A Sequent Calculus for the Lesniewskian Modal Logic

*Dedicated for the memory of the
late Professor S. Maehara*

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Introduction.

A binary relation R on a non-empty set W is called *Lesniewskian*, if the following three conditions hold :

- (0.1) aRb implies aRa , for every $a, b \in W$.
- (0.2) aRb and bRc imply aRc , for every $a, b, c \in W$.
- (0.3) aRb and bRc imply bRa , for every $a, b, c \in W$.

The notion of Lesniewskian relation, though the naming is new, has been introduced and extensively investigated by Ishimoto [1] in the study of the propositional fragment of Lesniewski's ontology. Examples of Lesniewskian relations are "is a one-element subset of" for sets (Takano [6]), and "is a prime divisor of" for positive integers (N. Kanai).

By the *Lesniewskian modal logic* we mean the normal propositional modal logic which is characterized by the class of Kripke models with Lesniewskian accessibility relation.

The purpose of this paper is to give a sequential calculus L^* for the Lesniewskian modal logic L enjoying a kind of the subformula property, and as an application of this property to show the L -version of the Craig interpolation lemma. So, this article forms an L -version of our previous paper [7], and hence will be better appreciated if readers are acquainted with it. (The same result as Main Theorem for $S5^*$ in [7] is seen in Mints [4, Theorem 2 (p. 238)] as well.)

§1. Formulation of Main Theorem.

In this section, we will describe the sequent calculus L^* for the Lesniewskian modal logic L and then formulate our main theorem.

We will, first of all, explain some notions and notations. *Formulas* are con-

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structed from the propositional constant $\perp$ (falsity) and propositional variables by the help of the logical symbols $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\sim$ (negation) and $\Box$ (necessity). *Subformulas* of a formula are defined as usual, and $\text{Sub}(\alpha)$ will designate a sequence (ordered by some alphabetical order) of all subformulas of a formula α . A *box-formula* is that of the form $\Box\gamma$ for some formula γ .

Let $\Sigma = (\gamma_1, \dots, \gamma_n)$ be a sequence of formulas. We let $\text{Sub}(\Sigma)$ represent a sequence of all subformulas of $\gamma_1, \dots, \gamma_{n-1}$ or γ_n , while $\Box\Sigma$ and $\Sigma \supset \alpha$ the sequences $(\Box\gamma_1, \dots, \Box\gamma_n)$ and $(\gamma_1 \supset \alpha, \dots, \gamma_n \supset \alpha)$, respectively. In the meantime, Σ_α denotes the sequence which is obtained from Σ by deleting all the occurrences of α . A pair $\langle \Phi; \Psi \rangle$ is a *partition* of Σ , if $\Phi = (\gamma_{i_1}, \dots, \gamma_{i_l})$ and $\Psi = (\gamma_{j_1}, \dots, \gamma_{j_m})$ for some numbers $i_1, \dots, i_l, j_1, \dots, j_m$ such that $i_1 < \dots < i_l, j_1 < \dots < j_m, \{i_1, \dots, i_l\} \cap \{j_1, \dots, j_m\} = \emptyset$ and $\{i_1, \dots, i_l\} \cup \{j_1, \dots, j_m\} = \{1, \dots, n\}$.

From lack of suitable notations, we will apply those for sets such as $\gamma \in \Phi, \gamma \in \Phi \cap \Psi, \Phi \cap \Psi \neq \emptyset$ and $\Phi \subseteq \Psi$ also for sequences; thus for example, the notation $\Phi \subseteq (\Psi, \alpha)$, where Φ and Ψ represent sequences of formulas while α a formula, means that any formula occurring in Φ either occurs in Ψ or is identical with α , irrespective of the order and the number of the occurrences.

It is well-known that the scheme $\Box\alpha \supset \Box\Box\alpha$ is complete for the condition (0.2), namely the transitivity of the accessibility relation; meanwhile, it is noted in Lemmon [2, p. 72] that $\alpha \vee (\Box\alpha \supset \Box\perp)$ is complete for (0.1). On the other hand, it can be shown by the standard method that the scheme $\alpha \vee \Box(\Box\alpha \supset \Box\perp)$ is complete for the condition (0.3). Hence we have the following characterization of the Lesniewskian modal logic.

PROPOSITION 1.1. *The Lesniewskian modal logic is axiomatized as the smallest normal propositional modal logic K with the additional axiom schemes $\alpha \vee (\Box\alpha \supset \Box\perp)$, $\Box\alpha \supset \Box\Box\alpha$ and $\alpha \vee \Box(\Box\alpha \supset \Box\perp)$.*

Now, the sequent calculus L^* is defined to be the propositional fragment of the system LK (based on the language of modal logic) supplemented with the additional beginning sequent $\perp \rightarrow$ for $\perp$, and for each formulas α and β the additional beginning sequent $\Box\alpha \rightarrow \alpha, \Box\beta$ for $\Box$, as well as the additional rule:

$$(\rightarrow \Box): \frac{\Gamma, \Box\Gamma \rightarrow \Box\Theta, \alpha \quad \Box\perp, \Gamma \rightarrow \alpha}{\Box\Gamma \rightarrow \Box\Theta, \Box\alpha}.$$

The structural rules “weakening”, “contraction” and “exchange” are called generically as *weak inferences*. In describing (formal) proofs, we will often neglect applications of weak inferences.

