Cut-Elimination in the Intuitionistic Many-Valued Logic Based on a Partial Order

Mitio TAKANO

Niigata University

The purpose of this paper is to prove the following cut-elimination theorem for the intuitionistic many-valued logic based on a partial order, which was claimed in Takano [1, Theorem 14] :

THEOREM. *Assume that $(T, \leq)$ forms a lower semi-lattice. Then every provable matrix is cut-free provable provided that it is permissible.*

Readers are supposed to be familiar with our previous paper [1].

§ 1. Recall that a matrix K is *permissible* iff for every formula A the following conditions **P1**[A] and **P2**[A] hold :

P1[A]. If $\langle \lambda, A \rangle \in K$, then either $\lambda^- \times \{A\} \subseteq K$ or $\lambda^+ \times \{A\} \subseteq K$.

P2[A]. If the set $\{\lambda \mid \langle \lambda, A \rangle \notin K\}$ is not empty, then it has the least element $\min \{\lambda \mid \langle \lambda, A \rangle \notin K\}$.

We show some simple properties of permissible matrices, which we shall use without mention.

PROPOSITION 1. *Suppose $\lambda_0 = \min \{\lambda \mid \langle \lambda, A \rangle \notin K\}$. Then $\lambda_0 \leq \mu$ if and only if $\bigcap [\mu^- \times \{A\} \subseteq K]$.*

PROOF. If $\lambda_0 \leq \mu$, then $\langle \lambda_0, A \rangle$ belongs to $\mu^- \times \{A\}$ but does not to K , so $\bigcap [\mu^- \times \{A\} \subseteq K]$. To prove the converse, suppose $\bigcap [\mu^- \times \{A\} \subseteq K]$. Then $\lambda \leq \mu$ but $\langle \lambda, A \rangle \notin K$ for some λ . Hence $\lambda_0 \leq \lambda \leq \mu$, so $\lambda_0 \leq \mu$. $\square$

PROPOSITION 2. *Suppose $\lambda_0 = \min \{\lambda \mid \langle \lambda, A \rangle \notin K\}$ and $\lambda_0 \in U \subseteq T$. Then $(K \cup (U \times \{A\}))^\circ = K^\circ$.*

PROOF. It suffices to prove that $\langle \mu, A \rangle \in K^\circ$ whenever $\langle \mu, A \rangle \in (K \cup (U \times \{A\}))^\circ$. If $\langle \mu, A \rangle \in (K \cup (U \times \{A\}))^\circ$, that is, if $\mu^- \times \{A\} \subseteq K \cup (U \times \{A\})$, then $\bigcap (\lambda_0 \leq \mu)$ since $\langle \lambda_0, A \rangle \notin K \cup (U \times \{A\})$, so $\mu^- \times \{A\} \subseteq K$, that is, $\langle \mu, A \rangle \in K^\circ$. $\square$

PROPOSITION 3. 1°) *Assume that $(T, \leq)$ forms a lower semi-lattice. Then the empty matrix $\emptyset$ is permissible.*

2°) *If K is permissible, so are K° , $K \cup (\mu^- \times \{A\})$ and $K \cup (\mu^+ \times \{A\})$.*

PROOF. 1°) By the assumption T has the least element $\min(T)$, and $\min(T)$

$= \min \{ \lambda \mid \langle \lambda, B \rangle \in \emptyset \}.$

2°) Suppose that K is permissible. If $\langle \lambda, B \rangle \in K^\circ$, then $\lambda^- \times \{B\} \subseteq K^\circ$; on the other hand, if $\{ \lambda \mid \langle \lambda, B \rangle \in K^\circ \} \neq \emptyset$, then $\{ \lambda \mid \langle \lambda, B \rangle \in K \} \neq \emptyset$ and

$$\min \{ \lambda \mid \langle \lambda, B \rangle \in K \} = \min \{ \lambda \mid \langle \lambda, B \rangle \in K^\circ \};$$

