***A Meta-Physical Investigation for the Special
Theory of Relativity***
— Preliminary Remarks —

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The term Meta-Physics is to be explained. An adjective “meta” is used on one hand in a similar sense to the one used in the term metamathematics, and at the same time we mean by “meta-physics” a survey of the foundation of physics. In other words we will attempt a metatheoretical and philosophical investigation for the special theory of relativity. In this paper preliminary remarks for such purpose are presented.

§ 1. Introduction.

It is usually considered that Einstein’s special theory of relativity is based on the following two principles:

1) The special principle of relativity, which says that “the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good” [1], and

2) the principle of constancy of light velocity which says that “light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body” [1]. And it is also considered that the Lorentz transformation can be derived from these principles. Indeed various methods for such derivation have been proposed, but it is a question whether the Lorentz transformation can be derived from these principles only. In many derivations some additional assumptions are usually proposed in the course of the deduction. These additional conditions are considered as the natural ones from the common sense of physicists, but strictly speaking the physical foundation of such conditions is not necessarily clear. Therefore we will now attempt to propose a new method of derivation for the Lorentz transformation without such “unreasonable” conditions. In the next section we will show some examples of such “unreasonable” conditions. Our assumptions for such approach base only on physically verified facts and some further conventional assumptions whose meanings are already clear. The purpose of this paper is to clarify the foundation of the special theory of relativity and its limit of application. In other words we will verify directly the “truth” of the Lorentz transformation in opposition to the

usual belief that the “truth of the Lorentz transformation” is assured indirectly by the fact that the Lorentz invariant theories fit exactly to many experimental facts. Namely, we will survey the foundation of a theory by the heuristic considerations and the logical analysis of the theory. In this sense this investigation will be regarded as a meta-physical approach for the theory of special relativity.

In such approach the relation between the theoretical concepts used in the deduction and the experimental results taken as verified should be well considered. Because the special theory of relativity is quite a new theory different from the old one, therefore in this theory old ready-made theoretical concepts must not be used unconditionally. Moreover “the relativistic effects” are very small, then the accuracy of the experiments must be examined carefully.

§ 2. Some problems in the usual derivations.

As is stated in the foregoing section it is usually understood that the Lorentz transformation can be derived from the fundamental principles of the special theory of relativity. But usually in almost every derivations some assumptions independent of the fundamental principles are proposed for “a priori” reasons. We will clarify these circumstances further in the following.

Generally speaking the derivation of some consequence is a deductive reasoning written schematically as follows:

$$S_1 S_2 \cdots S_k \rightarrow S_l, \quad (1)$$

where S_l on the right hand side is the conclusion to be derived, and S_i ($i=1, \dots, k$) on the left hand side is proposition assumed already verified. In this scheme the “truth” of the proposition on the left hand side of the formula is considered to be assured by the one of the following conditions:

- 1) The direct conclusion from the fundamental principle,
- 2) the truth of mathematics and logic,
- 3) experimentally verified facts,
- 4) conventions or definitions, or
- 5) logical conclusion derived from the propositions situated on the left side of the proposition.

On the other hand when we investigate the usual derivations of the Lorentz transformation, we find that some assumptions which are not of the above nature are used in the course of derivation. We will show some examples of such procedure in detail, but we remark some general problems beforehand. The most important of them will be how to select the propositions having the nature stated in 1).

The special principle of relativity requires that physical laws should be covariant under a certain coordinate transformation connecting the inertial frames which move uniformly each other. In this connection the problems to be considered

are

- i) the equivalence of some kind of reference systems, and
- ii) the kinematical relation of such reference system, i.e.
- i') the problem how to define "the inertial system", and
- ii') what is the relation between them.

Therefore the first problem to be considered is how to lay out the coordinate system in an inertial system. After this procedure of the layout the equivalent frame can be layed out in this system by the same method. That will be at least the sufficient condition for the first problem. Since the second problem is the kinematical one so the question will be how to connect two mutually moving systems by what kind of coordinate transformation.