PROPOSITION 1.2. *The sequent calculus L^* represents the Lesniewskian modal logic L .*

PROOF. By Proposition 1.1, it suffices to show the equivalence between L^* and the calculus GL which is obtained from the propositional fragment of LK by adding the sequents of the forms :

$$\perp \rightarrow ; \Box \alpha \rightarrow \alpha, \Box \perp ; \Box \alpha \rightarrow \Box \Box \alpha ; \text{ and } \rightarrow \alpha, \Box(\Box \alpha \supset \Box \perp),$$

as beginning sequents, and the rule :

$$(\rightarrow \Box)_K : \frac{\Gamma \rightarrow \alpha}{\Box \Gamma \rightarrow \Box \alpha},$$

as an additional rule.

It is easy to show that any sequent provable in GL is also provable in L^* ; the proof in L^* of the beginning sequent $\rightarrow \alpha, \Box(\Box \alpha \supset \Box \perp)$ of GL is obtained from the one described in Proof of Proposition 1.3 by substituting α for p throughout. Conversely, proofs in GL of the beginning sequents of L^* are immediately given. So we will mention only the rule $(\rightarrow \Box)$ of L^* . An application :

$$\frac{\Gamma, \Box \Gamma \rightarrow \Box \Theta, \alpha \quad \Box \perp, \Gamma \rightarrow \alpha}{\Box \Gamma \rightarrow \Box \Theta, \Box \alpha},$$

of $(\rightarrow \Box)$ is justified in GL by the following figure :

$$\text{cuts } \frac{\frac{P_\theta \text{ for every } \theta \in \Theta \quad Q}{\Box \Gamma \rightarrow \Box \Theta, \Box \alpha} \text{ cuts}}{\Box \Gamma \rightarrow \Box \Theta, \Box \alpha},$$

where P_θ and Q represent the following figures, respectively :

$$\begin{array}{c} \frac{\Box \theta \rightarrow \Box \Box \theta \quad \Box \perp \rightarrow \Box \perp}{\Box \Box \theta \supset \Box \perp, \Box \theta \rightarrow \Box \perp} \\ \frac{\Box \Box \theta \supset \Box \perp \rightarrow \Box \theta \supset \Box \perp}{\Box \Box \theta \supset \Box \perp \rightarrow \Box \theta \supset \Box \perp} \\ \text{cut } \frac{\rightarrow \Box \theta, \Box(\Box \Box \theta \supset \Box \perp) \quad \Box(\Box \Box \theta \supset \Box \perp) \rightarrow \Box(\Box \theta \supset \Box \perp)}{\rightarrow \Box \theta, \Box(\Box \theta \supset \Box \perp)} ; \\ \frac{(\rightarrow \Box)_K \frac{\Gamma, \Box \Gamma \rightarrow \Box \Theta, \alpha \quad \Box \perp, \Gamma \rightarrow \alpha}{\Box \Theta \supset \Box \perp, \Gamma, \Box \Gamma \rightarrow \alpha}}{(\rightarrow \Box)_K \frac{\Box \Theta \supset \Box \perp, \Gamma, \Box \Gamma \rightarrow \alpha}{\Box(\Box \Theta \supset \Box \perp), \Box \Gamma, \Box \Box \Gamma \rightarrow \Box \alpha}} \quad \square \end{array}$$

Suppose that $\Pi \rightarrow \Sigma$ and $\Gamma \rightarrow \Theta$ are (any one of) the upper sequent(s) and the lower sequent, respectively, of an application of the rule R of our calculus L^* . If R is neither $(\rightarrow \Box)$ nor the cut rule, then $(\Pi, \Sigma) \subseteq \text{Sub}(\Gamma, \Theta)$; and, if R is $(\rightarrow \Box)$, then $(\Pi, \Sigma) \subseteq \text{Sub}(\Gamma, \Theta, \Box \perp)$. On the other hand, if R is the cut rule, similar properties do not hold in general. However, if L^* enjoyed the cut-elimination property, this would not cause any trouble. But unfortunately, this is not the case.

PROPOSITION 1.3. *The sequent calculus L^* lacks the cut-elimination property.*

PROOF. We cannot eliminate the application of the cut rule in the following proof, where p is any propositional variable :

$$(\rightarrow\Box) \frac{\frac{\Box p \rightarrow \Box p}{\rightarrow\Box p, \Box p \supset \Box \perp} \quad \frac{\Box \perp \rightarrow \Box \perp}{\Box \perp \rightarrow \Box p \supset \Box \perp}}{\text{cut} \frac{\rightarrow\Box p, \Box(\Box p \supset \Box \perp) \quad \Box p \rightarrow p, \Box(\Box p \supset \Box \perp)}{\rightarrow p, \Box(\Box p \supset \Box \perp)}}. \quad \square$$

