On Numerations of a Formal System

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The results presented here are extensions of S. Feferman; Arithmetization of Metamathematics in a General Setting. Fund. Math. 49 (1960), 35–92.

As to the notations we shall follow him.†

Proposition A

Let $\mathcal{A} = \langle A, K \rangle \supseteq \mathcal{F}$ be a consistent axiom system which is binumerable (resp. numerable) in $\mathcal{A}$.

There is a formula $R(x)$ of $K_0$ such that

$$\vdash \mathcal{A} \models R(\bar{n})$$

for every natural number $n$ and

$$\vdash \mathcal{A} \models \forall x R(x)$$

if and only if there is an $a^*$ which binumerates (resp. numerates) $A$ in $\mathcal{A}$ and

$$\vdash \mathcal{A} \models \text{Con}_{a^*}.$$  

Proof

If-part: Since $a^*$ numerates $A$, we have

$$\vdash \mathcal{A} \models \text{Prf}_{a^*}(\bar{\phi}_0 \land \neg \bar{\phi}_0, \bar{n})$$

for every $n$.

And $\vdash \mathcal{A} \models \text{Prf}_{a^*}(\bar{\phi}_0 \land \neg \bar{\phi}_0, x)$ means that $\mathcal{A}$ is $\omega$-inconsistent.

† $\langle A, K \rangle$ represents an axiom system with a set of non-logical axioms $A$ and with a language $K$.

$\mathcal{F}$ is the Peano's arithmetic with the language $K_0$. $\mathcal{A}$ is a particular extension of $\mathcal{F}$, which can intensionally binumerate all recursive functions and predicates which appear in our propositions.

$\bar{n}$ is the numeral corresponding to a natural number $n$. Let $E$ be a Gödelized metamathematical expression. Then $E$ denotes its binumeration in $\mathcal{A}$.

$E^1(\xi)$ is an expression in $\mathcal{F}$ which corresponds to an expression $E$ in $\mathcal{A}$ and which has roughly the same effect in $\mathcal{F}$ as $E$ has in $\mathcal{A}$. $\text{Prf}_{\mathcal{A}}(x, y)$ is a numeric expression of "$y$ is a proof of $x$ in an axiom system $\mathcal{A} = \langle A, K \rangle$". $\text{Prf}_{\mathcal{A}}(x, y)$ is defined by $\forall y \text{Prf}_{\mathcal{A}}(x, y)$.

If $a$ numerates $A$, then, roughly speaking, $\text{Prf}_{a}(x, y)$ is a formula numerating $\text{Prf}_{\mathcal{A}}(x, y)$ in a natural way in $\mathcal{F}$. $\text{Prf}_{a}(x)$ is defined by $\forall y \text{Prf}_{a}(x, y)$.

Let $\bar{\chi}$ be a Gödel-number of a contradiction. $\text{Con}_{a}$ is defined by $\vdash \text{Prf}_{a}(\bar{\chi})$.

$\text{Fm}_{K}$, $\text{Vr}$ and $v_{xy}$ are the numeric expressions for the set of formulas in $K$, the set of variables and the $y$-th variable respectively.

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Clearly, \( \text{Prf}_{a^*}(\tilde{\phi}_0 \land \neg \tilde{\phi}_0, x) \) is a formula of \( K_0 \).

Only if-part: Let \( a \) be an arbitrary binumeration (resp. numeration) of \( A \) in \( \mathcal{A} \). Put

\[
a^*(x) = a(x) \lor \forall y [x \approx \neg \land \lor y \approx \lor y \land R(y)].
\]

Moreover we put the numeral corresponding to \( \neg \land \lor y \approx \lor y \) to be \( \bot_y \).

We show, at first, that \( a^* \) binumerates (resp. numerates) \( A \) in \( \mathcal{A} \).

**Lemma 1.** For every \( n \)

\[
\vdash_{\mathcal{A}} (\tilde{n} \approx \bot_y)^{(\mathcal{A})} \to y \leq \tilde{n}.
\]

**Lemma 2.** For every \( n \)

\[
\vdash_{\mathcal{A}} \forall y [y \leq \tilde{n} \to R(y)].
\]

1. \( \vdash_{\mathcal{A}} \forall y [(\tilde{n} \approx \bot_y)^{(\mathcal{A})} \land R(y)] \) \text{ for every } \( n \) \text{ lemma 1, 2}

2. \( \vdash_{\mathcal{A}} a^*(\tilde{n}) \to a(\tilde{n}) \) \text{ for every } \( n \)

3. \( \vdash_{\mathcal{A}} a^*(\tilde{n}) \leftrightarrow a(\tilde{n}) \) \text{ for every } \( n \)

Therefore we have \( a^* \) binumerating (resp. numerating) \( A \) in \( \mathcal{A} \).

Now we prove \( \vdash_{\mathcal{A}} \sim \text{Con}_{a^*} \).

We put

\( \mathcal{A}' = \langle \text{Prf}_{a^*} \cap \text{Fm}_{K_0}, K_0 \rangle \).