so K° is permissible. If $\lambda \in \mu^\wedge$, that is, if $\lambda \neq \mu$, then $\lambda^+ \subseteq \mu^\wedge$ or $\lambda^- \subseteq \mu^\wedge$ according as $\neg(\lambda \leq \mu)$ or $\neg(\mu \leq \lambda)$; on the other hand, if $\{ \lambda \mid \langle \lambda, A \rangle \in K \cup (\mu^\wedge \times \{A\}) \} \neq \emptyset$, then $\mu = \min \{ \lambda \mid \langle \lambda, A \rangle \in K \cup (\mu^\wedge \times \{A\}) \}$; so $K \cup (\mu^\wedge \times \{A\})$ is permissible. If $\lambda \in \mu^+$, then $\lambda^+ \subseteq \mu^+$; on the other hand, if $\{ \lambda \mid \langle \lambda, A \rangle \in K \cup (\mu^+ \times \{A\}) \} \neq \emptyset$, then $\{ \lambda \mid \langle \lambda, A \rangle \in K \} \neq \emptyset$ and

$$\min \{ \lambda \mid \langle \lambda, A \rangle \in K \} = \min \{ \lambda \mid \langle \lambda, A \rangle \in K \cup (\mu^+ \times \{A\}) \};$$

so $K \cup (\mu^+ \times \{A\})$ is also permissible. $\square$

§ 2. In this section we shall explain that, to prove Theorem it suffices to show the following lemma:

LEMMA. Assume that $(T, \leq)$ forms a lower semi-lattice. If K is permissible, $\vdash K \cup (U_1 \times \{A\}), \vdash K \cup (U_2 \times \{A\})$ and if (U_1, U_2) is a pair of disjoint subsets of T , then $\vdash K$.

Recall that the cut inference has the following form:

Cut inference:

$$\frac{K \cup \{ \langle \mu, A \rangle \} \quad K \cup \{ \langle \nu, A \rangle \}}{K},$$

where $\mu \neq \nu$.

If we replace the cut inference by the new inference rule described below, we obtain another formal system from one developed in [1, § 4]:

Cut inference in the second form:

$$\frac{K \cup (\mu^\wedge \times \{A\}) \text{ for every } \mu \text{ in } T}{K}.$$

Let $\vdash^* K$ denote that K is provable in the new system.

PROPOSITION 4. If $\vdash K$, then $\vdash^* K$.

PROOF. It suffices to prove that, if $\vdash^* L \cup \{ \langle \lambda_1, A \rangle \}, \vdash^* L \cup \{ \langle \lambda_2, A \rangle \}$, and if $\lambda_1 \neq \lambda_2$, then $\vdash^* L$. Suppose $\vdash^* L \cup \{ \langle \lambda_1, A \rangle \}, \vdash^* L \cup \{ \langle \lambda_2, A \rangle \}$ and $\lambda_1 \neq \lambda_2$. For every μ in T , either $\mu \neq \lambda_1$ or $\mu \neq \lambda_2$, that is, either $\lambda_1 \in \mu^\wedge$ or $\lambda_2 \in \mu^\wedge$, so in view of the fact that

$$(1) \quad \vdash^* L_2, \text{ whenever } \vdash^* L_1 \text{ and } L_1 \subseteq L_2,$$

we obtain $\vdash^* L \cup (\mu^\wedge \times \{A\})$ in either case. Hence by the cut inference in the second form, we have $\vdash^* L$. $\square$

PROPOSITION 5. Suppose that $\mathfrak{P}$ is a proof-figure of the new system. If the

end-matrix of $\mathfrak{P}$ is permissible, so are all matrices contained in it.

PROOF. If the conclusion of the cut inference in the second form is permissible, so are premises. Similarly for left or right inferences. $\square$

PROPOSITION 6. Assume that a formula B is given. Assume further that for every permissible matrix K and for every pair (U, V) of disjoint subsets of T , if $\vdash K \cup (U \times \{B\})$ and $\vdash K \cup (V \times \{B\})$, then $\vdash K$. If L is permissible and if $\vdash L \cup (\mu \wedge \times \{B\})$ for every μ in T , then $\vdash L$.