REMARK 1.4. Lack of the cut-elimination property of L^* is not due to the additional beginning sequents for $\Box$. For, though they can be formulated naturally as the rule :

$$(\Box \rightarrow) : \frac{\alpha, \Gamma \rightarrow \Theta}{\Box \alpha, \Gamma \rightarrow \Theta},$$

where at least one box-formula occurs in Θ , the counterexample described above still works, provided that the following proof of the right upper sequent of the cut is supplemented :

$$(\Box \rightarrow) \frac{\frac{p \rightarrow p}{p \rightarrow p, \Box(\Box p \supset \Box \perp)}}{\Box p \rightarrow p, \Box(\Box p \supset \Box \perp)}. \quad \square$$

So we put the following definition after Smullyan [5] and Takano [7] ; a cut (namely, an application of the cut rule) :

$$\frac{\Gamma \rightarrow \Theta, \alpha \quad \alpha, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda},$$

is called *acceptable*, when $\alpha \in \text{Sub}(\Gamma, \Delta, \Theta, \Lambda)$.

By a *suitable proof*, we mean a proof in L^* such that every cut applied in it, if any, is acceptable.

Evidently, if $\Pi \rightarrow \Sigma$ and $\Gamma \rightarrow \Theta$ are any one of the upper sequents and the lower sequent, respectively, of an acceptable cut, then $(\Pi, \Sigma) \subseteq \text{Sub}(\Gamma, \Theta)$. So, a suitable proof of $\Gamma \rightarrow \Theta$ is constituted from those formulas in $\text{Sub}(\Gamma, \Theta, \Box \perp)$ alone.

Then our main theorem, which claims a kind of the subformula property of L^* and so works as a substitute for the cut-elimination property, is formulated as follows.

MAIN THEOREM. *Every proof in the sequent calculus L^* can be transformed into a suitable proof with the same endsequent.*

The proof of this theorem will be given in the next section.

According to this theorem as well as Proposition 1.2, a formula α is provable in

L , if and only if the sequent $\rightarrow\alpha$ has a proof in L^* consisting solely of formulas in $\text{Sub}(\alpha, \Box\perp)$. Hence a decision procedure for L follows from this immediately.

COROLLARY 1.5. *The Lesniewskian modal logic is decidable.*

§2. Proof of Main Theorem.

This section is devoted to the proof of our main theorem.

LEMMA 2.1. 1) *If $\Gamma\rightarrow\Theta$ has a suitable proof, so too has $\Gamma\rightarrow\Theta_\perp$.*
2) *If $\Gamma\rightarrow\Theta$ has a suitable proof, so too has $\Gamma\rightarrow\Box\beta, \Theta_{\Box\perp}$ for any formula β .*

PROOF. Since the proofs are similar, we will merely prove 2). The proof is carried out by induction on the length of the given suitable proof; we show only the typical cases. We denote by S and $S^{(\beta)}$ the sequents $\Gamma\rightarrow\Theta$ and $\Gamma\rightarrow\Box\beta, \Theta_{\Box\perp}$, respectively.

CASE 1. S is the beginning sequent $\gamma\rightarrow\gamma$. If $\gamma\neq\Box\perp$, then $S^{(\beta)}$ is $\gamma\rightarrow\Box\beta, \gamma$; so it has a suitable proof clearly. Hence we suppose $\gamma=\Box\perp$, so that $S^{(\beta)}$ is $\Box\perp\rightarrow\Box\beta$. Then $S^{(\beta)}$ is obtainable from two occurrences of the beginning sequent $\perp\rightarrow$ by $(\rightarrow\Box)$; so it has a suitable proof.

CASE 2. S is the beginning sequent $\Box\gamma\rightarrow\gamma, \Box\delta$. If $\gamma\neq\Box\perp$, then $S^{(\beta)}$ is $\Box\gamma\rightarrow\Box\beta, \gamma, \Box(\delta_\perp)$; so it has a suitable proof clearly. Hence we suppose $\gamma=\Box\perp$, so that $S^{(\beta)}$ is $\Box\Box\perp\rightarrow\Box\beta, \Box(\delta_\perp)$. Then $S^{(\beta)}$ is obtainable by an acceptable cut from the beginning sequent $\Box\Box\perp\rightarrow\Box\perp, \Box\perp$ and the sequent $\Box\perp\rightarrow\Box\beta$, which in turn is obtainable from $\perp\rightarrow$ by $(\rightarrow\Box)$; so it has a suitable proof.

CASE 3. S is the lower sequent $\Pi^1, \Pi^2\rightarrow\Sigma^1, \Sigma^2$ of an acceptable cut with the upper sequents $\Pi^1\rightarrow\Sigma^1, \gamma$ and $\gamma, \Pi^2\rightarrow\Sigma^2$, where $\gamma\in\text{Sub}(\Pi^1, \Pi^2, \Sigma^1, \Sigma^2)$. Suppose first that $\gamma\in\text{Sub}(\Box\perp)$. Then $S^{(\beta)}$ is obtainable by an acceptable cut from $\Pi^1\rightarrow\Box\beta, \Sigma_{\Box\perp}^1, \gamma$ and $\gamma, \Pi^2\rightarrow\Box\beta, \Sigma_{\Box\perp}^2$, which have suitable proofs by the hypothesis of induction; so it has a suitable proof. Hence we suppose $\gamma\in\text{Sub}(\Box\perp)$. Then $\Pi^1\rightarrow\Box\beta, \Sigma_{\Box\perp}^1$ has a suitable proof by the hypothesis of induction and by 1) when $\gamma=\perp$, and $S^{(\beta)}$ is obtainable from this by weak inferences. So $S^{(\beta)}$ has a suitable proof in this case too.

