A Note on the Length of Proofs

Tsuyoshi YUKAMI

§1. Introduction.

Let PA be Peano arithmetic whose function symbols are $'$, $+$ and $\cdot$. PA^* is the corresponding system to PA with $\cdot$ replaced by a ternary predicate symbol P and the axiom saying that P represents a function. The length of a proof is the maximal length of threads of proof. (For the definition of 'thread', see [16, p. 14].) For terms s and t , $s < t$ is an abbreviation of $\exists x(s + x' = t)$ for some appropriate bound variable x . For a natural number n , $\bar{n}$ denotes the term having $'$ applied n times to 0.

In this paper the language of LK is that of PA^* . For a natural number m , a formula A and a set S of sentences, $LK \mid^m S \rightarrow A$ means that there exists a proof, in LK with length $\leq m$, of a sequent $\Gamma \rightarrow A$ for some finite sequence Γ of sentences in S .

S_0 is the set of sentences of the following forms (1), (2), ..., (13).

- (1) $\forall x(x = x)$ (2) $\forall x \forall y(x = y \supset x' = y')$
- (3) $\forall x \forall y \forall z \forall w(x = y \wedge z = w \supset x + z = y + w)$
- (4) $\forall x \forall y \forall z(x = y \wedge y = z \supset x = z)$
- (5) $\forall x \forall y \forall z(x = y \wedge x = z \supset y = z)$ (6) $\forall x P(x, 0, 0)$
- (7) $\forall x \forall y \forall z(P(x, y, z) \supset P(x, y', z + x))$
- (8) $\forall x \forall y \forall z \forall w \forall u \forall v(x = y \wedge z = w \wedge u = v \wedge P(x, z, u) \supset P(y, w, v))$
- (9) $\bar{k} = \bar{m} + \bar{n}$, where m plus n is equal to k .
- (10) $0 = 0 + \cdots + 0$
- (11) $\forall x((\overset{m}{x} + \cdots + \overset{m}{x}) + \bar{m} = \overset{m}{x}' + \cdots + \overset{m}{x}')$ (12) $\bar{n} + \cdots + \bar{n} = \bar{m} \times \bar{n}$
- (13) $P(\bar{m}, 0, 0 + \cdots + 0) \wedge \forall x(P(\bar{m}, x, \overset{m}{x} + \cdots + \overset{m}{x}) \supset P(\bar{m}, x', \overset{m}{x}' + \cdots + \overset{m}{x}')) \supset \forall x P(\bar{m}, x, \overset{m}{x} + \cdots + \overset{m}{x})$

For a primitive recursive function $f(n_1, \dots, n_\nu)$, $A_f(x_1, \dots, x_\nu, x)$ denotes the standard formula of PA^* which expresses the graph of f . For the precise definition of A_f , see [7, pp. 243-244].

Semi-RE-formulas are defined inductively as follows. 1. If $f(n_1, \dots, n_\nu)$ is a

primitive recursive function then $A_f(x_1, \dots, x_\nu, 0)$ and $\neg A_f(x_1, \dots, x_\nu, 0)$ are semi-RE-formulas. 2. If A and B are semi-RE-formulas then so are $A \vee B$ and $A \wedge B$. 3. If $A(x)$ is semi-RE-formula then so is $\exists x A(x)$. 4. If t is a term, x does not occur in t and $A(x)$ is a semi-RE-formula then $\forall x(x < t \supset A(x))$ is a semi-RE-formula.

Σ_1 -formulas of PA^* are defined by the same way as Σ_1 -formulas of PA are defined in [6, p. 188]. $I\Sigma_1$ denotes the set of the universal closures of induction for Σ_1 -formulas of PA^* . S_1 is the set of sentences $\forall x \forall y (x = y \supset y = x)$, $\forall x \forall y \forall z \forall w (P(x, y, z) \wedge P(x, y, w) \supset z = w)$, (1), (2), (3), (4), (6), (7), (8) in the definition of S_0 and the transformations into the language of PA^* of the first twelve sentences in the list of the axioms of P^- is [6, p. 188].

THEOREM. *There is a sentence A_0 which satisfies the following properties (1), (2) and (3).*

(1) A_0 is provable in the subtheory $I\Sigma_1 \cup S_1$ of PA^* .

