Information-Extraction Through Reduction Methods In Some Formal Systems

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Introduction

In many formal systems which are based on predicate calculi, an (essentially) existential quantifier in a derivation can be regarded as an abbreviated expression of some occurrences of terms or abstracts, and such an abbreviation makes the “contraction” possible. There are, however, occasions when one wishes to extract some information about the objects of existence and their dependences on eigenvariables by some concrete mechanism. In a sense a mechanism of this sort can be considered as the representation of a formal system, and the information produced by it as that of a derivation in the system.

In this article, we present a few examples of information-extraction along this idea. The methods used are closely related to various reduction processes such as cut-eliminations and consistency proofs. We start with first order predicate calculi (Section 1) and proceed to systems of elementary number theory (Section 2), the cut-free part of G^1LC (Section 3) and second order systems which admit certain reduction methods (Section 4). For Section 1, the cut-elimination process of LK or LJ , which is well-known and relatively simple, is presumed. For second order systems, the comprehension abstracts are transformed into those of first order, and they are determined through reduction processes. In any of the cases treated, the mechanism of the information-extraction is represented by the (generalized) functional which interprets the accessibility of the order system underlying the reduction method.

§1. First order predicate calculi.

Definition 1.1. Instances of sequents.

1) Let $\mathcal{L}$ be a language of first order predicate calculus (allowing function constants). We shall work with LK as well as LJ in this language, where the initial sequents are assumed to consist of atomic sequent formulas; see [2] for these systems. We also prepare two countable sets of new symbols, say $\mathscr{V}_1$ and $\mathscr{V}_2$, both

of which are assumed to be arranged in ω -type. The symbols in $\mathcal{V}_1$ are called eigenvariables, and those in $\mathcal{V}_2$ are called denotation-variables. The quantifier-free part of the language $\mathcal{L}$ augmented by $\mathcal{V}_1$ will be called $\mathcal{L}_1$, and the one augmented by $\mathcal{V}_1$ and $\mathcal{V}_2$ will be called $\mathcal{L}_2$.

For a later convenience, we assume that $\mathcal{L}$ has a designated individual constant c_0 .

2) Let S be an $\mathcal{L}$ -sequent. By a quasiformula in S we mean a subformula-like expression in S which may contain some free occurrences of bound variables. We define the positive or the negative occurrence of a quasiformula in S , and the universal or the existential occurrence of a quantifier in it.

(1) A sequent formula of S is positive or negative in S according as it occurs in the succedent or antecedent.

(2) Let B be a quasiformula in S . If B is of one of the forms $\forall xF(x)$, $\exists xF(x)$, $C \wedge D$ and $C \vee D$, then $F(x)$, C and D in B are positive or negative in accordance with the occurrence of B . Suppose B is of one of the forms $\neg C$ and $C \vdash D$. Then C is positive or negative according as B is negative or positive, while D is positive or negative in accordance with B .

(3) The $\forall x$ in $\forall xF(x)$ is universal or existential in S according as $\forall xF(x)$ is positive or negative in S . The $\exists x$ in $\exists xF(x)$ is universal or existential in S according as $\exists xF(x)$ is negative or positive in S .

3) An existential quantifier q_1 is said to be dependable on a universal quantifier q_2 if q_2 is not in the scope of q_1 .

4) For each universal quantifier in S , we pick up a finite (non-empty) set of eigenvariables, and for each existential one a finite set of denotation-variables, so that distinct quantifiers are associated with distinct new variables. Such a finite set of variables as well as each variable in it is called an assignment to quantifier under concern. Let $\mathcal{A}$ denote a set of assignments to the universal quantifiers in S and let $\mathcal{B}$ denote one to the existential quantifiers in S . Then $\mathcal{C} \equiv (\mathcal{A}, \mathcal{B})$ will be called an assignment for S .

5) Let B be any quasiformula in S . We define ground matrices and (general) matrices of B relative to an assignment $\mathcal{C}$ by knocking off quantifiers in B and by replacing the remaining bound variables by corresponding assignments, so that they will be (quantifier-free) formulas of $\mathcal{L}_2$.

(1) Suppose $\{B_1, \dots, B_k\}$ is a maximal set of *distinct* ground matrices of B satisfying that each quantifier in B is assigned distinct variables from $\mathcal{C}$ in different $B_1, \dots, B_k$ while any bound variable occurring free in B is assigned a same variable in all of $B_1, \dots, B_k$. Then $B_1 \vee \dots \vee B_k$ or $B_1 \wedge \dots \wedge B_k$ (abbreviated to $[B]$) is a (general) matrix of B according as B is positive or negative in S . Each of $B_1, \dots, B_k$ is called a constituent of the general matrix.