4. \( \vdash_{\mathcal{A'}} \forall u (\text{Prf}(u) \approx \text{Prf}(u)) \)

5. \( \vdash_{\mathcal{A'}} \forall v (\text{Prf}(v)) \)

6. \( \vdash_{\mathcal{A'}} \text{Fm}_K (\bot) \)

7. \( \vdash_{\mathcal{A'}} \forall y \text{Prf}(\sim \bot_y) \)

8. \( \vdash_{\mathcal{A'}} \forall y \text{Prf}(\bot_y \to \bot_y) \)

9. \( \vdash_{\mathcal{A'}} \text{Prf}(x) \land \text{Prf}(x \to y) \to \text{Prf}(y) \)

10. \( \vdash_{\mathcal{A'}} \forall y [\text{Prf}(\bot_y) \to \text{Prf}(\bot_y)] \)

11. \( \vdash_{\mathcal{A'}} \forall y [x \approx \bot_y \land R(y)] \)

12. \( \vdash_{\mathcal{A'}} \forall y [\forall (x \approx \bot_y \land R(y)) \land \forall (x \approx \bot_y)] \)

13. \( \vdash_{\mathcal{A'}} \forall y [a^*(x) \land x \approx \bot_y] \)
Let $A = \langle A, K \rangle$ be $\omega$-consistent with $A \supseteq \mathcal{P}$ and let $\mathcal{P} \subseteq \zeta \subseteq A$. Then, $A$ is finite if and only if there is a PR (resp. RE) numeration $a_0$ of $A$ in $\zeta$ such that for every PR (resp. RE) numeration $a_1$ of $A$ in $\zeta$

$$\vdash_{A} \forall x [a_0(x) \rightarrow a_1(x)]$$

holds.

**Proof**

Only if-part: Let $A$ be

$$(k_1, k_2, \ldots, k_n).$$

Then $x \equiv k_1 \lor x \equiv k_2 \lor \cdots \lor x \equiv k_n$ is the desired $a_0(x)$.

If-part: Suppose $A$ is infinite. Then, for any natural number $n$ and for any PR (resp. RE) numeration $a$ of $A$ in $\zeta$

1. $\vdash_{\zeta} \forall \gamma [\bar{a} \leq y \land a(y)].$

From the $\omega$-consistency of $\mathcal{A}$ and 1 we have

2. $\vdash_{\zeta} \forall \gamma [x \leq y \rightarrow \neg a(y)].$

Therefore for every PR (resp. RE) numeration $a$ of $A$ in $\zeta$

3. $\mathcal{A} = \mathcal{A} \cup \{ \forall \gamma [x \leq y \land a(y)] \}$ is consistent.

Now define

$$\text{Prf}_a(x, y) = \lor_w [w \leq y \land \text{Prf}_a(x, w)]$$

And put

$$a_1(x) = a_0(x) \land \neg \text{Prf}_a(\bot, x),$$

where $\bot$ means $\bot \vDash$ in the proof of the proposition $A$. Since for every number $n$
4. \( \vdash_S \sim \Prf_{\mathcal{A}_0}(\perp, \overline{\mathcal{A}}) \),

\( a_j \) is PR (resp. RE) formula numerating \( A \) in \( \mathcal{A} \).

5. \( \vdash \mathcal{A} \Prf_{\mathcal{A}_0}(\perp, x) \to \bigwedge_y [x \leq y \to \Prf_{\mathcal{A}_0}(\perp, y)] \)

6. \( \vdash \mathcal{A} \Prf_{\mathcal{A}_0}(\perp, x) \to \bigwedge_y [x \leq y \to \sim a_j(y)] \)

7. \( \vdash \mathcal{A}_0 \Prf_{\mathcal{A}_0}(\perp, x) \to \bigvee_z [x \leq z \land a_0(z)] \)

8. \( \vdash \mathcal{A}_0 \Prf_{\mathcal{A}_0}(\perp, x) \to \bigvee_z [a_0(z) \land \sim a_1(z)] \)

9. \( \vdash \mathcal{A}_0 \bigvee_x \Prf_{\mathcal{A}_0}(\perp, x) \to \bigvee_z [a_0(z) \land \sim a_1(z)] \)

10. \( \vdash \mathcal{A}_0 \bigwedge_x [a_0(z) \to a_j(z)] \to \bigwedge_x \sim \Prf_{\mathcal{A}_0}(\perp, x) \)

Let \( \alpha^*(x) \) be

\[ a_0(x) \lor x \equiv \bigwedge_z \bigvee_y [x \leq y \land a_0(y)]. \]

Clearly \( \alpha^* \) is a PR (resp. RE) numeration of

\[ A \cup [\bigwedge_z \bigvee_y (x \leq y \land a_0(y))] \]

in \( \mathcal{A} \). We can moreover formulate the relative consistency of \( \mathcal{A}_0 \) (i.e. the discussion leading to 3) in \( \mathcal{A} \). That is,

11. \( \vdash \mathcal{A} \Con_{\mathcal{A}_0} \to \Con_{\alpha^*} \)

12. \( \vdash \mathcal{A}_0 \bigwedge_x [a_0(x) \to a_j(x)] \to \Con_{\alpha^*} \)

13. \( \vdash \mathcal{A}_0 \Con_{\alpha^*} \)

Gödel’s second incompleteness theorem

14. \( \vdash \mathcal{A}_0 \bigwedge_x [a_0(x) \to a_j(x)] \)

15. \( \vdash \mathcal{A}_0 \bigwedge_x [a_0(x) \to a_j(x)] \)

This completes the proof of if-part.

Q.E.D.