PROOF. We put $L(S) = L \cup (S \times \{B\})$ and $n(S) = \text{Card} \{ \lambda \mid \langle \lambda, B \rangle \in L(S) \}$ for every subset S of T , where $\text{Card}(W)$ denotes the cardinality of the set W . Suppose that L is permissible and $\vdash L(\mu \wedge)$ for every μ . To show $\vdash L$, that is, to show $\vdash L(\emptyset)$, it suffices to prove that $L(S)$ is cut-free provable whenever it is permissible, which we shall prove by induction on $n(S)$.

CASE 1. $n(S) = 0$. $L(S)$ is basic since it contains $T \times \{B\}$, so $\vdash L(S)$.

CASE 2. $n(S) = 1$. Since $L(S) = L(\mu \wedge)$ for some μ , by the assumption $\vdash L(S)$.

CASE 3. Otherwise. In view of the fact that $L(S)$ is permissible, we put $\lambda_0 = \min \{ \lambda \mid \langle \lambda, B \rangle \in L(S) \}$. We take λ_1 such that $\langle \lambda_1, B \rangle \in L(S)$ and $\lambda_1 \neq \lambda_0$. Then both $L(S) \cup (\lambda_1^+ \times \{B\})$ and $L(S) \cup ((T - \lambda_1^+) \times \{B\})$ are permissible; the latter claim follows from the facts that $\lambda^- \subseteq T - \lambda_1^+$ whenever $\lambda \in T - \lambda_1^+$ and that

$$\lambda_1 = \min \{ \lambda \mid \langle \lambda, B \rangle \in L(S) \cup ((T - \lambda_1^+) \times \{B\}) \}.$$

Hence by the induction hypothesis they are cut-free provable, so by the assumption of the proposition, we have $\vdash L(S)$. $\square$

PROOF THAT LEMMA IMPLIES THEOREM. Suppose that Lemma holds. Suppose further that a permissible matrix K is provable. Then $\vdash^* K$ by Proposition 4, so we let $\mathfrak{P}$ be the proof-figure of the new system having K as its end-matrix. By Proposition 5, every matrices contained in $\mathfrak{P}$ is permissible. Hence, in view of the fact that

(2) $\vdash L$, whenever L is permissible and $\vdash L \cup (\mu \wedge \times \{B\})$ for every μ in T ,

which follows from Lemma and Proposition 6, each matrices contained in $\mathfrak{P}$ is cut-free provable; especially $\vdash K$. $\square$

§ 3. The purpose of this section is to prove Lemma. It suffices to show, on the assumption of Lemma, that if K is permissible, $\vdash_{n_1} K \cup (U_1 \times \{A\})$, $\vdash_{n_2} K \cup (U_2 \times \{A\})$ and if $U_1 \cap U_2 = \emptyset$, then $\vdash K$, which we shall prove by induction on $\omega \cdot d(A) + n_1 + n_2$, where $d(A)$ denotes the number of occurrences of connectives in A .

Suppose that K is permissible, $\vdash_{n_1} K \cup (U_1 \times \{A\})$, $\vdash_{n_2} K \cup (U_2 \times \{A\})$ and that $U_1 \cap U_2 = \emptyset$. We shall deduce $\vdash K$. We further suppose $\{ \lambda \mid \langle \lambda, A \rangle \in K \} \neq \emptyset$; otherwise, K contains $T \times \{A\}$ and so is basic, hence $\vdash K$. We put $\lambda_0 = \min$

$\{\lambda \mid \langle \lambda, A \rangle \in K\}$ and $K_i = K \cup (U_i \times \{A\})$ for $i=1, 2$.

By the induction hypothesis and by Proposition 6 we have the fact that

- (3) for every permissible matrix L and for every formula B , if $\vdash L \cup (\mu \wedge \times \{B\})$
for every μ and if $d(B) < d(A)$, then $\vdash L$.