Laying stress on these problems we will next show in some typical papers how to treat these problems. The first paper which we will refer is written by Einstein in 1905 [1]. He begins his paper with the heuristic considerations how to make the principle of the constancy of the light velocity consistent with such concepts as the length of the uniformly moving rigid rod or the synchronization of clocks situated at different places.

After that he searches for the form of the coordinate transformation. In the course of his reasoning he put as the mathematically important assumption that the transformation is linear. He writes as the reason of this linearity assumption as follows: "wegen Homogenitätseigenschaften, welche wir Raum und Zeit beilegen" [1]. However the meaning of the term, "Homogenität" of the space and time is not clear and moreover it is also not clear how the linearity of the transformation originates from this homogeneous character of the space and time. From our point of view this linearity condition cannot be considered well founded. Further the obscurity of this assumption is pointed out later by several authors [2], [3]. Therefore such condition is to be excluded.

Next we will cite the paper written by Severi [4]. This paper gives us another appropriate example for our discussion. The problematic point on this paper is the assumption that the length of the uniformly moving rigid rod seems to be of constant length for the observer in the rest system. The correctness of this assumption will be verified, if we succeed the direct experimental observation of the moving rod. However in the present we have no method having enough exactness. Therefore, though the length of the moving rod may be "observable" in principle, this assumption is, in this sense, also undesirable from our point of view.

Examples of the same kind also appears in the text book written by Panofsky and Phillip [5]. They assume that parallel measuring sticks oriented perpendicular to their direction of relative motion has the same length. They also put this assumption because of the homogeneity of the space. This assumption is also

unacceptable from the same reason. Pauli [6] also takes the same kind of assumption that the length of the uniformly moving rods have the same value independent of the directions of motion if they moves lengthwise. His reason is the symmetry of the space.

In other papers some authors assume that the transformation makes a group, or some mathematical conditions at infinity are assumed for the a priori reasons. These are equally of undesirable nature from the standpoint stated above. Hereupon we will make some comments concerning the meaning of the term "observable". Though we said in the above that we should not assume unobservable facts as self-evident, the inverse problem whether we can assume that fact as self-evident only because of its "observable" nature. To make the contents of the discussion precise it is necessary to clarify the meaning of the term "observable". Generally speaking some discrepancies exist as for the intension of the term between philosophers and physicists. Detailed discussions will be reserved to the next section concerning the criterion of meaning. Among physicists the term is used in narrow sense or in wide sense. In the latter sense it is used as "observable in principle", in this sense the results of the observation are not necessarily known. On the other hand in the narrow sense we use only when the results can be practically obtained. We will adopt the narrow sense of the term. Therefore "the length of a moving rigid scale" is not observable in our above sense, though it is "observable in principle".

§ 3. Our standpoint concerning the criterion of the meaning.

At first we will propose a question whether the theory of natural science must be unique and necessary. Further we will investigate whether such a question is the meaningful one or not. In this respect we will reconsider the theory proposed by Kant.

As is well known, Kant insists that the authentic and necessary character of the theory of natural science is due to the fact that the propositions in scientific theories have synthetic a priori character. Basing on this ground, he build the scheme of his theory, i.e. he shows that the system of all principles of pure understanding can be derived through the reasoning of the transcendental analytic. Following this line of thought we may be enlightened on many points, at the same time it seems to me difficult to make the theory consistent with the modern physics. Nevertheless the discussions appeared in Transcendental Aesthetic seems to me consistent with the concepts of space and time in the theory of relativity. But a technical problem such as the derivation of Lorentz transformation cannot be followed from the discussions of such a priori nature.