(2) B is quantifier-free. Then any figure obtained from B by replacing each

bound variable in it by an assignment (in $\mathcal{E}$) is a ground matrix of B .

Notice that in this case, by definition, for any two distinct ground matrices of B , there must be distinct assignments to a bound variable occurring free in B and hence each ground matrix is the sole matrix involving it.

In the following, B is not quantifier-free.

(3) B is of the form $\forall xF(x)$ or $\exists xF(x)$. Any general matrix of $F(x)$ is a ground matrix of B .

(4) B is $C \wedge D$. For C any (general) matrix of C and D any matrix of D , such that the free occurrence of a bound variable in B is assigned a same variable in C and D , $C \wedge D$ is a ground matrix of B .

Similarly for other propositional connectives.

6) Let S be of the form $A_1, \dots, A_m \rightarrow B_1, \dots, B_n$, and let $A_{i1}, \dots, A_{ip_i}$ be all the (distinct) ground matrices of A_i , $p_i \geq 1$, $i = 1, \dots, m$, and let $B_{j1}, \dots, B_{jq_j}$ be all the ground matrices of B_j , $q_j \geq 1$, $j = 1, \dots, n$, relative to $\mathcal{E}$ an assignment for S . Then

$$\begin{aligned} &A_{i1}, \dots, A_{ip_i}, \dots, A_{i1}, \dots, A_{ip_i}, \dots, A_{m1}, \dots, A_{mp_m} \\ &\rightarrow B_{11}, \dots, B_{1q_1}, \dots, B_{j1}, \dots, B_{jq_j}, \dots, B_{n1}, \dots, B_{nq_n} \end{aligned}$$

will be called a matrix of S (relative to $\mathcal{E}$). This may be briefly written as $[S]$, and each of A_{il} and B_{jk} is called a constituent of $[S]$.

Notice that since there are no free occurrences of bound variables in the sequent formulas, distinct constituents of $[S]$ do not contain common variables of $\mathcal{V}_1$ and $\mathcal{V}_2$.

7) Let δ be a denotation-variable and let a be an eigenvariable or a free variable (of $\mathcal{V}$) occurring in $[S]$. $\delta \Rightarrow a$ (δ is dependable on a) will denote any relation between δ and a (relative to $[S]$) which satisfies the subsequent conditions.

(1) For any eigenvariable a and denotation-variable δ both occurring in a ground matrix in $[S]$, $\delta \Rightarrow a$ holds only if the corresponding existential quantifier is dependable on the corresponding universal quantifier in S (see 3) above).

(2) Let B be a quasiformula of the form $\forall xF(x)$ or $\exists xF(x)$ whose outermost quantifier is universal in S . Then for any δ a denotation-variable in any matrix of B whose corresponding quantifier occurs in B and for any a an assignment to x , $\delta \Rightarrow a$.

(3) Let γ and δ be two denotation-variables in $[S]$. $\gamma > \delta$ will denote a relation between γ and δ which holds if there is an eigenvariable a such that

(*) $\delta \Rightarrow a$ and $\gamma \nRightarrow a$ (γ is not dependable on a).

Then we require that there be no cycle with respect to $>$ among the denotation-variables in $[S]$.

(4) If a is a free variable, then $\delta \Rightarrow a$.

Notice that $\Rightarrow$ is not defined unless both δ and a occur in $[S]$.

8) If a dependability relation $\Rightarrow$ can be defined for $[S]$ satisfying (1)~(4) above, then we say that $[S]$ is $\Rightarrow$ -consistent.

9) Suppose $[S]$ is $\Rightarrow$ -consistent. An instance of S (or of $[S]$) (relative to $\Rightarrow$) is obtained from $[S]$ by replacing the denotation-variables in it by $\mathcal{L}_1$ -terms in a manner that the term to replace a δ can contain a free or eigenvariable a only if $\delta \Rightarrow a$. Such an instance will be denoted by $\{S\}$.

Corollary. 1) A matrix of B contains an $\mathcal{L}$ -term if and only if B does.

2) An instance of S is an $\mathcal{L}_1$ -sequent (which is quantifier-free).

3) (3) in 7) above implies the following fact.

(**) Let B be a quasiformula of the form $\forall x F(x)$ or $\exists x F(x)$ where the outermost quantifier is existential in S . Then there is an enumeration of the ground matrices of B , say $C_1, \dots, C_p$, such that for any δ an assignment to x in C_l and any a an assignment in C_k to a universal quantifier in B (so that $\forall x$ or $\exists x$ is not dependable on it), $\delta \Rightarrow a$ only if $k < l$.