For the cut-free provable matrix K_i ($i=1, 2$), one of the following six cases occurs :

- I. K_i is basic.
- II. K_i is the conclusion of a left inference, and the principal matrix is contained in K .
- III. K_i is the conclusion of a left inference, but the principal matrix is not contained in K .
- IV. K_i is the conclusion of a right inference, the principal matrix is contained in K , but $\lambda_0 \notin U_i$.
- V. K_i is the conclusion of a right inference, the principal matrix is contained in K , and $\lambda_0 \in U_i$.
- VI. K_i is the conclusion of a right inference, but the principal matrix is not contained in K .

We shall show $\vdash K$ by cases.

CASE 1. Either K_1 or K_2 is of case I. We suppose, without loss of generality, that K_1 is the case. Then $T \times \{B\} \subseteq K_1$ for some B .

SUBCASE 1.1. B is distinct from A . Then $T \times \{B\} \subseteq K$, so K is basic, so $\vdash K$.

SUBCASE 1.2. Otherwise. If $\mu \in U_2$, then $\mu \in U_1$ and so $\langle \mu, A \rangle \in K$; hence $U_2 \times \{A\} \subseteq K$. So $K_2 = K$, hence $\vdash K$.

CASE 2. Either K_1 or K_2 is of case II. We suppose that K_1 is the case. Suppose that $\vdash_{<n_1} K_1 \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\})$ for every $\nu_1, \dots, \nu_h$ falsifying $g(\nu_1, \dots, \nu_h) \leq \nu$, and that $\nu^- \times \{g(B_1, \dots, B_h)\} \subseteq K$. If $\neg [g(\nu_1, \dots, \nu_h) \leq \nu]$, then

$$\vdash_{<n_1} K \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\}) \cup (U_1 \times \{A\})$$

and

$$\vdash_{n_2} K \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\}) \cup (U_2 \times \{A\}),$$

so by the induction hypothesis $\vdash K \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\})$. Hence by the left inference $(g, \nu)^-$, we obtain $\vdash K$.

CASE 3. Either K_1 or K_2 is of case IV. We suppose that K_1 is the case. Suppose that $\vdash_{<n_1} (K_1)^\circ \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\})$ for every $\nu_1, \dots, \nu_h$ falsifying $\nu \leq g(\nu_1, \dots, \nu_h)$, and that $\nu^+ \times \{g(B_1, \dots, B_h)\} \subseteq K$ but $\lambda_0 \notin U_1$. Then $\vdash K^\circ \cup (\nu_1 \wedge \times \{B_1\}) \cup \dots \cup (\nu_h \wedge \times \{B_h\})$ for every $\nu_1, \dots, \nu_h$ falsifying $\nu \leq g(\nu_1, \dots, \nu_h)$, since $(K_1)^\circ = K^\circ$. Hence by the right inference $(g, \nu)^+$, we obtain $\vdash K$.

CASE 4. Both K_1 and K_2 are either of case III or of case V. We suppose that

K_i is of case III, and derive $\lambda_0 \in U_i$; when K_i is of case V, immediately $\lambda_0 \in U_i$. Suppose that $\vdash_{<n_i} K_i \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$ for every $\mu_1, \dots, \mu_k$ falsifying $f(\mu_1, \dots, \mu_k) \leq \mu$, and that $\mu^- \times \{f(A_1, \dots, A_k)\}$ is contained in K_i but is not in K . Then $f(A_1, \dots, A_k)$ is identical with A and so $\lambda_0 \leq \mu$, so $\langle \lambda_0, A \rangle \in \mu^- \times \{f(A_1, \dots, A_k)\} \subseteq K_i$, so $\lambda_0 \in U_i$. Hence, if K_i is either of case III or of case V, then $\lambda_0 \in U_i$. So, Case 4 does not occur since $U_1 \cap U_2 = \emptyset$.

CASE 5. One of K_1 and K_2 is of case III, while another of case VI. We suppose that K_1 is of case III, while K_2 of case VI. We also suppose that A has the form $f(A_1, \dots, A_k)$. Suppose that $\vdash_{<n_1} K_1 \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$ for every $\mu_1, \dots, \mu_k$ falsifying $f(\mu_1, \dots, \mu_k) \leq \mu$, and that $\mu^- \times \{A\}$ is contained in K_1 but is not in K . Suppose further that $\vdash_{<n_2} (K_2)^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$ for every $\mu_1, \dots, \mu_k$ falsifying $\nu \leq f(\mu_1, \dots, \mu_k)$, and that $\nu^+ \times \{A\}$ is contained in K_2 but is not in K . By the argument given in Case 4 we have $\lambda_0 \in U_1$ and $\lambda_0 \leq \mu$. Hence $\lambda_0 \notin U_2$ and so $(K_2)^\circ = K^\circ$.