On the other hand there exists the theory of empiricist criterion of meaning. In this paper we expect to lay the foundation of the authenticity and necessity of

the theory on such empiricist considerations. Although the empiricist criterion of meaning have proposed by many authors in several different forms, we mean here the following vague sense that we can decide whether any proposition has empirical meaning or not in connection with the fact that it is an observation proposition. Therefore we will attempt to build a theory of natural science based only on such empirically meaningful sentences and reasonings using some mathematical propositions whose truth are already clear. If such an attempt can be established, we can advocate the authentic character of the theory through the reasoning of the theory. We will be thus able to anticipate that the "authenticity" of the theory is verified through such procedure. Though our paper is only a preliminary one for such aim, and the results accomplished are limmited in small region, such an attempt may serve as a cheque for the validity of the empiricist criterion of meaning. For such purpose it may be useful to take some simple physical theory from the outside region of the classical ones. The so-called special relativistic kinematics may be very useful for such purpose, and at the same time such survey can be regarded as a metatheory of the special theory of relativity.

§ 4. Preliminary considerations for a thought experiment.

As is stated in the above sections we will start our approach with the enumeration of physically verified facts and concepts which are useful for our purpose. Using the concepts formulated in such a way we will lay out the coordinate system in a rest system.

1. Rest system. In almost every papers of the derivation of Lorentz transformation the concept of rest system is regarded as well established one. Then the next problem is how to describe the motion in this system. In our approach following conditions are presupposed:

- i) The space is the three-dimensional Euclidean.
- ii) Any observer in this space has a clock and a scale.
- iii) He is equipped with some experimental devices, with which he can establish some physical experiments. The results which he can establish are shown in the next subsection.
- iv) He can lay out a coordinate system with his clock and scale or with his equipments. Detailed discussions will be made in the next subsection.
- v) As the results of the above condition coordinates of any body is given at any time t as

$$\mathbf{x}_p = \{x_p(t), y_p(t), z_p(t)\} .$$

2. Some experimental results concerning the propagation of light.

- i) The constancy of the velocity of light. In Einstein's original paper, the

so-called principle of the constancy of the velocity of light which says that "light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body" [1] is regarded as a conjecture. On the other hand we have many experimental results which clarify that the constancy of the light velocity is not a mere assumption but a verified experimental result. Therefore we take it as follows: In vacuum light propagates with the constant velocity c independent of the velocity both of its source and receiver, the path of which describes the straight line, where c is a universal constant independent of the place, direction and time.

ii) The rate of a moving clock. At first we will define the rate of the moving clock. Let us now suppose an observer M , moving with constant velocity along a straight line in a rest system, carries a clock with him, and let A, B are two point on the straight line l ;

t_1 =the reading of the clock at A when the moving observer passes A ;

t_2 = " " " at B " " passes B ;

and T' =the number of seconds ticked off by the clock carried by M during the motion from A to B . We will define the observed rate r at which the clock carried by M is running with respect to the clocks at rest in the rest system as:

$$r = \frac{T'}{t_2 - t_1}.$$

On the other hand the so-called Ives-Stilwell experiment concerning the transverse Doppler effect is reported, and we can conclude from their result that: The frequency of a moving atomic source is altered by the factor $\sqrt{1 - v^2/c^2}$, where v is the velocity of the source with respect to the observer in the rest system.

Taking account of this result, we can conclude that the observed rate r of the moving clock is in the above case

$$r = \sqrt{1 - v^2/c^2}.$$

3. Some remarks on "light geometry". Before we proceed to the next step, some concepts in the framework of light geometry are to be introduced. In this geometry proposed by Reichenbach light is regarded "to be the only physical entity used for the measurement of space", [7]. The fundamental principles of this geometry consist of the definition of the synchronization of the clocks situated at different places and that of spatial distance between two points by means of light ray. As is easily seen these definitions are consistent with usual ones in a rest system because of the above stated facts concerning the propagation of light. The problems now arise in the layout of the coordinate system in a moving system. We propose, for this purpose, in the next section a thought experiment following the line of thought proposed by Huntington [8].

§ 5. A thought experiment based on which Lorentz transformation is derived.

The thought experiment which will be explained in the following was partly introduced at the first time by Einstein [1] and since then was adopted in various form repeatedly by many authors [8], [9]. Among them the scheme proposed by Huntington is useful for our purpose. Though his standpoint is different from ours, and the assumptions used by him are not same as ours, we can adopt his method after some alterations.