CASE 4. S is the lower sequent $\Box\Pi\rightarrow\Box\Sigma, \Box\gamma$ of an application of $(\rightarrow\Box)$ with the upper sequents $\Pi, \Box\Pi\rightarrow\Box\Sigma, \gamma$ and $\Box\perp, \Pi\rightarrow\gamma$. First suppose $\gamma\neq\perp$, so that $S^{(\beta)}$ is $\Box\Pi\rightarrow\Box\beta, \Box(\Sigma_\perp), \Box\gamma$. Then $\Pi, \Box\Pi\rightarrow\Box\beta, \Box(\Sigma_\perp), \gamma$ has a suitable proof by the hypothesis of induction, and $S^{(\beta)}$ is obtainable from this together with $\Box\perp, \Pi\rightarrow\gamma$ by $(\rightarrow\Box)$. So $S^{(\beta)}$ has a suitable proof. Next we suppose $\gamma=\perp$, so that $S^{(\beta)}$ is $\Box\Pi\rightarrow\Box\beta, \Box(\Sigma_\perp)$. By the hypothesis of induction $\Pi, \Box\Pi\rightarrow\Box\beta, \Box(\Sigma_\perp), \perp$

has a suitable proof, and so too has Π , $\Box\Pi \rightarrow \Box\beta$, $\Box(\Sigma_\perp)$ by 1); and $S^{(\beta)}$ is obtainable from the latter together with the beginning sequents $\Box\pi \rightarrow \pi$, $\Box\beta$ for each $\pi \in \Pi$ by acceptable cuts. Hence $S^{(\beta)}$ has a suitable proof in this case too. $\square$

The following lemma is a version of Takano [7, Lemma 3.1], and is easily proved by induction on the length of the sequence Ξ . But we will give a definition beforehand.

For any formula α , an α -suitable proof is a proof in L^* such that every cut applied in it, if any, is acceptable or has a subformula of α or of $\Box\perp$ as its cut formula.

LEMMA 2.2. *Suppose $\Xi \subseteq \text{Sub}(\Gamma, \Theta, \alpha, \Box\perp)$. If $\Phi, \Gamma \rightarrow \Theta, \Psi$ has an α -suitable proof for every partition $\langle \Phi; \Psi \rangle$ of Ξ , then so too has $\Gamma \rightarrow \Theta$.*

The following lemma forms the L -version of Takano [7, Lemma 3.2], and serves as a principal lemma to our main theorem.

LEMMA 2.3. *If $\Box\Gamma \rightarrow \Box\Theta$ has a suitable proof with an application of $(\rightarrow\Box)$ for its lowest inference, then $\Box\Gamma \rightarrow \Box(\Theta_\alpha)$, α , $\Box\beta$ has an α -suitable proof for any formula β .*

PROOF. Let P be a suitable proof of $\Box\Gamma \rightarrow \Box\Theta$ with an application K of $(\rightarrow\Box)$ for its lowest inference.

If $\Box\alpha \in \text{Sub}(\Box\Gamma, \Box(\Theta_\alpha))$, then $\Box\Gamma \rightarrow \Box(\Theta_\alpha)$, α , $\Box\beta$ is obtainable from $\Box\Gamma \rightarrow \Box\Theta$ and the beginning sequent $\Box\alpha \rightarrow \alpha$, $\Box\beta$ by an acceptable cut; so it has an α -suitable proof in this case. Hence we assume $\Box\alpha \notin \text{Sub}(\Box\Gamma, \Box(\Theta_\alpha))$.

Meanwhile, if the lower sequent of a cut contains the cut formula, it is obtainable from one of the upper sequents by weak inferences. Hence we suppose from the outset that for every (acceptable) cut in P , the cut formula does not occur in the lower sequent. It follows that none of the sequents of P contain $\Box\alpha$ in its antecedent.

Let Ξ be a sequence (ordered by some alphabetical order) of all formulas except $\Box\alpha$ which occur in the lower sequent of some application in P of $(\rightarrow\Box)$. It follows that Ξ consists solely of box-formulas, and $\Xi \subseteq \text{Sub}(\Box\Gamma, \Box(\Theta_\alpha), \alpha, \Box\perp)$. Hence to show that $\Box\Gamma \rightarrow \Box(\Theta_\alpha)$, α , $\Box\beta$ has an α -suitable proof, it suffices by Lemma 2.2 to show that, for every partition $\langle \Box\Phi; \Box\Psi \rangle$ of Ξ :

$$(2.1) \quad \Box\Phi, \Box\Gamma \rightarrow \Box(\Theta_\alpha), \alpha, \Box\beta, \Box\Psi \text{ has an } \alpha\text{-suitable proof.}$$

If $\Gamma \cap \Psi \neq \phi$ or $\Theta_\alpha \cap \Phi \neq \phi$, then (2.1) holds clearly. Hence we assume other-

wise. Since $\Box\Gamma \subseteq \Xi$ and $\Box(\Theta_\alpha) \subseteq \Xi$, it follows $\Gamma \subseteq \Phi$ and $\Theta_\alpha \subseteq \Psi$.