SUBCASE 5.1. $\nu \leq \mu$. Since the pair (ν, λ_0) has an upper bound μ , it has the least upper bound $\nu \vee \lambda_0$. To show $\vdash K$, it suffices to prove

$$(4) \quad \vdash K^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}) \text{ for every } \mu_1, \dots, \mu_k \text{ falsifying } \nu \vee \lambda_0 \leq f(\mu_1, \dots, \mu_k),$$

and

$$(5) \quad (\nu \vee \lambda_0)^+ \times \{A\} \subseteq K,$$

since from (4) in view of (5) we obtain $\vdash K$ by the right inference $(f, \nu \vee \lambda_0)^+$.

For the proof of (4), suppose $\neg[\nu \vee \lambda_0 \leq f(\mu_1, \dots, \mu_k)]$. Then either $\neg[\nu \leq f(\mu_1, \dots, \mu_k)]$ or $\neg[\lambda_0 \leq f(\mu_1, \dots, \mu_k)]$. We suppose $\neg[\lambda_0 \leq f(\mu_1, \dots, \mu_k)]$; otherwise, $\vdash K^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$ immediately. If $\neg[f(\nu_1, \dots, \nu_k) \leq f(\mu_1, \dots, \mu_k)]$, then $f(\nu_1, \dots, \nu_k) \neq f(\mu_1, \dots, \mu_k)$, so $\nu_j \neq \mu_j$ and so

$$K^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}) \cup (\nu_1^\wedge \times \{A_1\}) \cup \dots \cup (\nu_k^\wedge \times \{A_k\})$$

contains $T \times \{A_j\}$ for some j ($j=1, \dots, k$), hence it is basic and so is cut-free provable. Moreover, if $\lambda \leq f(\mu_1, \dots, \mu_k)$, then $\neg(\lambda_0 \leq \lambda)$, so $\lambda^- \times \{A\} \subseteq K$, that is, $\langle \lambda, A \rangle \in K^\circ$; hence $f(\mu_1, \dots, \mu_k)^- \times \{A\} \subseteq K^\circ$. Hence by the left inference $(f, f(\mu_1, \dots, \mu_k))^-$, we obtain $\vdash K^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$.

Now we shall prove (5). We obtain $\langle \nu \vee \lambda_0, A \rangle \in (\mu^- \times \{A\}) \cap (\nu^+ \times \{A\}) \subseteq K_1 \cap K_2 = K$. So either $(\nu \vee \lambda_0)^- \times \{A\} \subseteq K$ or $(\nu \vee \lambda_0)^+ \times \{A\} \subseteq K$, since K is permissible. But $\neg[(\nu \vee \lambda_0)^- \times \{A\} \subseteq K]$, since $\langle \lambda_0, A \rangle$ belongs to $(\nu \vee \lambda_0)^- \times \{A\}$ but does not to K . Hence $(\nu \vee \lambda_0)^+ \times \{A\} \subseteq K$.

SUBCASE 5.2. Otherwise. To show $\vdash K$, it suffices to prove

$$(6) \quad \vdash K \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}) \text{ for every } \mu_1, \dots, \mu_k \text{ in } T,$$

since from (6) in view of the facts that $d(A_1), \dots, d(A_k) < d(A)$ we obtain $\vdash K$ by

the repeated use of (3).