1. Coordinate system in a rest system. Laying out a coordinate system, observer in a rest system can assign to each event a pair of real numbers called its coordinates, cartesian coordinates. We adopt the usual method of the procedure for the layout of a coordinate system in a rest system. Because of the experimental results verified by the optical method stated above this procedure is consistent with the definitions in terms of the light geometry. We will resume some fundamental results. At first we will define the observed velocity. The observers in a rest system can now define the velocity of an object which moves across the position situated by the observer. In our present paper only the uniform motions are treated. For this purpose it is necessary to define the straightness of the motion and the constancy of the velocity. We will call the motion as straight if its locus is given by the straight line in the space and is represented using the coordinates as usual. This condition of the straightness is also consistent with the behavior of the propagation of light. Next the uniformity of the velocity is assured by the following facts. We consider a motion in (x, y) plane, for simplicity's sake, and take two arbitrary observers A and B on the locus of the moving body. Observer A , at (x_1, y_1) , notes that the body passes his position when his clock reads t_1 , and observer B , at (x_2, y_2) , notes that it passes his place when his clock reads t_2 . The distance between A and B is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and the "difference in clock readings" is given by

$$t_2 - t_1.$$

Then the ratio of the former by the latter is called the observed velocity of the body with respect to the rest system:

$$v = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{t_2 - t_1}.$$

If this value is the same for any arbitrary taken two observers on the locus, the motion of the body is called uniform. Using this definition, we can say that the light is propagated straightly in this system with a uniform velocity.

2. Moving coordinate system. In this subsection we will propose a method for obtaining some results for a moving observers. We will use these results for laying out the moving system of coordinates. Let an observer O' move along the x -axis in the positive direction with a constant observed velocity v with respect the rest system. We consider four positions of the observer O' , namely:

- (1) when O' is opposite O ;
- (2) when a light-signal starts from O' along x -axis to any observer A' on x -axis;
- (3) when this light-signal arrives at A' and is immediately reflected to O' ;
- (4) when the signal returned to O' .

Let the clocks of O and O' read zero in the position (1). When the clock at

Fig. 1

O' reads t'_0 , the signal starts from O' and the point at which O' is opposite has an abscissa x_0 and a clock-reading t_0 . In this position (2) the relations between the coordinates of O' and that of the opposite point are given as follows. (See Fig. 2)

Fig. 2

Since the observed rate of the clocks at O' is given by

$$r = \sqrt{1 - v^2/c^2},$$

we have $t'_0 = rt_0$. And since the observed velocity of O' is v , x_0 is given by $x_0 = vt_0$.

In position (3), when the light-signal has reached A' , the reading of the clock carried by A' is t' , and let the abscissa of the opposite point on x -axis is x and the clock-reading t . Fig. 3 shows this position.

Fig. 3

In position (4), when the return signal has reached O' , the clock at O' read t_2' , and let x_2 and t_2 be the abscissa and clock-reading of the point opposite O' at that instant. Then $t_2' = rt_2$, and $x_2 = vt_2$.

Fig. 4

3. Derivation of the transformation equation. By definition the observed velocity of the light-signal is given by

$$c = \frac{x - vt_0}{t - t_0} \quad (1)$$

in the interval between Position (2) and Position (3). Similarly, we have

$$-c = \frac{vt_2 - x}{t_2 - t} \quad (2)$$

in the interval between Position (3) and Position (4).

Further if we synchronize the clock at A' with the one at O' , we have, by definition,

$$t' = (1/2)(t_0' + t_2') = (r/2)(t_0 + t_2). \quad (3)$$

This reading coincides with the one synchronized in a usual way on a rest system.