We now come to our main theme.

Theorem 1. If S is an $\mathcal{L}$ -sequent and is LK-provable, then there is an instance of S which is (cut-free) provable in the propositional calculus in the language $\mathcal{L}_1$.

Furthermore, such an instance and its cut-free derivation can be primitive recursively found from a cut-free derivation of S .

Note. The theorem necessarily implies that an assignment and a dependence relation can be defined for S .

Proof. Let P be a cut-free derivation of S . We define the desired derivation Q by induction on the number of inferences in P . In the following we agree that should there arise identical ground matrices of a sequent formula, contractions be applied so that all the surviving constituents be distinct.

If P consists of $D \rightarrow D$ alone (which consists of atomic formulas), then $Q \equiv P$ will do.

Suppose P is compound, and let I be the last inference in P . Note that P is either of the form

$$\frac{\begin{array}{c} \vdots \\ S' \end{array}}{S} I$$

or

$$\frac{\begin{array}{c} \vdots \\ S_1 \end{array} \quad \begin{array}{c} \vdots \\ S_2 \end{array}}{S} I.$$

The induction step is to be dealt with according to I . Since most of the cases are self-evident, we confine ourselves to some crucial ones.

1°. I is $\forall$ in the succedent. Let S' be $\Gamma \rightarrow \Delta, F(b)$ and S be $\Gamma \rightarrow \Delta, \forall x F(x)$. Let $[S']$ be of the form

$$\Gamma^* \rightarrow \Delta^*, F_1^*(b), \dots, F_q^*(b),$$

and let a be a new eigenvariable. Notice that b occurs exclusively at the indicated places. Then

$$\Gamma^* \rightarrow \Delta^*, F_1^*(a) \vee \dots \vee F_q^*(a)$$

can be regarded as $[S]$. The dependence relation for $[S']$ can be employed as a dependence relation for $[S]$, since in $[S']$ every denotation-variable is dependable on b by virtue of (4), while in $[S]$ a denotation-variable being dependable on a is consistent by virtue of (2). We have a $Q'(b)$ for

$$\{S'\}: \Gamma' \rightarrow \Delta', F_1(b), \dots, F_q(b),$$

from which Q can be obtained as below.

$$\begin{array}{c} Q'(a) \\ \Gamma' \rightarrow \Delta', F_1(a), \dots, F_q(a) \\ \hline \Gamma' \rightarrow \Delta', F_1(a) \vee \dots \vee F_q(a) \end{array}$$

2°. I is $\forall$ in the antecedent. Let S' be $F(t)$, $\Gamma \rightarrow \Delta$ and S be $\forall x F(x)$, $\Gamma \rightarrow \Delta$. Let $[S']$ be

$$F_1^*(t), \dots, F_p^*(t), \Gamma^* \rightarrow \Delta^*,$$

and let $F_i^*(\delta)$ be obtained from $F_i^*(t)$ by replacing the indicated occurrences of t by δ a new denotation-variable. Then

$$F_1^*(\delta) \wedge \dots \wedge F_p^*(\delta), \Gamma^* \rightarrow \Delta^*$$

can be regarded as $[S]$. The dependability relation for $[S']$ augmented by the conditions that δ be dependable on the free variables occurring in $[S]$ and that it be non-dependable on any eigenvariable can be taken as the dependability relation for $[S]$. Let s be obtained from t by replacing all the free variables in it which do not occur in S by the constant c_0 . We have a Q' for

$$\{S'\}: F_1(t), \dots, F_p(t), \Gamma' \rightarrow \Delta'.$$

By the same replacement as for t , we obtain Q from Q' as follows.

$$\begin{array}{c} F_1(s), \dots, F_p(s), \Gamma' \rightarrow \Delta' \\ \hline \{S\}: F_1(s) \wedge \dots \wedge F_p(s), \Gamma' \rightarrow \Delta' \end{array}$$

The intuitionistic version of the theorem above can be proved by a similar method.

Theorem 2. If S is an $\mathcal{L}$ -sequent and is LJ-provable, then there is a matrix of S , $[S]$, such that for any positive quasiformula in S , there is a sole ground matrix in $[S]$ corresponding to it, and an instance $\{S\}$ defined from $[S]$, so that $\{S\}$ is provable in the intuitionsitic propositional calculus.

Remark. The theorem above, in particular, explains the following fact.

Consider an LJ-provable formula of the form

$$\exists x \forall y A(x, y) \vdash \exists u \forall v B(u, v),$$

where A and B are quantifier-free. Its provable instance will have a form (in a simplified notation)

$$A^*(a, f(a, b)) \vdash B(g(a), b),$$

for an appropriate quantifier-free A^* , since $\exists u \forall v B(u, v)$ is positive and $\exists u$ is not dependable on $\forall v$.