Now, draw attention to one of the uppermost application :

$$K^1 \frac{\Gamma^1, \Box\Gamma^1 \rightarrow \Box\Theta^1, \gamma^1 \quad \Box\perp, \Gamma^1 \rightarrow \gamma^1}{\Box\Gamma^1 \rightarrow \Box\Theta^1, \Box\gamma^1},$$

of $(\rightarrow\Box)$ in P , such that the antecedent and the succedent of the lower sequent are contained in $\Box\Phi$ and $(\Box\Psi, \Box\alpha)$, respectively ; such an application exists certainly, since the antecedent and the succedent of the lower sequent $\Box\Gamma \rightarrow \Box\Theta$ of K are contained in $\Box\Phi$ and $(\Box\Psi, \Box\alpha)$, respectively. Then it follows $\Gamma^1 \subseteq \Phi$ and $(\Theta^1, \gamma^1) \subseteq (\Psi, \alpha)$; and, if $\Box\Pi \rightarrow \Box\Sigma$ is the lower sequent of an application of $(\rightarrow\Box)$ which is applied above K^1 , either $\Pi \cap \Psi \neq \phi$ or $\Sigma_\alpha \cap \Phi \neq \phi$.

Suppose we have proved the proposition :

(2.2) *For every sequent $\Pi \rightarrow \Sigma$ which lies above the lower sequent of K^1 , the sequent $\Pi, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Sigma_{\Box\alpha}$ has an α -suitable proof.*

Now, apply (2.2) to the left upper sequent of K^1 . It follows that $\Gamma^1, \Box\Gamma^1, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Box(\Theta_\alpha^1), \gamma^1$ has an α -suitable proof. Hence, if $\gamma^1 = \alpha$, (2.1) follows immediately from this together with the beginning sequents $\Box\gamma \rightarrow \gamma, \Box\beta$ for each $\gamma \in \Gamma^1$ by acceptable cuts ; on the other hand, if $\gamma^1 \neq \alpha$, (2.1) follows by making use of $(\rightarrow\Box)$ instead of acceptable cuts.

So we shall prove (2.2), by induction on the number of sequents which are situated above $\Pi \rightarrow \Sigma$. We will show only the typical cases, and denote by S and $S^\#$ the sequents $\Pi \rightarrow \Sigma$ and $\Pi, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Sigma_{\Box\alpha}$, respectively.

CASE 1. S is the beginning sequent $\gamma \rightarrow \gamma$. Then $S^\#$ denotes $\gamma, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \gamma$ since $\gamma \neq \Box\alpha$; so it clearly has an α -suitable proof.

CASE 2. S is the beginning sequent $\Box\gamma \rightarrow \gamma, \Box\delta$. Then $S^\#$ denotes $\Box\gamma, \Phi \rightarrow \Box\Psi, \Box\beta, \gamma, \Box(\delta_\alpha)$ since $\gamma \neq \Box\alpha$; so it has an α -suitable proof.

CASE 3. S is the lower sequent $\Pi^1, \Pi^2 \rightarrow \Sigma^1, \Sigma^2$ of an acceptable cut with the upper sequents $\Pi^1 \rightarrow \Sigma^1, \gamma$ and $\gamma, \Pi^2 \rightarrow \Sigma^2$, where $\gamma \in \text{Sub}(\Pi^1, \Pi^2, \Sigma^1, \Sigma^2)$. Since $\gamma \neq \Box\alpha$, either $\gamma \in \text{Sub}(\alpha)$ or $\gamma \notin \text{Sub}(\Box\alpha)$. Then $S^\#$ is obtainable by a cut from $\Pi^1, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Sigma_{\Box\alpha}^1, \gamma$ and $\gamma, \Pi^2, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Sigma_{\Box\alpha}^2$, which have α -suitable proofs by the hypothesis of induction, and the cut has the subformula γ of α as its cut formula or is acceptable according as $\gamma \in \text{Sub}(\alpha)$ or $\gamma \notin \text{Sub}(\Box\alpha)$; so it has an α -suitable proof.

CASE 4. S is the lower sequent $\Box\Pi \rightarrow \Box\Sigma$ of an application of $(\rightarrow\Box)$, so that $S^\#$ is $\Box\Pi, \Box\Phi \rightarrow \Box\Psi, \Box\beta, \Box(\Sigma_\alpha)$. Then $S^\#$ has an α -suitable proof clearly, since either $\Pi \cap \Psi \neq \phi$ or $\Sigma_\alpha \cap \Phi \neq \phi$.

Thus we have completed the proof of (2.2), and so of (2.1) for every partition $\langle \Box\Phi ; \Box\Psi \rangle$ of Ξ . □

Now, we can prove our main theorem.