Finally the abscissa of A' can be determined in terms of light-geometry as

$$x' = c(t' - rt_0).$$

We can obtain x' and t' as functions of x and t from the above four equations. Namely

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \quad (5)$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \quad (6)$$

A similar kind of thought experiment will give us for a moving point not on x -axis as its y coordinate

$$y = y'. \quad (7)$$

By another similar thought experiment we can prove that the moving system thus constructed has the Euclidean character.

4. Remarks on the results obtained. The equations (5) to (7) give the Lorentz transformation. In this way we can establish the layout of a coordinate system in a uniformly moving system. Each step of the above reasoning can easily be verified by means of light geometrical procedure. Namely the distance between two points in a moving system calculating with the above equations has the same value as that defined light geometrically. The results obtained in the above subsection thus give us the method of the layout of a coordinate system in a uniformly moving system. In the above derivation we need not assume further a priori conditions as adopted in other usual derivations. Namely a constant length defined by

$$\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}$$

in a moving system is observed to be constant in the original rest system. This result is in most derivation an assumption to derive the transformation equation.

We have thus complete the layout of a moving coordinate system starting from the one in a rest system. We will explain in this place some properties of the obtained coordinate system and also some conclusions which can be derived from this transformations. These equations have quite the same form as the ordinary Lorentz transformation. Therefore every conclusions derived from the usual Lorentz equation (strictly speaking special Lorentz transformation) are also appropriate in our case. Further as is easily seen, many kinds of assumptions layed in other derivations, the physical meaning of which is not necessarily clear, can be in our case derived as the results of the transformation equations. We enumerate the examples of such assumptions in the following:

- i) The transformation is linear,
- ii) the length of a moving rigid scale is constant for an observer in a rest system,
- iii) the transformations form group, or
- iv) some mathematical conditions at infinity.

Thus we have obtained the so-called Lorentz transformation without using the assumptions the meaning of which is not physically clear. This method of derivation has also clarified the question whether we obtain another form of equation

if we do not use any one of the above assumptions.

§ 6. Concluding Remarks.

In this section we will comment on some problems which are not argued in the foregoing sections. The following three problems are among them especially important.

1. The scale and the clock. At first it must be reminded that the constancy of the velocity of light is an experimentally verified fact in a rest system (or in a so-called laboratory system), while in a moving system the constancy is assured only as a matter of conventional fact. Namely the distance between two spatial points is determined by light geometrical method. The definition of the clock and scale is, on the other hand, not given explicitly in the rest system. The usual concepts of them are used in our case. Therefore we understand implicitly that the following conditions, say, the coincidence of the reading of the atomic clock and that of the mechanical clock or the agreement of the reading of the rigid scale and that of the one made of the light geometrical method, are satisfied. In other words, the appropriateness of both "light axioms" and "matter axioms" in terminology proposed by Reichenbach [7] is considered as proved in a rest system. Then we shall consider these circumstances in our case of a moving system. We lay out a moving coordinate system light geometrically, namely, the metrical concepts are introduced by means of the clock and the light signal. Such concept as rigid scale has not yet been introduced. Now we will consider the motion of a rigid body.

In a rest system we can define a uniform translatory motion of a rigid body as usual. Namely the velocity of any point on a rigid body is independent of its position and the time. Let the x -axis of the rest system be the direction of this motion and the x' -axis of the moving system be coincide with the x -axis and the velocity of the origin of the moving system be same as the one of the rigid body. Then, according to the above given coordinate transformation any point on the rigid body is rest relatively at the moving system. The shape and the size of the body is therefore constant in the moving system. In this case the problem is whether the size of the body thus obtained is the same one observed in the rest system when the body is at rest relative to the latter system. Consequently it is to be remarked that the appropriateness of the matter axioms are not verified in a moving system. Namely the property of the rigid body in a moving system must be researched directly by experiments.