(2) for each primitive recursive function $f(n_1, \dots, n_\nu)$, there exists a natural number m such that

$$LK \mid \frac{m}{S_0 \cup \{A_0\}} \rightarrow A_f(\bar{n}_1, \dots, \bar{n}_\nu, \bar{n})$$

for each natural numbers $n_1, \dots, n_\nu$ and n with $f(n_1, \dots, n_\nu)$ equal to n .

(3) For each semi-RE-formula $A(x_1, \dots, x_\nu)$, there exists a natural number m such that

$$LK \mid \frac{m}{S_0 \cup \{A_0\}} \rightarrow A(\bar{n}_1, \dots, \bar{n}_\nu)$$

for each natural numbers $n_1, \dots, n_\nu$ with $A(\bar{n}_1, \dots, \bar{n}_\nu)$ true.

In the literature there are some sentences which are true but not provable in PA^* . The transformations into the language of PA^* of $A_q(\bar{q})$ in [7, p. 208], *Consis* in [7, p. 210] and Paris-Harrington principle in [14, p. 1134] are typical examples of such sentences. Notice that the transformations of $A_q(\bar{q})$, *Consis* and Paris-Harrington principle are of the forms $\forall x B_1(\bar{q}, x)$, $\forall x B_2(\bar{r}, x)$ and $\forall x \forall y \forall z B_3(x, y, z)$, respectively, where $B_1(y, x)$, $B_2(y, x)$ and $B_3(x, y, z)$ are semi-RE-formulas and $B_1(\bar{q}, \bar{n})$, $B_2(\bar{r}, \bar{n})$ and $B_3(\bar{n}_1, \bar{n}_2, \bar{n}_3)$ are true for each natural numbers n_1, n_2, n_3 and n . Hence the following corollary is immediate

COROLLARY. *Let A_0 be any sentence satisfying the property (3) in Theorem.*

Then there is a natural number m such that $LK \mid \frac{m}{S_0 \cup \{A_0\}} \rightarrow B_1(\bar{q}, \bar{n}) \wedge B_2(\bar{r}, \bar{n}) \wedge B_3(\bar{n}_1, \bar{n}_2, \bar{n}_3)$

for each natural numbers n_1, n_2, n_3 and n .

Note that the sentences (1), (2), ..., (8) in the definition of S_0 are provable in PA^* and all sentences of the form (13) are provable in PA^* with length 1. Further note that all equations of the forms (9) and (10) are provable in PA^* with length ≤ 25 (see [15] and Fact 4 in [17]). Hence the following problem is very interesting.

PROBLEM. *Is there a natural number m such that all equations of the forms (11) and (12) are provable in $PA(PA^*)$ with length $\leq m$?*

(In the introduction of [17] we proposed this problem and pointed out that the affirmative answer to the problem implies the negative answer to Kreisel's conjecture. For the literature related to Kreisel's conjecture, see [1], [2], [8], [9], references in [9], [12], [13], [18] and [19].)

In §2 we shall prove Theorem. In §3 we shall give three remarks. In the first remark we shall point out a relation between Theorem in [18] and the proof of Theorem in §2. The purpose of this remark is to explain that Theorem in [18] shows an importance of 'detour' in a proof for making the length of the proof short. The second remark is to explain that Theorem shows a proof-theoretic meaning of MRDP theorem. (MRDP theorem is the theorem asserting that every recursive enumerable predicate is diophantine.) The last remark is related to the k-provability problem of PA . (For the literature related to the k-provability problems of LK and PA , see [3], [5], [8], [9], [11] and [13].)

§2. Proof of Theorem.

First of all we define E-formulas inductively as follows. 1. Atomic formulas are E-formulas. 2. If A and B are E-formulas then so are $A \vee B$ and $A \wedge B$. 3. If $A(x)$ is an E-formula then so is $\exists x A(x)$.

$A_1, A_2, \dots, A_8$ are the sentences (1), (2), ..., (8) in the definition of S_0 . $\varphi(m)$ and $\psi(k, m)$ are the functions defined as follows. 1. $\varphi(0)$ is 4. 2. $\varphi(m+1)$ is $\varphi(m) + m^2 + 5 \times m + 18$. 3. $\psi(0, m)$ is $\max(\varphi(m) + m^2 + 10 \times m + 27, 3 \times m + 99)$. 4. $\psi(k+1, m)$ is $\psi(k, m) + m^2 + 16 \times k^2 + 4 \times k \times m + 36 \times k + m + 3$.