PROOF OF MAIN THEOREM. Given a proof in L^* , we first replace all unacceptable cuts applied in it by mixes, where mix (of the mix formula α) is the rule :

$$\frac{\Gamma \rightarrow \Theta \quad \Delta \rightarrow \Lambda}{\Gamma, \Delta_{\alpha \rightarrow \Theta \cap \Lambda}},$$

where $\alpha \in \Theta \cap \Delta$. Then it suffices for our purpose to show the following :

(2.3) *Any proof P with a mix for its lowest inference, and not containing any other mix (nor unacceptable cut) can be transformed into a suitable proof with the same endsequent.*

We prove (2.3) by double induction on the degree and the rank of P as usual. But here the *degree* of a formula α is defined to be the number of occurrences of propositional variables and logical symbols in α . An effect of this modification is that, whenever $\alpha \neq \perp$, every formula in $\text{Sub}(\alpha, \Box \perp)$ has a smaller degree than $\Box \alpha$, which fact is utilized in Subcase 2.2 below.

We will mention only the typical few cases of our system ; for the case where at least one of the upper sequents of the mix is the lower sequent of an acceptable cut, one can easily lower the rank of the mix (see Case 1 of Proof of Main Theorem for $S5^*$ in Takano [7]).

CASE 1. The left upper sequent of the mix is the beginning sequent $\Box \alpha \rightarrow \alpha$, $\Box \beta$.

SUBCASE 1.1. The mix formula is α . Then the end of P runs as follows :

$$\frac{\Box \alpha \rightarrow \alpha, \Box \beta \quad \Delta \rightarrow \Lambda}{\Box \alpha, \Delta_{\alpha \rightarrow (\Box \beta)_{\alpha}}, \Lambda},$$

where $\alpha \in \Delta$. In this case, the endsequent is obtainable from the upper sequents of the mix by an acceptable cut ; so it has a suitable proof.

SUBCASE 1.2. The mix formula is not α but $\Box \beta$. Since the right rank can be easily reduced in other cases, we suppose that the right upper sequent of the mix is either an additional beginning sequent for $\Box$ or the lower sequent of an application of $(\rightarrow \Box)$. Then the end of P runs as follows :

$$\frac{\Box \alpha \rightarrow \alpha, \Box \beta \quad \Delta \rightarrow \Lambda, \Box \gamma}{\Box \alpha, \Delta_{\Box \beta \rightarrow \alpha}, \Lambda, \Box \gamma},$$

where $\Box \beta \in \Delta$. In this case, the endsequent is obtainable from the beginning sequent $\Box \alpha \rightarrow \alpha$, $\Box \gamma$ by weak inferences ; so it has a suitable proof.

CASE 2. The left upper sequent of the mix is the lower sequent of an application of $(\rightarrow \Box)$, while the right upper sequent of the mix is an additional beginning

sequent for $\Box$. In this case, the end of P runs as follows :

$$\frac{\Box\Gamma \rightarrow \Box\Theta \quad \Box\alpha \rightarrow \alpha, \Box\beta}{\Box\Gamma \rightarrow \Box(\Theta_a), \alpha, \Box\beta},$$

where $\Box\alpha$ is the mix formula and $\alpha \in \Theta$.

SUBCASE 2.1. $\alpha = \perp$. The endsequent has a suitable proof by Lemma 2.1.2) and weak inferences.

SUBCASE 2.2. $\alpha \neq \perp$. It follows that every formula in $\text{Sub}(\alpha, \Box\perp)$ has a smaller degree than $\Box\alpha$. The endsequent has an α -suitable proof by Lemma 2.3, and so has a suitable proof by the hypothesis of induction.

CASE 3. The left upper sequent of the mix is the lower sequent $\Box\Gamma \rightarrow \Box\Theta, \Box\alpha$ of an application of $(\rightarrow\Box)$, and the right one too is the lower sequent of another application of $(\rightarrow\Box)$.

SUBCASE 3.1. The mix formula is $\Box\alpha$. In this case, the end of P runs as follows :

$$\frac{\Gamma, \Box\Gamma \rightarrow \Box\Theta, \alpha \quad \Box\perp, \Gamma \rightarrow \alpha \quad \Delta, \Box\Delta \rightarrow \Box\Lambda, \gamma \quad \Box\perp, \Delta \rightarrow \gamma}{I \frac{\Box\Gamma \rightarrow \Box\Theta, \Box\alpha \quad \Box\Delta \rightarrow \Box\Lambda, \Box\gamma}{\Box\Gamma, \Box(\Delta_a) \rightarrow \Box(\Theta_a), \Box\Lambda, \Box\gamma}},$$

where $\alpha \in \Delta$. Since another case is similar, we suppose $\alpha \in \Theta$. We transform P into the proof :

$$(\rightarrow\Box) \frac{Q \quad \frac{\Box\perp, \Gamma \rightarrow \alpha \quad \Box\perp, \Delta \rightarrow \gamma}{\Box\perp, \Gamma, \Delta_a \rightarrow \gamma} J_1}{\Box\Gamma, \Box(\Delta_a) \rightarrow \Box(\Theta_a), \Box\Lambda, \Box\gamma},$$

where Q denotes the proof :