The property of the clock in a moving system is also to be considered in a like manner. We have adopted the so-called Ives-Stilwell experiment to obtain the observed rate of a moving clock. This means that the clock we used in the moving system is the atomic one in principle. In a rest system we have many

evidences that the atomic clock and the ordinary mechanical one have the same rate. In our laboratory system considered as a rest system we can synchronize a clock mechanically regulated, say, schematically speaking, a harmonic oscillator, with an atomic clock. Do we have such coincidence in a moving system? Although, through the reasoning stated above, the observed rates of such clocks at rest system observed at the moving system are same each other, we must have a direct evidence for such coincidence in the moving clocks observed at the moving system relatively rest at the clocks. Namely, the result of Ives-Stilwell experiment teaches us the observed rate of a moving atomic clock, whereas we have no such experimental evidence for the rate of the mechanical clock. The conclusions of the above problems may also be given in harmony with the transformation properties of mechanical laws. But our considerations does not enter such steps of the theory.

2. Interpretations of some optical experiments. Some optical experiments explained in the following may serve to solve the above stated problems. The so-called Michelson-Moreley experiment is usually explained as showing the null effect for the existence of the rest aether or abstractly the non-existence of the absolute motion. But phenomenologically speaking the result gives nothing but the null effect for the difference of two light path. For our purpose the following interpretation due to Robertson [10] is useful:

M-M: The total time required for light to traverse, in free space, a distance l and to return is independent of its direction. His interpretation of Kennedy-Thorndike experiment:

K-T: The total time required for light to traverse a closed path in S is independent of the velocity v of S relative to Σ , (where S is any moving system and Σ is a rest system.) is also adopted. We consider that these two experimental results together with the one obtained from the γ -decay of a certain kind of elementary particles consist the evidences for the constancy of the light velocity.

The characteristic differences between the rest system and the moving system must be now considered. Our investigation has started with the standpoint as if our laboratory system is a rest system. According to the ordinary interpretation of the M-M experiment the concept of "absolutely rest" has no physical meaning. Our interpretations have been made based only on direct experimental evidences. Therefore from such a standpoint we can not conclude the existence of the aether. We must here examine the meaning of the words "rest" and "moving" in our above usage. Robertson [10] pointed out, however, in the explanation of his interpretation of K-T experiment that "velocity of a point on the earth's surface undergoes a diurnal change because of the earth's rotation and a much larger annual change because of the revolution of the earth about the sun". Whereas we explained the experimental results were obtained as if the experiments were performed in a rest

system, therefore our seemingly “rest system” has various velocities with respect to, say, the stellar system. Then if the coincidence of the readings of the optical clock and scale with the mechanical ones exists at a certain time, this coincidence should hold in any moving system. This conclusion suggest that the problem proposed at the end of the last subsection have been solved by the indirect evidences. In this sense if we interpret such facts extensively, the evidence for the special principle of relativity can be found in these line of thoughts.

3. The criterion of meaning (Continued.) We declared at the begining that the theory must be constructed starting only from directly verified facts, and further that the reasoning must be performed after the nature of every conventions have been clarified. Now we will compare our initial standpoint with the results obtained. Some problematic points are pointed out in the above two subsections. In this subsection we shall state the remaining ones especially with respect to the criterion of meaning.

Following our initial programme strictly, we must construct the “theory” using only the observational language and the propositions which state only directly verified facts.

Ordinarily physicists explain empirically (or experimentally) verified results in terms of observation language and theoretical language. In almost every cases they cannot do without the use of theortical language. Namely many of the experimental values are read directly or indirectly from the indicator of the instruments, which can be interpreted after some theortical reductions. Together with many physicists we cannot but adopt such results as empirically (or experimentally) verified. Therefore we use the term “empirically verified” in such wide sense. Then our initial thesis must be interpreted in such an extended meaning, and theories thus adopted in the premises in such manners may be chequed its adequacy based on the standpoint stated above. Consequently theories appeared in such steps of the reasoning may not consistent with each other, though the former one may be regarded as an approximate one to the latter one. In such cases the theory construction cannot be regarded as closed and moreover it may involve some kinds of reasoning in a circle. Concerning this problem detailed analysis must be performed carefully. In this paper we have stated only an outline of the problem and left it without further discussions.

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