Theta^*,$$

where $D(c)$ and $A(c, t_1^*, \dots, t_n^*)$ are contained in Ψ_2 .

5.152.321. The case where $\varepsilon xA(x, t_1, \dots, t_n)$ is contained in $\Gamma \rightarrow \Theta$:

By the hypothesis of the induction, S_2^* is (ν) -provable. In this case the formulas in Ψ_2 are the same formulas that are in Ψ . Hence by possibly several interchanges and two contractions, the sequent

$$\Gamma^*, \Psi \rightarrow \Theta^*$$

is (ν) -provable. By n thinnings, S^* is (ν) -provable.

5.152.322. The case where $\varepsilon xA(x, t_1, \dots, t_n)$ is not contained in $\Gamma \rightarrow \Theta$:

The only one open ε -term with rank ν that is contained in S_2 but not in S is $\varepsilon xA(x, t_1, \dots, t_n)$. Hence Ψ_2 is obtained from Ψ by adding the two formulas $D(c)$ and $A(c, t_1^*, \dots, t_n^*)$. By the hypothesis of the induction S_2^* is (ν) -provable. Therefore, using possibly several interchanges and two contractions, the sequent

$$D(c), A(c, t_1^*, \dots, t_n^*), \Gamma^*, \Psi \rightarrow \Theta^*$$

is (ν) -provable. The free variable c does not occur in the sequent (cf. 5.133)

$$(1) \quad \exists xA(x, t_1^*, \dots, t_n^*), \Gamma^*, \Psi \rightarrow \Theta^*.$$

The result of substituting c for x in $A(x, t_1^*, \dots, t_n^*)$ (cf. 1.3) is identical with $A(c, t_1^*, \dots, t_n^*)$. Hence, by an introduction of $\exists$ in antecedent, the sequent (1) is (ν) -provable. On the other hand, by the hypothesis of the induction S_1^* is (ν) -provable. Using a cut which has the sequents (1) and S_1^* as the upper sequents, the sequent

$$\Gamma^*, \Psi_1, \Gamma^*, \Psi \rightarrow \Theta^*$$

is (ν) -provable. Hence by possibly several interchanges, contractions and thinnings, the sequent S^* is (ν) -provable.

5.152.4. When I is an inference of other kind:

The treatment is similar to the above or simpler, hence the proof is omitted. We only notice that for the proof it is essential that P contains no cuts (cf. condition b)).

5.2. Proof of Proposition 2.

Let P be the given LEJ' -proof with positive rank ν which satisfies the conditions:

- a) the endsequent contains no ε -symbol;
- b) it contains no cuts;
- c) it contains no open ε -terms with rank ν .

We prove the fact that there is an LEJ' -proof P' which satisfies the conditions:

- a') the endsequent is the same as P ;
- b') the rank is not greater than ν ;
- c') it contains no open ε -terms with rank ν ;
- d') the number of the distinct closed ε -terms with rank ν is less than that in P .

5.21. P contains no formulas with rank ν (cf. 5.11).

5.22. Let $\varepsilon x_1 E_1(x_1)$ be one of the closed ε -terms with rank ν which are not subterms of any other ε -term with rank ν in P .

Let c_1 be a free variable not contained in P .

5.23. The formal expression obtained from a formal expression U by replacing all of $\varepsilon x_1 E_1(x_1)$ in U by c_1 is denoted by U^* .

When $\forall x F(x)$, $\exists x F(x)$ or $\varepsilon x F(x)$ is a formula or term of such a form and t is an arbitrary term, $F^*(x)$ means $(F(x))^*$ and $F^*(t)$ means the result of substituting t for x in $F^*(x)$ (cf. 1.3).

When Γ stands for a finite sequence of formulas $A_1, \dots, A_l$, Γ^* means the sequence $A_1^*, \dots, A_l^*$.

5.231. If U is $\varepsilon x_1 E_1(x_1)$, U^* is c_1 .

5.232. If U is an ε -term and is not $\varepsilon x_1 E_1(x_1)$, U^* is also an ε -term and the rank of U^* is equal to that of U (cf. 3.122).

5.233. If a is a free variable not identical with c_1 , and a is contained in U^* , then it is contained in U .

5.234. $(E_1(\varepsilon x_1 E_1(x_1)))^*$ is $E_1(c_1)$, for $\varepsilon x_1 E_1(x_1)$ is not a subterm of any other ε -term with rank ν .

5.235. If V is an ε -subterm of $E_1(c_1)$ and is of rank ν , V does not contain c_1 .

5.24. Let S be a sequent

$$\Gamma \rightarrow \Theta$$

in P . S^* means the sequent

$$\Gamma^*, \Psi \rightarrow \Theta^*,$$

where Ψ is a sequence of formulas $D(c_1)$, $E_1(c_1)$ or void according as S contains $\varepsilon x_1 E_1(x_1)$ or not.

5.241. If S is a sequent in P , the ε -terms with rank ν in S^* are those which occur in S but are not $\varepsilon x_1 E_1(x_1)$ (cf. 5.231, 5.232, 5.234, 5.235).

5.242. If S is a sequent in P , S^* contains no ε -terms with rank greater than ν (cf. 5.231, 5.232).

5.25. A sequent S is said to be *P -provable without $\varepsilon x_1 E_1(x_1)$* when there exists an *LEJ'*-proof which satisfies the following conditions:

- a) the endsequent is S ;
- β) the rank is not greater than that of P ;
- γ) the ε -terms with rank ν occur in P , but are not $\varepsilon x_1 E_1(x_1)$.

To show the existence of an *LEJ'*-proof P' which satisfies the previously stated conditions a'), b'), c') and d'), it is sufficient to prove the following

LEMMA 3. *If S is a sequent in P , then S^* is P -provable without $\varepsilon x_1 E_1(x_1)$. Because, if S is the endsequent of P , then S^* is identical with S itself and therefore S is P -provable without $\varepsilon x_1 E_1(x_1)$.*

The proof of this lemma is carried out in a similar way as that of Lemma 2 (cf. 5.15), and it is omitted.

References

- [1] G. Gentzen, Untersuchungen über das logische Schliessen. Math. Zeitschr., 39, 176–210, 405–431 (1935).
- [2] D. Hilbert and P. Bernays, Grundlagen der Mathematik, Bd. 2. Berlin, 1939.
- [3] S. Maehara, The predicate calculus with ε -symbol. J. Math. Soc. Japan, 7. 323–344 (1955).
- [4] S. Maehara, Equality axiom on Hilbert's ε -symbol. J. Fac. Sci, Univ. Tokyo, Sec. I, 7. 419–435 (1957).
- [5] S. Maehara, A general theory of completeness proofs. Ann. Jap. Ass. Philos. Sci, 3. 54–68 (1970).