Definition 1.2. Decidable calculus.

1) Let $\mathcal{L}_a$ be the language $\mathcal{L}$ where the constants are all decidable in a natural interpretation and there are sufficient function constants so that quantifier free formulas can be represented by the characteristic functions in the language.

2) In the definition of matrices in 5) of Definition 1.1, ground matrices suffice, and we call them simply matrices. So, if S is of the form

$$A_1, \dots, A_m \rightarrow B_1, \dots, B_n,$$

then a matrix of S , say $[S]$, will have the form

$$A_1, \dots, A_m \rightarrow B_1, \dots, B_n.$$

3) Dependability relations are not to be defined for LK, and hence no restrictions on eigenvariables are imposed for the instances, while for LJ we define dependability relations as in Definition 1.1.

The decidable version of the preceding theorems can be easily established.

Theorem 3. 1) If S is an $\mathcal{L}_a$ -sequent and is LK-provable, then there is an instance of S , $\{S\}$, which is valid (in the natural interpretation of constants).

2) Let S be an $\mathcal{L}_a$ -sequent which is LJ-provable. Then there is a valid instance of S , say $\{S\}$. Furthermore, for any positive occurrence of the quantifier $\exists x$ in S , if x is replaced by δ and if a is an eigenvariable occurring in $[S]$, then the term to replace δ in $\{S\}$ can have a only if $\delta \Rightarrow a$.

The following theorem and the subsequent remarks have some bearing on Theorem 2 in [8]; see 1.1 and Section 3 there.

Theorem 4. 2) of Theorem 3 above holds for the cut-free part of LJ augmented by the inferences $\exists \vdash$ and $\exists \top$ (for first order formulas), listed below.

$$\exists \vdash \frac{\Gamma \rightarrow A \vdash \exists x F(x)}{\Gamma \rightarrow \exists x (A \vdash F(x))},$$

where all the quantifiers in A are existential (in the sequent).

$$\exists \top \frac{\Gamma \rightarrow \top \forall x F(x)}{\Gamma \rightarrow \exists x \top F(x)},$$

where all the quantifiers in $F(x)$ are existential (in the sequent).

Proof. We can take the same instances for the conclusions as for the premises.

Remark. 1) The restrictions on the quantifiers in the inferences above are necessary. For, suppose A is of the form $\exists yA(y)$ and $\Gamma' \rightarrow A(a) \vdash F(f(a))$ is the instance of the upper sequent obtained in the proof. f can contain a since there is no restriction on eigenvariables in such a case. For the lower sequent, however, an instance should have the form $\Gamma'' \rightarrow A(a) \vdash F(r)$, where r cannot contain a (see (1) of 7) in Definition 1.1). For $\exists \neg$, a similar remark should be made.

2) As for the axiom of choice,

$$\frac{\Gamma \rightarrow \forall x \exists y F(x, y)}{\Gamma \rightarrow \exists \chi \forall x F(x, \chi(x))},$$

the χ can be regarded as a term which is constructed for the upper sequent depending on x .

§2. Systems of elementary number theory.

This section is a supplement for [9]. We quote numberings there by affixing asterisks; Definition 2.3* will indicate Definition 2.3 in [9].

Definition 2.1. Let N denote the formal system of classical first order arithmetic and let J denote one of intuitionistic arithmetic (see Sections 1* and 4*). Here, however, we assume all the connectives for N also. Let S be a closed sequent of either system. We can define matrices and dependence relations for S as we did in Definition 1.1, except that free variables are not involved here and that with each universal quantifier exactly one eigenvariable is associated, so that there be a single ground matrix associated with a quasiformula whose outermost connective is a universal quantifier. A matrix is essentially the same as a quantifier-free form defined in Definition 2.3*.

Let n be any sequence of numerals corresponding to the universal quantifiers in S , and let $[S]n$ denote a matrix of S associated with n . Let t be any sequence of closed terms corresponding to the denotation-variables in $[S]n$. That is, with each n , one set of t is associated. Then $[S](n, t)$ will denote a figure obtained from $[S]n$ by replacing eigenvariables by n and denotation-variables by t . $[S](n, t)$ is a quantifier-free closed sequent of N . $\{S\}$ will denote the family of $[S](n, t)$ for all n (and one t for each n), and will be called a family of instances of S .

We then have a re-styled version of Theorem 3* (in Section 2*).