$$\frac{J_2 \quad \Gamma, \Box\Gamma \rightarrow \Box\Theta, \alpha \quad \Box\Delta \rightarrow \Box\Lambda, \Box\gamma \quad \Box\Gamma \rightarrow \Box\Theta, \Box\alpha \quad \Delta, \Box\Delta \rightarrow \Box\Lambda, \gamma}{J_4 \quad \frac{\Gamma, \Box\Gamma, \Box(\Delta_a) \rightarrow \Box(\Theta_a), \alpha, \Box\Lambda, \Box\gamma \quad \Box\Gamma, \Delta, \Box(\Delta_a) \rightarrow \Box(\Theta_a), \Box\Lambda, \gamma}{\Gamma, \Delta_a, \Box\Gamma, \Box(\Delta_a) \rightarrow \Box(\Theta_a), \Box\Lambda, \Box\gamma, \gamma} J_3},$$

where those parts which are above the topmost sequents of these figures remain the same. Since the mixes J_2 and J_3 of $\Box\alpha$ have the same degree with and smaller ranks than I , and the mixes J_1 and J_4 of α have smaller degrees than I , the endsequent has a suitable proof by the hypothesis of induction.

SUBCASE 3.2. The mix formula is not $\Box\alpha$. Then the end of P runs as follows :

$$\frac{\Gamma, \Box\Gamma \rightarrow \Box\Theta, \alpha \quad \Box\perp, \Gamma \rightarrow \alpha}{I \quad \frac{\Box\Gamma \rightarrow \Box\Theta, \Box\alpha \quad \Box\Delta \rightarrow \Box\Lambda}{\Box\Gamma, \Box(\Delta_\beta) \rightarrow \Box(\Theta_\beta), \Box\alpha, \Box\Lambda}},$$

where $\Box\beta$ is the mix formula, $\beta \neq \alpha$ and $\beta \in \Theta \cap \Delta$. We transform P into the proof :

$$(\rightarrow \Box) \frac{J \frac{\Gamma, \Box \Gamma \rightarrow \Box \Theta, \alpha \quad \Box \Delta \rightarrow \Box \Lambda}{\Gamma, \Box \Gamma, \Box (\Delta_\beta) \rightarrow \Box (\Theta_\beta), \alpha, \Box \Lambda} \quad \Box \perp, \Gamma \rightarrow \alpha}{\Box \Gamma, \Box (\Delta_\beta) \rightarrow \Box (\Theta_\beta), \Box \alpha, \Box \Lambda}.$$

Since the mix J of $\Box\beta$ has the same degree with and a smaller rank than I , the endsequent has a suitable proof by the hypothesis of induction. $\square$

§3. The Craig Interpolation Property.

One application of Main Theorem is the decidability of our logic L (Corollary 1.5). As another application of our main theorem, the Craig interpolation property for L is shown in this section.

THEOREM 3.1. *If $\alpha \supset \beta$ is provable in the Lesniewskian modal logic L , then there is a formula γ such that $\alpha \supset \gamma$ and $\gamma \supset \beta$ are provable in L , and every propositional variable occurring in γ occurs both in α and β .*

For the proof of this theorem, it suffices by Proposition 1.2 and Main Theorem to prove Lemma 3.2 below.

The pair $\langle \Gamma^1 \rightarrow \Theta^1; \Gamma^2 \rightarrow \Theta^2 \rangle$ is called a *partition* of $\Gamma \rightarrow \Theta$, when $\langle \Gamma^1; \Gamma^2 \rangle$ and $\langle \Theta^1; \Theta^2 \rangle$ are partitions of Γ and Θ , respectively. A formula γ is an *interpolant* of $\langle \Gamma^1 \rightarrow \Theta^1; \Gamma^2 \rightarrow \Theta^2 \rangle$, if $\Gamma^1 \rightarrow \Theta^1$, γ and $\gamma, \Gamma^2 \rightarrow \Theta^2$ are provable in L^* , and if every propositional variable occurring in γ occurs both in some formula in (Γ^1, Θ^1) and some in (Γ^2, Θ^2) .

LEMMA 3.2. *If a sequent S has a suitable proof, every partition of S has an interpolant.*

PROOF. We prove this, after Maehara [3], by induction on the length of the given suitable proof; we will show only the typical cases of our calculus.

CASE 1. S is the beginning sequent $\perp \rightarrow$. Then $\perp$ and $\sim \perp$ are interpolants of $\langle \perp \rightarrow; \rightarrow \rangle$ and $\langle \rightarrow; \perp \rightarrow \rangle$, respectively.

CASE 2. S is the beginning sequent $\Box \alpha \rightarrow \alpha, \Box \beta$. Then $\perp, \Box \perp, \alpha, \Box \alpha, \sim \Box \alpha, \sim \alpha, \sim \Box \perp$ and $\sim \perp$ are interpolants of $\langle \Box \alpha \rightarrow \alpha, \Box \beta; \rightarrow \rangle, \langle \Box \alpha \rightarrow \alpha; \rightarrow \Box \beta \rangle, \langle \Box \alpha \rightarrow \Box \beta; \rightarrow \alpha \rangle, \langle \Box \alpha \rightarrow; \rightarrow \alpha, \Box \beta \rangle, \langle \rightarrow \alpha, \Box \beta; \Box \alpha \rightarrow \rangle, \langle \rightarrow \alpha; \Box \alpha \rightarrow \Box \beta \rangle, \langle \rightarrow \Box \beta; \Box \alpha \rightarrow \alpha \rangle$ and $\langle \rightarrow; \Box \alpha \rightarrow \alpha, \Box \beta \rangle$, respectively.