For the proof of (6), let $\mu_1, \dots, \mu_k$ be given. Then either $\neg[f(\mu_1, \dots, \mu_k) \leq \mu]$ or $\neg[\nu \leq f(\mu_1, \dots, \mu_k)]$. If $\neg[f(\mu_1, \dots, \mu_k) \leq \mu]$, then

$$\vdash_{<n_1} K \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}) \cup (U_1 \times \{A\})$$

and

$$\vdash_{n_2} K \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}) \cup (U_2 \times \{A\}),$$

so by the induction hypothesis, we obtain $\vdash K \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$. Supposing $\neg[\nu \leq f(\mu_1, \dots, \mu_k)]$, we obtain the same result similarly.

CASE 6. One of K_1 and K_2 is of case V, while another of case VI. We suppose that K_1 is of case V, while K_2 of case VI. We also suppose that A has the form $f(A_1, \dots, A_k)$. Suppose that $\vdash_{<n_1} (K_1)^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\})$ for every $\nu_1, \dots, \nu_h$ falsifying $\nu \leq g(\nu_1, \dots, \nu_h)$, and that $\nu^+ \times \{g(B_1, \dots, B_h)\} \subseteq K$ and $\lambda_0 \in U_1$. Suppose further that $\vdash_{<n_2} (K_2)^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\})$ for every $\mu_1, \dots, \mu_k$ falsifying $\mu \leq f(\mu_1, \dots, \mu_k)$, and that $\mu^+ \times \{A\}$ is contained in K_2 but is not in K . It follows $\lambda_0 \notin U_2$, so $(K_2)^\circ = K^\circ$. To show $\vdash K$, it suffices to prove

$$(7) \quad \vdash K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\}) \text{ for every } \nu_1, \dots, \nu_h \text{ falsifying } \nu \leq g(\nu_1, \dots, \nu_h),$$

since from (7) we obtain $\vdash K$ by the right inference $(g, \nu)^+$.

With a view to proving (7), suppose $\neg[\nu \leq g(\nu_1, \dots, \nu_h)]$. If $\neg[\mu \leq f(\mu_1, \dots, \mu_k)]$, then

$$\vdash_{<n_2} (K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\}) \cup (\mu^+ \times \{A\}))^\circ \cup (\mu_1^\wedge \times \{A_1\}) \cup \dots \cup (\mu_k^\wedge \times \{A_k\}),$$

since

$$(K_2)^\circ = K^\circ = (K^\circ)^\circ \subseteq (K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\}) \cup (\mu^+ \times \{A\}))^\circ.$$

Hence by the right inference $(f, \mu)^+$, we obtain

$$\vdash_{n_2} K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\}) \cup (\mu^+ \times \{A\}).$$

Now to show $\vdash K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\})$, it suffices to prove

$$(8) \quad (K_1)^\circ \subseteq K^\circ \cup ((T - \mu^+) \times \{A\}),$$

since from (8) it follows

$$\vdash_{<n_1} K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\}) \cup ((T - \mu^+) \times \{A\}),$$

from which and from the fact just proved, by the induction hypothesis we obtain $\vdash K^\circ \cup (\nu_1^\wedge \times \{B_1\}) \cup \dots \cup (\nu_h^\wedge \times \{B_h\})$.

To prove (8), it suffices to show that $\langle \lambda, A \rangle \in K^\circ \cup ((T - \mu^+) \times \{A\})$ whenever $\langle \lambda, A \rangle \in (K_1)^\circ$. Suppose, contrary to the conclusion, that $\langle \lambda, A \rangle$ belongs to $(K_1)^\circ$ but does not to $K^\circ \cup ((T - \mu^+) \times \{A\})$. Since $\langle \lambda, A \rangle \notin (T - \mu^+) \times \{A\}$ we have $\mu \leq$