Theorem 5. Let P be a closed derivation of N and let $o(P)$ be θ . Let S be the endsequent of P and let α be $\omega^{\omega^{\theta+1}}$. Then there is a computable functional σ of finite type (depending on α) which determines a family of instances of S and a quantifier-free derivation of each instance in the family. Furthermore, a term to replace a denotation-variable δ varies only in accordance with the changes in the assignment to the eigenvariables upon which δ is dependable.

See [5] and [8] for computable (primitive recursive) functionals of finite type.

The “complexity” of the type of σ will be discussed in the proof.

Proof. First construct the α -recursive ϕ claimed in Theorem 3*. Recall that this ϕ is defined by a single application of the “ α -recursion”. It then suffices to show that such a ϕ can be represented by a computable functional.

Let $<$ be the canonical order of α -type. There are primitive recursive functions u , v and τ such that

$$\begin{aligned}\phi(0, x) &= u(x); \\ \phi(k, x) &= v(k, x, \phi(\tau^*(k, x), x)) \text{ if } k > 0,\end{aligned}$$

where

$$\tau^*(k, x) = \begin{cases} \tau(k, x) & \text{if } \tau(k, x) < k, \\ 0 & \text{otherwise.} \end{cases}$$

(See Definition 2.4*. Here, for simplicity, ϕ is assumed to be binary and v ternary.)

We shall define e and π , computable functionals constructed from u , v and τ , so that

$$(1) \quad e(\pi(k, x), k, x) = \phi(k, x).$$

Then $\sigma \equiv \lambda k \lambda x e(\pi(k, x), k, x)$ will do.

Let $f \equiv \{f_n\}_n$ be a sequence of ordinals below α . A “modulus of finiteness” of f means a natural number m such that, for any $n \geq m$, f_n is empty. A “modulus of finiteness functional” is a functional which yields a modulus of finiteness of any decreasing sequence as above.

First define ϕ which satisfies

$$\begin{aligned}\phi(0, k, x) &= k, \\ \phi(n+1, k, x) &= \tau^*(\phi(n, k, x), x).\end{aligned}$$

ϕ is primitive recursive and, for each k and x $\{\phi(n, k, x)\}_n$ can be regarded as a $<$ -decreasing sequence of ordinals below α . So, if we can construct a “modulus of finiteness functional” χ as a computable functional so that $\chi(\lambda n \phi(n, k, x))$ be a modulus of finiteness for the sequence above, then the subsequent argument goes through. Put

$$\pi(k, x) = \chi(\lambda n \phi(n, k, x)),$$

and define e by

$$\begin{aligned}e(0, k, x) &= h(x), \\ e(l+1, k, x) &= g(\phi(\pi(k, x) - (l+1), k, x), x, e(l, k, x)).\end{aligned}$$

e is primitive recursive in χ . It can be easily checked that (1) holds with these e and π .

The construction of χ is, in its essence, contained in the course of the construction of N in Theorem 2 of [10], except that here we need not be concerned with the “uniformity”, since the h there, the height of α , is a fixed number here. In

outlining the construction of χ , we use the notations and the results in Section 3 of [10]. Let M_h be the functional defined in Proposition 3.3 there. Recall that it has the following property: for any decreasing sequence f (of ordinals below α) such that “the height of f ” is h , $M_h(f)$ is a decreasing sequence of the highest powers of some entries of f , and hence $\text{ht}(M_h(f)) = h - 1$, and the finiteness of $M_h(f)$ implies the same of f .

Now define $N_0, \dots, N_h$ successively.

$N_0(f) = f$, $N_{i+1}(f) = M_{h-i}(N_i(f))$, $i = 0, \dots, h - 1$.

(Notice that here $N_0, \dots, N_h$ are individually defined functionals since h is a fixed number, while in [10] h is a variable.) $N_h(f)$ consists of 0 alone if f is a sequence as above, and hence its modulus of finiteness can be defined to be 1. From the way M_i has been constructed, a modulus of finiteness of $M_i(g)$ for any appropriate g induces that of g , and hence, starting with 1, we can obtain a modulus of finiteness of f by returning from $\{0\}$ to f through $M_1, \dots, M_h$. This entire mechanism can be presented as χ . The types involved are the ones listed in Section 3 of [10], where $i \leq h + 1$.

Note. We shall call such a functional one of complexity h .

The theorem below is our present version of Theorem 4*.