LEMMA 1.

(1) *Let m be the number of function symbols in a term $t(x_1, \dots, x_\nu)$. For each natural numbers $n_1, \dots, n_\nu$; if $t(\overline{n_1}, \dots, \overline{n_\nu})$ represents a natural number n , then $\Gamma, A_4, A_3, A_2, A_1 \rightarrow \bar{n} = t(\overline{n_1}, \dots, \overline{n_\nu})$ is provable in LK with length $\leq \varphi(m)$ for some finite sequence Γ , with length $\leq m$, of equations of the form (9) is S_0 .*

(2) $\overline{n} + \dots + \overline{n} = (\overline{n \times m}), P(\overline{m}, 0, 0 + \dots + 0) \wedge \forall x (P(\overline{m}, x, x + \dots + x) \supset$

$P(\bar{m}, x', x' + \dots + x') \supset \forall x P(\bar{m}, x, x + \dots + x), 0 = 0 + \dots + 0, \forall x((x + \dots + x) + \bar{m} = x' + \dots + x'), A_8, A_7, A_6, A_1 \rightarrow P(\bar{m}, \bar{n}, \bar{m} \times \bar{n})$ is provable in **LK** with length 50.

(3) Let m be the number of function symbols in an E -formula $B(x_1, \dots, x_\nu)$ and k be that of logical symbols in it. For each natural numbers $n_1, \dots, n_\nu$; if $B(\bar{n}_1, \dots, \bar{n}_\nu)$ is true then $\Gamma, A_8, A_7, \dots, A_1 \rightarrow B(\bar{n}_1, \dots, \bar{n}_\nu)$ is provable in **LK** with length $\leq \psi(k, m)$ for some finite sequence Γ , with length $\leq m + 4 \times (k + 1)$, of sentences of the forms (9), (10), (11), (12) and (13) in S_0 .

PROOF.

(1) is proved by induction on the number of function symbols in $t(x_1, \dots, x_\nu)$.

(2) $0 = 0 + \dots + 0, A_8, A_6, A_1 \rightarrow P(\bar{m}, 0 + \dots + 0)$ is provable in **LK** with length 16. $\forall x((x + \dots + x) + \bar{m} = x' + \dots + x'), A_8, A_7, A_1 \rightarrow P(\bar{m}, a, a + \dots + a) \supset P(\bar{m}, a', a' + \dots + a')$ is provable in **LK** with length 24. Hence $P(\bar{m}, 0, 0 + \dots + 0) \wedge \forall x(P(\bar{m}, x, x + \dots + x) \supset P(\bar{m}, x', x' + \dots + x')) \supset \forall x P(\bar{m}, x, x + \dots + x), 0 = 0 + \dots + 0, \forall x((x + \dots + x) + \bar{m} = x' + \dots + x'), A_8, A_7, A_6, A_1 \rightarrow P(\bar{m}, \bar{n}, \bar{n} + \dots + \bar{n})$ is provable in **LK** with length 32 and so $\bar{n} + \dots + \bar{n} = (\bar{n} \times \bar{m}), P(\bar{m}, 0, 0 + \dots + 0) \wedge \forall x(P(\bar{m}, x, x + \dots + x) \supset P(\bar{m}, x', x' + \dots + x')) \supset \forall x P(\bar{m}, x, x + \dots + x), 0 = 0 + \dots + 0, \forall x((x + \dots + x) + \bar{m} = x' + \dots + x'), A_8, A_7, A_6, A_1 \rightarrow P(\bar{m}, \bar{n}, \bar{m} \times \bar{n})$ is provable in **LK** with length 50.

(3) is proved by induction on the number of logical symbols in $B(x_1, \dots, x_\nu)$. In the basis step, use (1) and (2).

Now Theorem is immediate from (3) in Lemma 1, the following lemma and the fact that each provable sentence in PA^* is true.

LEMMA 2. *There is a sentence A_0 which satisfies the following properties (1), (2) and (3).*

(1) A_0 is provable in the subtheory $I\Sigma_1 \cup S_1$ of PA^* .

(2) For each primitive recursive function $f(n_1, \dots, n_\nu)$, there exists an E -formula $B(x_1, \dots, x_\nu, x)$ such that $A_0 \rightarrow \forall x_1 \dots \forall x_\nu \forall x (A_f(x_1, \dots, x_\nu, x) \equiv B(x_1, \dots, x_\nu, x))$ is provable in **LK**.