CASE 3. S is the lower sequent $\Gamma, \Delta \rightarrow \Theta, \Lambda$ of an acceptable cut with the upper sequents $\Gamma \rightarrow \Theta, \alpha$ and $\alpha, \Delta \rightarrow \Lambda$, where $\alpha \in \text{Sub}(\Gamma, \Delta, \Theta, \Lambda)$. Let $\langle \Gamma^1; \Gamma^2 \rangle, \langle \Theta^1; \Theta^2 \rangle, \langle \Delta^1; \Delta^2 \rangle$ and $\langle \Lambda^1; \Lambda^2 \rangle$ be partitions of Γ, Θ, Δ and Λ , respectively, so that $\langle \Gamma^1, \Delta^1 \rightarrow \Theta^1, \Lambda^1; \Gamma^2, \Delta^2 \rightarrow \Theta^2, \Lambda^2 \rangle$ is a partition of S .

SUBCASE 3.1. $\alpha \in \text{Sub}(\Gamma^1, \Delta^1, \Theta^1, \Lambda^1)$. By the hypothesis of induction, the

partitions $\langle \Gamma^1 \rightarrow \Theta^1, \alpha ; \Gamma^2 \rightarrow \Theta^2 \rangle$ and $\langle \alpha, \Delta^1 \rightarrow \Lambda^1 ; \Delta^2 \rightarrow \Lambda^2 \rangle$ of the upper sequents have interpolants γ and δ , respectively. Then $\gamma \vee \delta$ is an interpolant of $\langle \Gamma^1, \Delta^1 \rightarrow \Theta^1, \Lambda^1 ; \Gamma^2, \Delta^2 \rightarrow \Theta^2, \Lambda^2 \rangle$.

SUBCASE 3.2. $\alpha \in \text{Sub}(\Gamma^2, \Delta^2, \Theta^2, \Lambda^2)$. By the hypothesis of induction, the partitions $\langle \Gamma^1 \rightarrow \Theta^1 ; \Gamma^2 \rightarrow \Theta^2, \alpha \rangle$ and $\langle \Delta^1 \rightarrow \Lambda^1 ; \alpha, \Delta^2 \rightarrow \Lambda^2 \rangle$ of the upper sequents have interpolants γ and δ , respectively. Then $\gamma \wedge \delta$ is an interpolant of $\langle \Gamma^1, \Delta^1 \rightarrow \Theta^1, \Lambda^1 ; \Gamma^2, \Delta^2 \rightarrow \Theta^2, \Lambda^2 \rangle$.

CASE 4. S is the lower sequent $\Box \Gamma \rightarrow \Box \Theta, \Box \alpha$ of an application of $(\rightarrow \Box)$ with the upper sequents $\Gamma, \Box \Gamma \rightarrow \Box \Theta, \alpha$ and $\Box \perp, \Gamma \rightarrow \alpha$. Let $\langle \Gamma^1 ; \Gamma^2 \rangle$ and $\langle \Theta^1 ; \Theta^2 \rangle$ be partitions of Γ and Θ , respectively. We must consider the partitions $\langle \Box \Gamma^1 \rightarrow \Box \Theta^1 ; \Box \Gamma^2 \rightarrow \Box \Theta^2, \Box \alpha \rangle$ and $\langle \Box \Gamma^1 \rightarrow \Box \Theta^1, \Box \alpha ; \Box \Gamma^2 \rightarrow \Box \Theta^2 \rangle$ of S . First, by the hypothesis of induction, the partitions $\langle \Gamma^1, \Box \Gamma^1 \rightarrow \Box \Theta^1 ; \Gamma^2, \Box \Gamma^2 \rightarrow \Box \Theta^2, \alpha \rangle$ and $\langle \Gamma^1 \rightarrow ; \Box \perp, \Gamma^2 \rightarrow \alpha \rangle$ of the upper sequents have interpolants γ and δ , respectively. Then it is easy to see that $\Box(\gamma \vee \Box \perp) \wedge \Box \delta$ is an interpolant of the former partition of S . Next, by the hypothesis of induction, the partitions $\langle \Gamma^1, \Box \Gamma^1 \rightarrow \Box \Theta^1, \alpha ; \Gamma^2, \Box \Gamma^2 \rightarrow \Box \Theta^2 \rangle$ and $\langle \Box \perp, \Gamma^1 \rightarrow \alpha ; \Gamma^2 \rightarrow \rangle$ of the upper sequents have interpolants γ' and δ' , respectively. Then $\sim \Box(\sim \gamma' \vee \Box \perp) \vee \sim \Box \sim \delta'$ is an interpolant of the latter partition of S . $\square$

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