Theorem 6. Let A be a closed theorem of N . Let

$$A: A(\delta_1, \dots, \delta_q, a_1, \dots, a_p)$$

be the matrix of A (in the sequent $\rightarrow A$) which consists of ground matrices alone, where $\delta_1, \dots, \delta_q$ is the list of the denotation-variables in A and $a_1, \dots, a_p$ is the list of eigenvariables in A . Then, for some h a natural number, there are computable functionals of complexity h , say $\phi_1, \dots, \phi_q$ such that for any p-tuple of numbers $(n_1, \dots, n_p)$ A becomes a true statement of arithmetic under the assignment

$$\begin{aligned} a_i &: n_i, i = 1, \dots, p, \\ \delta_j &: \phi_j(n_1, \dots, n_p), j = 1, \dots, q. \end{aligned}$$

Theorem 7. 1) The intuitionistic version of Theorem 5 holds for J (see 1 of Section 4*). As in Theorem 2, the matrix $[S]n$ in question complies with the restriction that for any positive quasiformula in S , there be a sole ground matrix in $[S]n$ corresponding to it.

2) The intuitionistic version of Theorem 6 holds for J with the restriction on the terms as in Theorem 5 (see also 2) of Theorem 3).

§3. Cut-free part of second order predicate calculus.

Here we prove a parallel of Theorem 1 for the cut-free part of G^1LC ; see

Definition 15.17 in [7] for G^1LC .

Definition 3.1. We can define the second order version of the notions defined in Definition 1.1. For example, we have to prepare two sorts of eigenvariables, of first order and of second order; similarly for denotation-variables. A matrix of a sequent S , $[S]$, is defined as before.

Let V be an abstract of the form $\{x\}A(x)$. By definition, $A(x)$ does not have a free occurrence of a bound variable except x . V is a meta-object and it occurs in S in the form $A(s)$, where s is a quasiterm (a term-like expression which may contain free occurrences of first order bound variables) which depends on the occurrence of V . Suppose $A(s')$ is a ground matrix in $[S]$ corresponding to $A(s)$. Then we write $[V]$ for $\{x\}A(x)$. $[V]$ depends on the occurrence of $A(s)$ in S as well as on the occurrence of $A(s')$ in $[S]$.

It is obvious that for any quasiformula B , a second order free variable X occurs in $[B]$ if and only if it occurs in B .

Dependability relations are not considered here. An instance of S is obtained from $[S]$ by replacing the denotation-variables in it by terms and abstracts of the quantifier-free language augmented by the eigenvariables.

The following lemma will play the basic role in our interpretations of second order systems.

Lemma. Let $F(X^1, \dots, X^n)$ be a quasiformula, where $X^1, \dots, X^n$ indicate all the occurrences of the second order free variable X in F , and let V be an abstract (of the original language). Suppose $F(V^1, \dots, V^n)$ is a quasiformula in a sequent S , where V^i indicates the V substituted for X^i . Let us further explain some notations. Let $[F(V^1, \dots, V^n)] (\equiv F)$ denote a matrix of $F(V^1, \dots, V^n)$ in $[S]$. For each V^i , we can locate the corresponding parts in the matrix (which may not be unique). Let $[V^i]_k$ be the k th (in some order) occurrence of such. We can define $[F(X^1, \dots, X^n)] (\equiv F')$ a matrix of $F(X^1, \dots, X^n)$ corresponding to F . The occurrence of X^i in F' corresponding to $[V^i]_k$ in F will be denoted by $[X^i]_k$. In general,

$$U([X^i]_k / [V^i]_k; k=1, \dots, l_i, i=1, \dots, n),$$

or $U(X^i / [V^i])$ for short, will represent the result of replacements of $[X^i]_k$ by $[V^i]_k$ in U for all k and i .

Now we claim

$$[F(V^1, \dots, V^n)] \equiv [F(X^1, \dots, X^n)](X^i / [V^i]),$$

or, for short,

$$[F(V)] \equiv [F(X)](X / [V]).$$

Theorem 8. If S is a sequent of G^1LC which is cut-free provable, then there is an instance of S which is (cut-free) provable in the propositional calculus.

Furthermore, such an instance and its derivation can be primitive recursively found from a cut-free derivation of S .

Proof. The proof of Theorem 1 nearly goes through. It suffices to deal with the case where I is a second order $\forall$ in the antecedent. Let S' and S be respectively

$$F(V^1, \dots, V^n), \Gamma \rightarrow \Delta$$

and

$$\forall \phi F(\phi^1, \dots, \phi^n), \Gamma \rightarrow \Delta,$$

where ϕ^i indicates the occurrence of ϕ at the place which corresponds to V^i . By the induction hypothesis, there is an $[S']$:

$$F_1^*, \dots, F_p^*, \Gamma' \rightarrow \Delta',$$

where F_j^* represents the j th $[F(V)]$, V denoting $V^1, \dots, V^n$ collectively. By the lemma above $[F(V)] \equiv [F(X)](X/[V])$, and hence F_j^* is of the form $F_j'(X)(X/V_j^*)$, $F_j'(X)$ denoting the j th $[F(X)]$ and V_j^* denoting the corresponding $[V]$ (collectively). We have a derivation for an instance of $[S']$,

$$\{S'\} : F_1(V_1), \dots, F_p(V_p), \Gamma' \rightarrow \Delta',$$

from which we can obtain a derivation for

$$\{S\} : F_1(V_1) \wedge \dots \wedge F_p(V_p), \Gamma' \rightarrow \Delta'.$$

§4. Reduction methods for second order systems.