(3) For each semi-RE-formula $A(x_1, \dots, x_\nu)$, there exists an E -formula $B(x_1, \dots, x_\nu)$ such that $A_0 \rightarrow \forall x_1 \dots \forall x_\nu (A(x_1, \dots, x_\nu) \equiv B(x_1, \dots, x_\nu))$ is provable in **LK**.

PROOF. This lemma is a careful formalization of the proof given in [4] of MRDP Theorem. For details, see §4 and §6 is [6] with the following two changes added. (1) Since we need the equivalence on the last line in [6, p. 198], we must work in $I\Sigma_1$ instead of $I\Sigma_0 + \text{exp}$. Notice that exp is provable in $I\Sigma_1$. Further notice that $\forall x_1 \dots \forall x_\nu \exists x A_f(x_1, \dots, x_\nu, x)$ is provable in $I\Sigma_1$ for each primitive recursive function $f(n_1, \dots, n_\nu)$. Hence $\neg A_f(x_1, \dots, x_\nu, 0) \equiv \exists x A_f(x_1, \dots, x_\nu, x')$ is

provable in $I\Sigma_1$. (2) To replace $I\Sigma_1$ by a single sentence we use the truth definition for Σ_1 -formulas insted of that for Σ_0 -formulas.

§3. Remarks.

REMARK 1. It seems that there exists a relation between Theorem in [18] and the proof of Theorem in §2. Though the relation is not rigorous, the author wants to explain it.

Roughly speaking, Theorem in [18] asserts that we can divide a short (in contrast with n) proof in PA of $A(\bar{n})$ into a proof in LK and 'purely number-theoretic' proof. The former proof proves $\forall x(B(x) \supset A(x))$ for some formula $B(x)$. The latter proof proves $B(\bar{n})$ and the length of it is short in contrast with n . For more precise details, see [18].

In the proof of Theorem in §2, we gave a short proof of true semi-RE formula $A(\bar{n})$ in the following manner. First we found an E-formula $B(x)$ and gave a proof in LK of $A_0 \rightarrow \forall x(B(x) \equiv A(x))$. Second we gave a short proof in LK of $\Gamma \rightarrow B(\bar{n})$ for some finite sequence Γ of sentences in S_0 . Finally, combining the above two proofs, we gave a short proof in LK of $\Gamma, A_0 \rightarrow A(\bar{n})$. The short proof of $\Gamma \rightarrow B(\bar{n})$ corresponds to the 'purely number-theoretic' proof in Theorem and the proof in LK of $A_0 \rightarrow \forall x(B(x) \equiv A(x))$ corresponds to the proof in LK in Theorem.

The reason why we cannot make the above relation rigorous is the fact that S_0 contains equations of the forms (11) and (12) in the definition of S_0 .

REMARK 2. Lemma 1 shows a proof-theoretic meaning of the notion 'diophantine'. Lemma 2 is a careful formalization of the proof given in [4] of MRDP theorem. Theorem shows a proof-theoretic meaning of MRDP theorem.

REMARK 3. We can conclude that the affirmative answer to Problem in the introduction implies the negative answer to the k-provability problem of PA , because the following proposition holds.

PROPOSITION. For each recursive enumerable subset X of ω , there exist a natural number m and a formula $A(x)$ such that for each natural number n the following propositions (1), (2) and (3) are equivalent.

(1) n belongs to X .

(2) $LK \vdash^m S_0 \rightarrow A(\bar{n})$.

(3) $A(\bar{n})$ is provable in PA^* .

PROOF. Take any recursive enumerable subset X of ω . Then, by MRDP theorem, there are polynomials $g(n, n_1, \dots, n_\nu)$ and $h(n, n_1, \dots, n_\nu)$ with natural

number coefficients for which the following two propositions are equivalent.

- (a) n belongs to X .
- (2) $g(n, n_1, \dots, n_\nu)$ is equal to $h(n, n_1, \dots, n_\nu)$ for some natural numbers $n_1, \dots, n_\nu$.

We can construct an E-formula $A(x)$ which expresses naturally the proposition (2) above. Now Proposition is immediate from (3) in Lemma 1 and the fact that each provable sentence in PA^* is true.

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