λ ; since $\langle \lambda, A \rangle \in K^\circ$, that is, $\neg[\lambda^- \times \{A\} \subseteq K]$ we have $\lambda_0 \leq \lambda$. Hence the pair (μ, λ_0) has the least upper bound $\mu \vee \lambda_0$. We obtain $\langle \mu \vee \lambda_0, A \rangle \in (\lambda^- \times \{A\}) \cap (\mu^+ \times \{A\}) \subseteq K_1 \cap K_2 = K$. So either $(\mu \vee \lambda_0)^- \times \{A\} \subseteq K$ or $(\mu \vee \lambda_0)^+ \times \{A\} \subseteq K$ since K is permissible. But if $(\mu \vee \lambda_0)^- \times \{A\} \subseteq K$, then $\langle \lambda_0, A \rangle \in (\mu \vee \lambda_0)^- \times \{A\} \subseteq K$, which is a contradiction. Now suppose $(\mu \vee \lambda_0)^+ \times \{A\} \subseteq K$. Since $\neg[\mu^+ \times \{A\} \subseteq K]$, we obtain $\mu \leq \mu'$, but $\langle \mu', A \rangle \notin K$ for some μ' . Then $\lambda_0 \leq \mu'$, so $\mu \vee \lambda_0 \leq \mu'$, so $\langle \mu', A \rangle \in (\mu \vee \lambda_0)^+ \times \{A\} \subseteq K$, which is also a contradiction. This completes the proof of (8) and hence of (7).

CASE 7. Both K_1 and K_2 are of case VI. We suppose, without loss of generality, that $\lambda_0 \in U_2$. We also suppose that A has the form $f(A_1, \dots, A_k)$. Suppose that $\vdash_{<n_1(K_1)^\circ \cup (\mu_1^+ \times \{A_1\}) \cup \dots \cup (\mu_k^+ \times \{A_k\})}$ for every $\mu_1, \dots, \mu_k$ falsifying $\mu \leq f(\mu_1, \dots, \mu_k)$, and that $\mu^+ \times \{A\}$ is contained in K_1 but is not in K . Suppose further that $\vdash_{<n_2(K_2)^\circ \cup (\mu_1^+ \times \{A_1\}) \cup \dots \cup (\mu_k^+ \times \{A_k\})}$ for every $\mu_1, \dots, \mu_k$ falsifying $\nu \leq f(\mu_1, \dots, \mu_k)$, and that $\nu^+ \times \{A\}$ is contained in K_2 but is not in K .

SUBCASE 7.1. The pair (μ, ν) has an upper bound. Then the pair has the least upper bound $\mu \vee \nu$. To show $\vdash K$, it suffices to prove

$$(9) \quad \vdash K^\circ \cup (\mu_1^+ \times \{A_1\}) \cup \dots \cup (\mu_k^+ \times \{A_k\}) \text{ for every } \mu_1, \dots, \mu_k \text{ falsifying } \mu \vee \nu \leq f(\mu_1, \dots, \mu_k),$$

and

$$(10) \quad (\mu \vee \nu)^+ \times \{A\} \subseteq K,$$

since from (9) in view of (10) we obtain $\vdash K$ by the right inference $(f, \mu \vee \nu)^+$.

For the proof of (9), suppose $\neg[\mu \vee \nu \leq f(\mu_1, \dots, \mu_k)]$. Then either $\neg[\mu \leq f(\mu_1, \dots, \mu_k)]$ or $\neg[\nu \leq f(\mu_1, \dots, \mu_k)]$. If $\neg[\mu \leq f(\mu_1, \dots, \mu_k)]$, then we obtain $\vdash K^\circ \cup (\mu_1^+ \times \{A_1\}) \cup \dots \cup (\mu_k^+ \times \{A_k\})$ by the similar method as the proof of (7); if $\neg[\nu \leq f(\mu_1, \dots, \mu_k)]$, then we obtain the same result since $(K_2)^\circ = K^\circ$.

Now we shall prove (10). If $\mu \vee \nu \leq \lambda$, then $\langle \lambda, A \rangle \in (\mu^+ \times \{A\}) \cap (\nu^+ \times \{A\}) \subseteq K_1 \cap K_2 = K$. Hence $(\mu \vee \nu)^+ \times \{A\} \subseteq K$.

SUBCASE 7.2. Otherwise. We obtain $\vdash K$ similarly to Subcase 5.2.

Thus we have proved Lemma, and hence the proof of Theorem is completed.

Reference

- [1] M. Takano, Extending the family of intuitionistic many-valued logics introduced by Rousseau, this ANNALS, Vol. 7, No. 1 (1986), 47-56.