Let us first introduce the tool we are to use in this section.

In Part II of [11] we developed the theory of hyper-principle (HP), by which the functional interpretation of the accessibility of ordinal diagrams was carried out (Parts I and III therein). That is, given a system of ordinal diagrams $O(I, A)$, there is a uniform method $\mathcal{A}$, called a hyper-functional, such that for any $i \in I$, $\alpha \in O(I, A)$ and f an i -decreasing sequence from $O(I, A)$ led by α , $\mathcal{A}(i, \alpha, f)$ ($\mathcal{A}$ operating on i, α and f) yields a modulus of finiteness of f . We can therefore prove the proposition below with the same kind of argument as in the proof of Theorem 5 (in Section 2).

Proposition 1. For any $i \in I$ and $\beta \in O(I, A)$, let $<$ be the order $<_i$ restricted to the diagrams below β (with respect to i). Any $<$ -recursive function where the $<$ -recursion is applied only once is represented by a hyper-functional in the theory of HP (determined by I, A, i and β). We do not elaborate on the concrete forms of such a hyper-functional, since it can be constructed from the hyper-terms in the proof of the theorem in Section 4, Part III of [11]. Let us make the following remark, however. In case I is finite, the inductive definition along I becomes superfluous; $G(i, \alpha)$ can be defined for each $i \in I$ as a formula in the language

without G , Ord and lss as constants (see Definitions 1.1 and 1.3 in Part I of [11]). $T\langle i \rangle$ is also defined individually in the language $\mathcal{L}_{tp}$ without T as a constant (see Definition 1.1 in Part II of [11]).

Our first concern in this section is to couple the method in Section 3 with the cut-elimination process of a subsystem of G^1LC . Since the argument goes the same with any system which admits a reduction method, we take up as an example the system of “semi-isolated proof-figures”, say S , where in the implicit second order $\forall$ in the antecedent,

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \phi F(\phi), \Gamma \rightarrow \Delta},$$

the principal formula $\forall \phi F(\phi)$ is “semi-isolated”, while the abstract V is arbitrary. This system is dealt with in 1 of Chapter 3 in [6]. (The result was later improved in [1].) With each derivation P in S , a natural number n (the order of P) is associated so that the cut-elimination is established for P with the system of ordinal diagrams $\mathbf{O}(n+3, 1)$. For a fixed n , let us denote the order $<_o$ of this system by $<$, and the set of derivations of order n by S_n .

Theorem 9. Let S be a theorem of S_n . Then there is an instance of S which is propositionally provable. Furthermore, there is a $<-$ recursive method to obtain such an instance and its (cut-free) derivation from a given derivation P of S (in S_n).

By Proposition 1 above, we can then conclude that such a method is a hyperfunctional (in HP) which interprets the ordinal diagram of P .

Proof. We follow the cut-elimination proof of S_n in [6]. That is, if P is a derivation of S in S_n , then P can be reduced to a Q_1 , or to two derivations Q_1 and Q_2 . Since the ordinal diagram of Q_i decreases, $i=1$ or 2 , the end-sequent of Q_i is cut-free derivable, and hence by Theorem 8 an instance and a derivation can be found for the end-sequent of Q_i , from which the desired objects can be found for S . In a manner similar to the proof of Theorem 5, this process can be defined as $<-$ recursive. We take up just two cases as examples.

(1) There is an explicit logical inference in the end-piece of P , and a lowest such, say I , is second order $\forall$ in the succedent.

$$\frac{\frac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow \Delta, \forall \phi F(\phi)} I}{\Gamma_0 \rightarrow \Delta_1, \forall \phi \tilde{F}(\phi), \Delta_2}$$

Q_1 is of the following form.

$$\frac{\frac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow F(\alpha), \Delta, \forall \phi F(\phi)}}{S': \Gamma_0 \rightarrow \tilde{F}(\alpha), \Delta_1, \forall \phi \tilde{F}(\phi), \Delta_2}$$

By the induction hypothesis (along $<$), there is a cut-free derivation of S' . By Theorem 8, we obtain an instance of S' , $\{S'\}$ and a propositional derivation of $\{S'\}$, from which we obtain the desired objects for S ; see below.

$$\begin{array}{c} \{S'\}: \Gamma'_0 \rightarrow F_1(\alpha), \dots, F_p(\alpha), \Delta'_1, G_1(\beta_1), \dots, G_q(\beta_q), \Delta'_2 \\ \hline \{S\}: \Gamma'_0 \rightarrow \Delta'_1, F_1(\alpha) \wedge \dots \wedge F_p(\alpha), G_1(\beta_1), \dots, G_q(\beta_q), \Delta'_2 \end{array}$$

(2) I is second order $\forall$ in the antecedent.

$$\frac{\frac{F(V), \Gamma \rightarrow \Delta}{\forall \phi F(\phi), \Gamma \rightarrow \Delta} I}{\Gamma_1, \forall \phi \tilde{F}(\phi), \Gamma_2 \rightarrow \Delta_0}$$

This can be reduced to the following.

$$\begin{array}{c} F(V), \Gamma \rightarrow \Delta \\ \hline \forall \phi F(\phi), \Gamma, F(V) \rightarrow \Delta \\ S': \Gamma_1, \forall \phi \tilde{F}(\phi), \Gamma_2, \tilde{F}(\tilde{V}) \rightarrow \Delta_0 \end{array}$$

There is a cut-free derivation of S' , to which Theorem 8 applies, and hence we obtain the following.

$$\begin{array}{c} \{S'\}: \Gamma'_1, G_1(U_1), \dots, G_q(U_q), \Gamma'_2, F_1(V_1), \dots, F_p(V_p) \rightarrow \Delta'_0 \\ \hline \{S\}: \Gamma'_1, G_1(U_1), \dots, G_q(U_q), F_1(V_1) \wedge \dots \wedge F_p(V_p), \Gamma'_2 \rightarrow \Delta'_0 \end{array}$$

This can be taken as $\{S\}$ by virtue of the lemma in Section 3.

Before we proceed to the next subject, let us introduce some (tentative) notations.

Consider the theory $O(\omega+1, 1)$. Let σ_n denote $(\omega, (\omega, \dots, (\omega, 0) \dots))$, where the nesting number (of $(\)$'s) is n . $<^n$ will denote the order $<_0$ of $O(\omega+1, 1)$ restricted to the diagrams below σ_n . Let HP_n be the subtheory of HP which is adequate to establish the accessibility of $<^n$ (see [11]).

Now, as an example of combinations of the method in Section 2 and the consistency proofs of second order systems of arithmetic, we take up the quantifier-free interpretation of INN, whose consistency proof and its improvement have been given in [6], [7] and [1]. We therefrom know that $O(\omega+1, 1)$ suffices.

Definition 4.1. Here a closed formula, abstract or sequent is one which does not contain first order free variables (but which may contain second order parameters). Reduct-forms of a closed sequent of INN can be defined similarly to Definition 1.2 of [9], with two additional cases. If A is of the form $\forall \phi F(\phi)$ and S is of the form $\Gamma_1, A, \Gamma_2 \rightarrow \theta$, then Σ is

$$\{\Gamma_1, F(V), A, \Gamma_2 \rightarrow \theta\}$$

for some closed V . If A is as above and S is of the form $\Gamma \rightarrow \theta_1, A, \theta_2$, then Σ is

$$\{\Gamma \rightarrow \theta_1, F(X), A, \theta_2\}$$

for some free variable X which does not occur in S .

A reduct-tree of a closed INN-derivation can be defined as in [9].

We can define a matrix of a closed sequent S as in Definitions 1.1 and 2.1, where to each first order universal quantifier, a single ground matrix corresponds. A family of instances $[S](n, t, V)$ is defined as before, except that second order denotation variables are replaced by some quantifier-free abstracts which may contain some second order eigenvariables occurring in the matrix. (t and V depend on n but are uniform in second order parameters.)

Proposition 2. Theorem 1 in [9] (the termination of branches in the reduct-tree) can be proved for INN by using the system $(O(\omega+1, 1), <_o)$.

Theorem 10. Let S be a closed theorem of INN with a derivation P . Then for some n there is an HP_n -(hyper-) functional which determines a family of instances of S and a quantifier-free derivation of each instance in the family.

Proof. The proof goes similarly to that of Theorems 5 and 9. For a fixed P , Proposition 2 above can be established by the accessibility of $<^n$ for some n , and hence the desired instances and derivations can be obtained by a $<^n$ -recursive method. So by Proposition 1, there is an HP_n -functional of the required sort.

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